

Exact Traveling Wave Solutions of Extended Kadomtsev–Petviashvili Equation via Modified Rational Sine–Cosine and New Extended Direct Algebraic Methods

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Abstract: This paper investigates the $(2 + 1)$ –dimensional extended Kadomtsev–Petviashvili equation (eKPE), importantly describing multidimensional nonlinear wave phenomena. Its exact traveling wave solutions are constructed using the modified rational sine–cosine method and the new extended direct algebraic method (NEDAM). The obtained solutions are expressed in diverse functional forms, including trigonometric, hyperbolic, rational, exponential, and complex-valued structures. Graphical representations of selected solutions are plotted to illustrate their physical characteristics, showing various nonlinear wave behaviors such as a singular periodic wave, a periodic wave solution, a kink soliton wave, and a singular kink-type solution. In addition, a linear stability analysis is performed to examine the modulation behavior of the model under different parameter choices. The results demonstrate that the proposed analytical approaches provide an efficient and systematic framework for generating a wide spectrum of exact solutions to the eKPE, thus enriching the understanding of its nonlinear wave dynamics.

Key-Words: - Exact solutions, Traveling wave solutions, Extended Kadomtsev–Petviashvili equation, Modified rational sine–cosine method, NEDAM, Stability analysis, Soliton solutions

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1 Introduction

Nonlinear partial differential equations (NPDEs) have been widely used to describe complex physical phenomena in engineering and physics, including electrostatics, [1], fluid mechanics, [2], [3], quantum mechanics, [4], optical fibers, [5], oceanology, [6], and finance, [7]. The derivation of exact solitary wave solutions for these physical phenomena typically involves traveling wave transformations, which facilitate the interpretation of the resulting solutions as various physical wave types, including shallow water and acoustic waves. In addition, nonlinear wave phenomena have been extensively investigated in geophysical fluid dynamics through Rossby solitary wave models governed by combined ZK–mZK equations, [8]. To construct exact traveling wave solutions of nonlinear evolution equations, numerous analytical techniques have been developed. Among the widely used approaches are the Ansatz method, [9], rational $(\frac{G'}{G})$ –expansion method, [10],

modified extended tanh function method, [11], $(\frac{G'}{G}, \frac{1}{G})$ –expansion method, [12], Lie symmetry method, [13], and the complete discrimination system method, [14]. These analytical frameworks have been successfully applied to a variety of NPDEs, revealing diverse nonlinear wave structures such as solitons, periodic waves, and singular solutions, [15], [16].

In recent years, the new extended direct algebraic method (NEDAM) has emerged as an efficient and systematic analytical tool for deriving exact solutions of nonlinear differential equations. The effectiveness of the NEDAM has been demonstrated for several nonlinear evolution models. For instance, new soliton solutions of the complex Ginzburg–Landau equation were obtained via the NEDAM, [17]. Subsequently, the NEDAM was applied to the coupled Konno–Oono equation for producing various closed–form solutions, [18]. More recently, the NEDAM has been successfully employed to construct soliton solutions of the Zakharov equation

arising in plasma physics, [19], and novel bright and kink wave solutions of nonlinear evolution equations, [20]. These studies confirm that the NEDAM provides a unified algebraic framework capable of generating broad families of exact solutions expressed in different functional forms.

Parallel to these developments, sine-cosine-based techniques have proven to be effective tools for constructing exact traveling wave solutions. In particular, modified rational sine-cosine formulations have been shown to generate diverse analytical wave structures, including periodic, solitary, and kink-type solutions, [21]. Related sine-cosine-type approaches have also been successfully applied to various nonlinear evolution equations and optical models, [22], [23], highlighting their flexibility and effectiveness.

Nonlinear evolution equations such as the Kadomtsev–Petviashvili (KP) equation have attracted considerable attention due to their important role in describing multidimensional nonlinear wave propagation phenomena. As a higher-dimensional extension of the Korteweg–de Vries (KdV) equation, the KP equation has been extensively applied to investigate weakly nonlinear dispersive waves and various nonlinear wave structures in fluid mechanics and related physical systems, [24], [25]. Analytical studies have produced multiple soliton and exact wave solutions for extended KP-type equations using modern techniques such as Painlevé analysis and Hirota’s bilinear method, [26]. Furthermore, generalized $(2 + 1)$ -dimensional KP models have been investigated to obtain exact solutions together with stability properties, [27], while lump solutions have been reported for extended KP-type equations, [28]. Recent studies have shown that combining exact analytical solutions with linear or modulation stability analysis provides additional insight into the physical relevance of nonlinear wave models. For example, stability properties have been investigated for several higher-dimensional nonlinear evolution equations in conjunction with exact solution structures, [29], [30].

Among KP-type generalizations, the $(2 + 1)$ -dimensional extended Kadomtsev–Petviashvili equation (eKPE) incorporates higher-order dispersion and nonlinear effects, enabling richer nonlinear wave dynamics. The $(2 + 1)$ -dimensional eKPE is given by, [26], [27], [28],

$$u_{xt} + \varepsilon(u_{xxxx} + 6u_x u_{xx}) + \eta u_{yy} + a u_{xx} + b u_{xy} = 0, \quad (1)$$

where $u = u(x, y, t)$ is the unknown function and ε, η, a , and b are real parameters associated with the physical characteristics of the system.

The main objective of this paper is to derive new exact traveling wave solutions of the $(2 +$

$1)$ -dimensional extended Kadomtsev–Petviashvili equation (eKPE) using the modified rational sine-cosine method and the new extended direct algebraic method (NEDAM). Although various analytical techniques have been applied to KP-type equations, the implementation of these two approaches for the eKPE has received little attention to date. The present study addresses this gap by providing a unified framework based on these two techniques. Furthermore, a linear stability analysis is carried out to examine the modulation behavior of the model.

The paper is organized as follows. Section 2 presents the modified rational sine-cosine method and the NEDAM. Section 3 derives exact traveling wave solutions of the eKPE. Section 4 illustrates graphs of selected solutions and analyzes their physical behaviors. Section 5 presents the stability analysis of the considered model. Section 6 summarizes the main conclusions.

2 Methodology

In this section, we present two effective analytical methods, the modified rational sine-cosine method and the new extended direct algebraic method (NEDAM), which are used to derive exact traveling wave solutions of Eq. (1). The details of each method are described below.

2.1 Traveling wave transformation

Consider the general nonlinear partial differential equation

$$P(u, u_t, u_x, u_y, u_{xt}, u_{yt}, u_{yx}, \dots) = 0, \quad (2)$$

where P is a polynomial function of the dependent variable $u = u(x, y, t)$ and its partial derivatives. Using the traveling wave transformation

$$u(x, y, t) = U(\xi), \quad \xi = c_1 x + c_2 y - \rho t, \quad (3)$$

reduces Eq. (2) to the ordinary differential equation

$$G(U, U', U'', U''', \dots) = 0. \quad (4)$$

2.2 The modified rational sine-cosine method

The main steps of the modified rational sine-cosine method are summarized as follows.

Step 1: Assume that the solutions of Eq. (4), based on the modified rational sine-cosine method introduced in, [21], can be expressed as

$$U(\xi) = \frac{1 + A \sin(\xi)}{B + F \cos(\xi)}, \quad \cos(\xi) \neq -\frac{B}{F}, \quad (5)$$

or

$$U(\xi) = \frac{1 + A \cos(\xi)}{B + F \sin(\xi)}, \quad \sin(\xi) \neq -\frac{B}{F}, \quad (6)$$

where A , B , and F are constants to be determined.

Step 2: Substituting Eq. (5) or Eq. (6) into Eq. (4), we collect terms with identical powers of $\sin(\xi)$ and $\cos(\xi)$ and equate their coefficients to zero. This yields a system of algebraic equations.

Step 3: Solving the algebraic system determines the constants A , B , and F . Substitution of these values into Eq. (5) or Eq. (6) provides the exact traveling wave solutions of Eq. (2).

2.3 The new extended direct algebraic method

The main steps of the NEDAM, [17], are summarized as follows.

Step 1: Assume that the solution of Eq. (4) can be expressed by a polynomial in $Q(\xi)$ as follows.

$$U(\xi) = \sum_{j=0}^N k_j Q^j(\xi), \quad k_N \neq 0, \quad (7)$$

where $Q(\xi)$ satisfies the ODE in the form

$$Q'(\xi) = \ln(A)(\alpha + \beta Q(\xi) + \sigma Q(\xi)^2), \quad (8)$$

where $A > 0$, $A \neq 1$ and $k_j (0 \leq j \leq N)$, α, β, σ are constants to be calculated later.

Step 2: The term N is a positive integer determined by balancing the highest-order derivative term with the highest-power nonlinear term in Eq. (4). Denoting the degree of $U(\xi)$ by

$$\text{Deg}(U(\xi)) = N, \quad (9)$$

the degree of the derivative term is defined as

$$\text{Deg}\left(\frac{d^q U(\xi)}{d\xi^q}\right) = N + q, \quad (10)$$

and the degree of the nonlinear term is given by

$$\text{Deg}\left[(U(\xi))^p \left(\frac{d^q U(\xi)}{d\xi^q}\right)^s\right] = Np + s(N + q), \quad (11)$$

where p , q , and s are positive integers. By using Eqs. (9)-(11) and equating the corresponding highest degrees, the value of N can be determined. This procedure is known as the homogeneous balance principle.

Step 3: Substituting Eq. (7) and Eq. (8) into Eq. (4), and gathering the terms that have like powers of $Q(\xi)$ and putting their coefficients equal to zero, we obtain a set of algebraic equations, which can be

solved using symbolic computation software such as Maple.

Step 4: The twelve cases of solutions of Eq. (8) are given as follows:

Case 1: If $\Delta = \beta^2 - 4\alpha\sigma < 0$ and $\sigma \neq 0$, then

$$Q_1(\xi) = \frac{-\beta}{2\sigma} + \frac{\sqrt{-\Delta}}{2\sigma} \tan_A \left(\frac{\sqrt{-\Delta} \xi}{2} \right), \quad (12)$$

$$Q_2(\xi) = \frac{-\beta}{2\sigma} - \frac{\sqrt{-\Delta}}{2\sigma} \cot_A \left(\frac{\sqrt{-\Delta} \xi}{2} \right), \quad (13)$$

$$Q_3(\xi) = \frac{-\beta}{2\sigma} + \frac{\sqrt{-\Delta}}{2\sigma} \left(\tan_A(\sqrt{-\Delta}\xi) \pm \sqrt{pq} \sec_A(\sqrt{-\Delta}\xi) \right), \quad (14)$$

$$Q_4(\xi) = \frac{-\beta}{2\sigma} + \frac{\sqrt{-\Delta}}{2\sigma} \left(\cot_A(\sqrt{-\Delta}\xi) \pm \sqrt{pq} \csc_A(\sqrt{-\Delta}\xi) \right), \quad (15)$$

$$Q_5(\xi) = \frac{-\beta}{2\sigma} - \frac{\sqrt{-\Delta}}{4\sigma} \left(\tan_A \left(\frac{\sqrt{-\Delta}\xi}{4} \right) - \cot_A \left(\frac{\sqrt{-\Delta}\xi}{4} \right) \right). \quad (16)$$

Case 2: If $\Delta = \beta^2 - 4\alpha\sigma > 0$ and $\sigma \neq 0$, then

$$Q_6(\xi) = \frac{-\beta}{2\sigma} - \frac{\sqrt{\Delta}}{2\sigma} \tanh_A \left(\frac{\sqrt{\Delta} \xi}{2} \right), \quad (17)$$

$$Q_7(\xi) = \frac{-\beta}{2\sigma} - \frac{\sqrt{\Delta}}{2\sigma} \coth_A \left(\frac{\sqrt{\Delta} \xi}{2} \right), \quad (18)$$

$$Q_8(\xi) = \frac{-\beta}{2\sigma} - \frac{\sqrt{\Delta}}{2\sigma} \left(\tanh_A(\sqrt{\Delta}\xi) \pm i\sqrt{pq} \text{sech}_A(\sqrt{\Delta}\xi) \right), \quad (19)$$

$$Q_9(\xi) = \frac{-\beta}{2\sigma} - \frac{\sqrt{\Delta}}{2\sigma} \left(\coth_A(\sqrt{\Delta}\xi) \pm i\sqrt{pq} \text{csch}_A(\sqrt{\Delta}\xi) \right), \quad (20)$$

$$Q_{10}(\xi) = \frac{-\beta}{2\sigma} - \frac{\sqrt{\Delta}}{4\sigma} \left(\tanh_A \left(\frac{\sqrt{\Delta}\xi}{4} \right) + \coth_A \left(\frac{\sqrt{\Delta}\xi}{4} \right) \right). \quad (21)$$

Case 3: If $\alpha\sigma > 0$ and $\beta = 0$, then

$$Q_{11}(\xi) = \sqrt{\frac{\alpha}{\sigma}} \tan_A(\alpha\sigma\xi), \quad (22)$$

$$Q_{12}(\xi) = -\sqrt{\frac{\alpha}{\sigma}} \cot_A(\alpha\sigma\xi), \quad (23)$$

$$Q_{13}(\xi) = \sqrt{\frac{\alpha}{\sigma}} \left(\tan_A(2\sqrt{\alpha\sigma}\xi) \pm \sqrt{pq} \sec_A(2\sqrt{\alpha\sigma}\xi) \right), \quad (24)$$

$$Q_{14}(\xi) = -\sqrt{\frac{\alpha}{\sigma}} \left(\cot_A(2\sqrt{\alpha\sigma}\xi) \pm \sqrt{pq} \csc_A(2\sqrt{\alpha\sigma}\xi) \right), \quad (25)$$

$$Q_{15}(\xi) = \frac{1}{2} \sqrt{\frac{\alpha}{\sigma}} \left(\tan_A \left(\frac{\sqrt{\alpha\sigma}\xi}{2} \right) - \cot_A \left(\frac{\sqrt{\alpha\sigma}\xi}{2} \right) \right). \quad (26)$$

Case 4: If $\alpha\sigma < 0$ and $\beta = 0$, then

$$Q_{16}(\xi) = -\sqrt{\frac{-\alpha}{\sigma}} \tanh_A(\sqrt{-\alpha\sigma}\xi), \quad (27)$$

$$Q_{17}(\xi) = -\sqrt{\frac{-\alpha}{\sigma}} \coth_A(\sqrt{-\alpha\sigma}\xi), \quad (28)$$

$$Q_{18}(\xi) = -\sqrt{\frac{-\alpha}{\sigma}} \left(\tanh_A(2\sqrt{-\alpha\sigma}\xi) \pm i\sqrt{pq} \operatorname{sech}_A(2\sqrt{-\alpha\sigma}\xi) \right), \quad (29)$$

$$Q_{19}(\xi) = -\sqrt{\frac{-\alpha}{\sigma}} \left(\coth_A(2\sqrt{-\alpha\sigma}\xi) \pm i\sqrt{pq} \operatorname{csch}_A(2\sqrt{-\alpha\sigma}\xi) \right), \quad (30)$$

$$Q_{20}(\xi) = -\frac{1}{2}\sqrt{\frac{-\alpha}{\sigma}} \left(\tanh_A\left(\frac{\sqrt{-\alpha\sigma}\xi}{2}\right) + \coth_A\left(\frac{\sqrt{-\alpha\sigma}\xi}{2}\right) \right). \quad (31)$$

Case 5: If $\sigma = \alpha$ and $\beta = 0$, then

$$Q_{21}(\xi) = \tan_A(\alpha\xi), \quad (32)$$

$$Q_{22}(\xi) = -\cot_A(\alpha\xi), \quad (33)$$

$$Q_{23}(\xi) = \tan_A(2\alpha\xi) \pm \sqrt{pq} \sec_A(2\alpha\xi), \quad (34)$$

$$Q_{24}(\xi) = -\cot_A(2\alpha\xi) \pm \sqrt{pq} \csc_A(2\alpha\xi), \quad (35)$$

$$Q_{25}(\xi) = \frac{1}{2} \left(\tan_A\left(\frac{\alpha\xi}{2}\right) - \cot_A\left(\frac{\alpha\xi}{2}\right) \right). \quad (36)$$

Case 6: If $\sigma = -\alpha$ and $\beta = 0$, then

$$Q_{26}(\xi) = \tanh_A(\alpha\xi), \quad (37)$$

$$Q_{27}(\xi) = \coth_A(\alpha\xi), \quad (38)$$

$$Q_{28}(\xi) = \tanh_A(2\alpha\xi) \pm i\sqrt{pq} \operatorname{sech}_A(2\alpha\xi), \quad (39)$$

$$Q_{29}(\xi) = \coth_A(2\alpha\xi) \pm i\sqrt{pq} \operatorname{csch}_A(2\alpha\xi), \quad (40)$$

$$Q_{30}(\xi) = \frac{1}{2} \left(\tanh_A\left(\frac{\alpha\xi}{2}\right) + \coth_A\left(\frac{\alpha\xi}{2}\right) \right). \quad (41)$$

Case 7: If $\beta^2 = 4\alpha\sigma$, then

$$Q_{31}(\xi) = -\frac{2\alpha(\beta\xi \ln(A) + 2)}{\beta^2\xi \ln(A)}. \quad (42)$$

Case 8: If $\beta = \lambda$, $\alpha = m\lambda$ ($m \neq 0$), and $\sigma = 0$, then

$$Q_{32}(\xi) = A^{\lambda\xi} - m. \quad (43)$$

Case 9: If $\beta = \sigma = 0$, then

$$Q_{33}(\xi) = \alpha\xi \ln(A). \quad (44)$$

Case 10: If $\beta = \alpha = 0$, then

$$Q_{34}(\xi) = \frac{-1}{\sigma\xi \ln(A)}. \quad (45)$$

Case 11: If $\alpha = 0$ and $\beta \neq 0$, then

$$Q_{35}(\xi) = -\frac{p\beta}{\sigma(\cosh_A(\beta\xi) - i\sinh_A(\beta\xi) + p)}, \quad (46)$$

$$Q_{36}(\xi) = -\frac{\beta(\cosh_A(\beta\xi) + i\sinh_A(\beta\xi))}{\sigma(\cosh_A(\beta\xi) + i\sinh_A(\beta\xi) + q)}. \quad (47)$$

Case 12: If $\beta = \lambda$, $\sigma = m\lambda$ ($m \neq 0$), and $\alpha = 0$, then

$$Q_{37}(\xi) = \frac{pA^{\lambda\xi}}{q - mpA^{\lambda\xi}}. \quad (48)$$

The special generalized trigonometric and hyperbolic functions, [17], are defined in Eq. (49).

$$\begin{aligned} \sin_A(\xi) &= \frac{pA^{i\xi} - qA^{-i\xi}}{2i}, \quad \csc_A(\xi) = \frac{2i}{pA^{i\xi} - qA^{-i\xi}}, \\ \cos_A(\xi) &= \frac{pA^{i\xi} + qA^{-i\xi}}{2}, \quad \sec_A(\xi) = \frac{2}{pA^{i\xi} + qA^{-i\xi}}, \\ \tan_A(\xi) &= -i\frac{pA^{i\xi} - qA^{-i\xi}}{pA^{i\xi} + qA^{-i\xi}}, \quad \cot_A(\xi) = i\frac{pA^{i\xi} + qA^{-i\xi}}{pA^{i\xi} - qA^{-i\xi}}, \\ \sinh_A(\xi) &= \frac{pA^\xi - qA^{-\xi}}{2}, \quad \csc_A(\xi) = \frac{2}{pA^\xi - qA^{-\xi}}, \\ \cosh_A(\xi) &= \frac{pA^\xi + qA^{-\xi}}{2}, \quad \operatorname{sech}_A(\xi) = \frac{2}{pA^\xi + qA^{-\xi}}, \\ \tanh_A(\xi) &= \frac{pA^\xi - qA^{-\xi}}{pA^\xi + qA^{-\xi}}, \quad \coth_A(\xi) = \frac{pA^\xi + qA^{-\xi}}{pA^\xi - qA^{-\xi}}, \end{aligned} \quad (49)$$

where p and q are constants greater than zero and called deformation parameters.

Remark: All obtained solutions (Cases 1–12) of Eq. (8) were verified using Maple, confirming their validity.

3 Implementation of the methods

This section presents the application of the modified rational sine-cosine method and the NEDAM to derive explicit exact solutions of Eq. (1).

Before applying the methods, the solution of Eq. (1) is assumed as the traveling wave form in Eq. (3). Substituting Eq. (3) into Eq. (1) yields the following nonlinear ordinary differential equation:

$$6\varepsilon c_1^3 U' U'' + (ac_1^2 + (bc_2 - \rho)c_1 + \eta c_2^2) U'' + \varepsilon c_1^4 U^{(4)} = 0. \quad (50)$$

Then integrating Eq. (50) once with respect to ξ and setting the constant of integration to zero, we obtain

$$3\varepsilon c_1^3 (U')^2 + (ac_1^2 + (bc_2 - \rho)c_1 + \eta c_2^2) U' + \varepsilon c_1^4 U''' = 0. \quad (51)$$

3.1 Application of the modified rational sine-cosine method

In this subsection, the modified rational sine-cosine and modified rational cosine-sine methods are utilized to obtain exact traveling wave solutions of Eq. (1). The following procedures are carried out to obtain these solutions.

3.1.1 The modified rational sine-cosine method

Suppose that the exact solution of Eq. (51) has the form

$$U(\xi) = \frac{1 + A \sin(\xi)}{B + F \cos(\xi)}, \quad \cos(\xi) \neq -\frac{B}{F}. \quad (52)$$

Substituting Eq. (52) into Eq. (51) and collecting the coefficients of $\sin(2\xi)$, $\sin^j(\xi)$ ($j = 1, 3$), and $\cos^i(\xi)$ ($i = 0, 1, 2, 3$), and setting each coefficient to zero, we obtain a system of seven algebraic equations as follows.

$$\begin{aligned} \sin(2\xi) : & 2BF^2\epsilon c_1^4 + 3ABF\epsilon c_1^3 + BF^2bc_1c_2 \\ & + BF^2ac_1^2 + BF^2\eta c_2^2 - BF^2\rho c_1 = 0, \\ \sin(\xi) : & -B^2F\epsilon c_1^4 + 5F^3\epsilon c_1^4 - B^2F\rho c_1 - F^3\rho c_1 \\ & + B^2Fac_1^2 + B^2F\eta c_2^2 + F^3ac_1^2 + F^3\eta c_2^2 \\ & + 6AF^2\epsilon c_1^3 + B^2Fbc_1c_2 + F^3bc_1c_2 = 0, \\ \sin^3(\xi) : & F^3\epsilon c_1^4 - F^3ac_1^2 - F^3bc_1c_2 - F^3\eta c_2^2 \\ & + F^3\rho c_1 = 0, \\ \cos^0(\xi) : & -4AB^2F\epsilon c_1^4 + 6AF^3\epsilon c_1^4 + 3A^2F^2\epsilon c_1^3 \\ & + AB^2Fac_1^2 + AB^2Fbc_1c_2 + 3F^2\epsilon c_1^3 \\ & + AB^2F\eta c_2^2 - AB^2F\rho c_1 = 0, \\ \cos(\xi) : & 2ABF^2ac_1^2 + 2ABF^2bc_1c_2 - AB^3\rho c_1 \\ & - AB^3\epsilon c_1^4 + 4ABF^2\epsilon c_1^4 + 6A^2BF\epsilon c_1^3 \\ & + AB^3ac_1^2 + AB^3bc_1c_2 + AB^3\eta c_2^2 \\ & + 2ABF^2\eta c_2^2 - 2ABF^2\rho c_1 = 0, \\ \cos^2(\xi) : & 2AB^2Fac_1^2 + 2AB^2Fbc_1c_2 - 2AB^2F\rho c_1 \\ & + 4AB^2F\epsilon c_1^4 - 4AF^3\epsilon c_1^4 + 3A^2B^2\epsilon c_1^3 \\ & + 2AB^2F\eta c_2^2 + AF^3ac_1^2 + AF^3bc_1c_2 \\ & + AF^3\eta c_2^2 - AF^3\rho c_1 - 3F^2\epsilon c_1^3 = 0, \\ \cos^3(\xi) : & -ABF^2\epsilon c_1^4 + ABF^2ac_1^2 + ABF^2bc_1c_2 \\ & + ABF^2\eta c_2^2 - ABF^2\rho c_1 = 0. \end{aligned} \quad (53)$$

Solving the algebraic system given in Eq. (53) with the aid of the Maple program, we obtain the parameters A , B , F , c_1 , c_2 and ρ as follows.

Result 1

$$\begin{aligned} A &= \pm F \sqrt{-\frac{1}{B^2 - F^2}}, \quad B = B, \quad F = F, \\ \rho &= bc_2 \pm i\eta c_2^2 \sqrt{B^2 - F^2} \mp \frac{i(B^2a - F^2a + \epsilon)}{(B^2 - F^2)^{\frac{3}{2}}}, \\ c_1 &= \mp \sqrt{-\frac{1}{B^2 - F^2}}, \quad c_2 = c_2. \end{aligned} \quad (54)$$

Substituting Eq. (54) into Eq. (52) and using Eq. (3), we obtain the solution of Eq. (1) as shown in Eq. (55).

$$u_1^\pm(x, y, t) = \frac{1 \pm F \sqrt{-\frac{1}{B^2 - F^2}} \sin(c_1x + c_2y - \rho t)}{B + F \cos(c_1x + c_2y - \rho t)}. \quad (55)$$

Result 2

$$\begin{aligned} A &= \pm 1, B = 0, F = \mp \frac{1}{c_1}, c_1 = c_1, c_2 = c_2, \\ \rho &= \frac{-\epsilon c_1^4 + ac_1^2 + bc_1c_2 + \eta c_2^2}{c_1}. \end{aligned} \quad (56)$$

Substituting Eq. (56) into Eq. (52) and employing Eq. (3), we get the solution of Eq. (1) as shown in Eq. (57).

$$u_2^\pm(x, y, t) = \mp \frac{(1 \pm \sin(c_1x + c_2y - \rho t))c_1}{\cos(c_1x + c_2y - \rho t)}. \quad (57)$$

3.1.2 The modified rational cosine-sine method

Suppose that the exact solution of Eq. (51) has the form

$$U(\xi) = \frac{1 + A \cos(\xi)}{B + F \sin(\xi)}, \quad \sin(\xi) \neq -\frac{B}{F}. \quad (58)$$

Substituting Eq. (58) into Eq. (51) and collecting the coefficients of $\sin(2\xi)$, $\sin^j(\xi)$ ($j = 1, 3$), and $\cos^i(\xi)$ ($i = 0, 1, 2, 3$), then setting each coefficient equal to zero, we obtain a system of seven algebraic equations:

$$\begin{aligned} \sin(2\xi) : & -2BF^2\epsilon c_1^4 + 3ABF\epsilon c_1^3 - BF^2bc_1c_2 \\ & - BF^2ac_1^2 - BF^2\eta c_2^2 + BF^2\rho c_1 = 0, \\ \sin(\xi) : & -2ABF^2ac_1^2 + AB^3\rho c_1 + 2ABF^2\rho c_1 \\ & + AB^3\epsilon c_1^4 - 4ABF^2\epsilon c_1^4 + 6A^2BF\epsilon c_1^3 \\ & - AB^3ac_1^2 - AB^3bc_1c_2 - AB^3\eta c_2^2 \\ & - 2ABF^2bc_1c_2 - 2ABF^2\eta c_2^2 = 0, \\ \sin^3(\xi) : & ABF^2\epsilon c_1^4 - ABF^2ac_1^2 - ABF^2bc_1c_2 \\ & - ABF^2\eta c_2^2 + ABF^2\rho c_1 = 0, \end{aligned}$$

$$\begin{aligned}\cos^0(\xi) : & -2AF^3\epsilon c_1^4 + 3A^2B^2\epsilon c_1^3 + 3A^2F^2\epsilon c_1^3 \\ & -3AB^2F\eta c_2^2 - AF^3ac_1^2 - AF^3bc_1c_2 \\ & - AF^3\eta c_2^2 + 3AB^2F\rho c_1 + AF^3\rho c_1 \\ & - 3AB^2Fac_1^2 - 3AB^2Fbc_1c_2 = 0, \\ \cos(\xi) : & B^2F\epsilon c_1^4 - 5F^3\epsilon c_1^4 + 6AF^2\epsilon c_1^3 - B^2Fac_1^2 \\ & + F^3\rho c_1 - B^2F\eta c_2^2 - F^3ac_1^2 - F^3bc_1c_2 \\ & - F^3\eta c_2^2 + B^2F\rho c_1 - B^2Fbc_1c_2 = 0, \\ \cos^2(\xi) : & 2AB^2Fac_1^2 + 2AB^2Fbc_1c_2 + 2AB^2F\eta c_2^2 \\ & + 4AB^2F\epsilon c_1^4 - 4AF^3\epsilon c_1^4 - 3A^2B^2\epsilon c_1^3 \\ & + AF^3ac_1^2 + AF^3bc_1c_2 - 2AB^2F\rho c_1 \\ & + AF^3\eta c_2^2 - AF^3\rho c_1 + 3F^2\epsilon c_1^3 = 0, \\ \cos^3(\xi) : & -F^3\epsilon c_1^4 + F^3ac_1^2 + F^3bc_1c_2 + F^3\eta c_2^2 \\ & - F^3\rho c_1 = 0.\end{aligned}\quad (59)$$

Solving the algebraic system given in Eq. (59) with the aid of the Maple program, we obtain the parameters A , B , F , c_1 , c_2 and ρ as follows.

Result 1

$$\begin{aligned}A &= \pm F\sqrt{-\frac{1}{B^2 - F^2}}, \quad B = B, \quad F = F, \\ \rho &= bc_2 \mp i\eta c_2^2\sqrt{B^2 - F^2} \pm \frac{i(B^2a - F^2a + \epsilon)}{(B^2 - F^2)^{\frac{3}{2}}}, \\ c_1 &= \pm\sqrt{-\frac{1}{B^2 - F^2}}, \quad c_2 = c_2.\end{aligned}\quad (60)$$

Substituting Eq. (60) into Eq. (58) and using the transformation in Eq. (3), we obtain the solution of Eq. (1) as shown in Eq. (61).

$$u_1^\pm(x, y, t) = \frac{1 \pm F\sqrt{-\frac{1}{B^2 - F^2}} \cos(c_1x + c_2y - \rho t)}{B + F \sin(c_1x + c_2y - \rho t)}. \quad (61)$$

Result 2

$$\begin{aligned}A &= \pm 1, B = 0, F = \pm \frac{1}{c_1}, c_1 = c_1, c_2 = c_2, \\ \rho &= \frac{-\epsilon c_1^4 + ac_1^2 + bc_1c_2 + \eta c_2^2}{c_1}.\end{aligned}\quad (62)$$

Substituting Eq. (62) into Eq. (58) and applying Eq. (3), we get the solution of Eq. (1) as expressed in Eq. (63).

$$u_2^\pm(x, y, t) = \pm \frac{(1 \pm \cos(c_1x + c_2y - \rho t))c_1}{\sin(c_1x + c_2y - \rho t)}. \quad (63)$$

3.2 Application of the NEDAM

The NEDAM is used to derive exact traveling wave solutions of Eq. (1). The implementation procedure is outlined below. Using the traveling wave transformation in Eq. (3), the homogeneous balance principle is applied to Eq. (51) to determine the value of the positive integer N . In Eq. (51), the dominant balance occurs between the highest-order derivative term $U'''(\xi)$, whose degree is

$$\text{Deg}(U''') = N + 3, \quad (64)$$

and the nonlinear term $(U'(\xi))^2$, whose degree is

$$\text{Deg}((U')^2) = 2(N + 1). \quad (65)$$

Balancing Eqs. (64) and (65) gives

$$N + 3 = 2(N + 1). \quad (66)$$

Solving Eq. (66), we obtain $N = 1$. Therefore, the solution is assumed in the form

$$U(\xi) = k_0 + k_1Q(\xi). \quad (67)$$

Substituting Eqs. (67) and (8) into Eq. (51), collecting the coefficients of like powers of $Q(\xi)$, namely $Q^i(\xi)$ ($i = 0, 1, 2, 3, 4$), and setting each coefficient equal to zero, we obtain a system of five algebraic equations:

$$\begin{aligned}Q^0(\xi) : & k_1 \ln(A) \alpha bc_1c_2 + 2\epsilon c_1^4 k_1 \ln(A)^3 \alpha^2 \sigma \\ & + 3\epsilon k_1^2 \ln(A)^2 c_1^3 \alpha^2 + k_1 \ln(A) \alpha ac_1^2 \\ & + k_1 \ln(A) \alpha \eta c_2^2 - k_1 \ln(A) \alpha \rho c_1 \\ & + \epsilon c_1^4 k_1 \ln(A)^3 \alpha \beta^2 = 0, \\ Q(\xi) : & 8\epsilon c_1^4 k_1 \ln(A)^3 \alpha \beta \sigma + 6\epsilon k_1^2 \ln(A)^2 c_1^3 \alpha \beta \\ & + k_1 \ln(A) \beta bc_1c_2 + k_1 \ln(A) \beta ac_1^2 \\ & + k_1 \ln(A) \beta \eta c_2^2 - k_1 \ln(A) \beta \rho c_1 \\ & + \epsilon c_1^4 k_1 \ln(A)^3 \beta^3 = 0, \\ Q^2(\xi) : & 8\epsilon c_1^4 k_1 \ln(A)^3 \alpha \sigma^2 + 7\epsilon c_1^4 k_1 \ln(A)^3 \beta^2 \sigma \\ & + 6\epsilon k_1^2 \ln(A)^2 c_1^3 \alpha \sigma + k_1 \ln(A) \sigma bc_1c_2 \\ & + 3\epsilon k_1^2 \ln(A)^2 c_1^3 \beta^2 + k_1 \ln(A) \sigma ac_1^2 \\ & + k_1 \ln(A) \sigma \eta c_2^2 - k_1 \ln(A) \sigma \rho c_1 = 0, \\ Q^3(\xi) : & 6\epsilon k_1^2 \ln(A)^2 c_1^3 \beta \sigma + 12\epsilon c_1^4 k_1 \ln(A)^3 \beta \sigma^2 \\ & = 0, \\ Q^4(\xi) : & 3\epsilon k_1^2 \ln(A)^2 c_1^3 \sigma^2 + 6\epsilon c_1^4 k_1 \ln(A)^3 \sigma^3 \\ & = 0.\end{aligned}\quad (68)$$

Solving the algebraic system given in Eq. (68) with the aid of the Maple program, we obtain the following values of the parameters A , c_1 , c_2 , ρ , k_0 , and k_1 .

Result

$$A = e^{\pm \frac{\sqrt{\varepsilon(4\alpha\sigma - \beta^2)(ac_1^2 + bc_1c_2 + \eta c_2^2 - \rho c_1)}}{\varepsilon(4\alpha\sigma - \beta^2)c_1^2}},$$

$$\rho = \rho, c_1 = c_1, c_2 = c_2,$$

$$k_0 = k_0, k_1 = \mp \frac{2\sqrt{ac_1^2 + (bc_2 - \rho)c_1 + \eta c_2^2}\sigma}{\sqrt{\varepsilon(4\alpha\sigma - \beta^2)c_1}}. \quad (69)$$

Substituting Eq. (69) and the solutions of Eq. (8), which are expressed in Eqs. (12)-(48), into Eq. (67), and using the traveling wave transformation in Eq. (3), we obtain the solutions of Eq. (1) as follows.

Case 1: If $\Delta = \beta^2 - 4\alpha\sigma < 0$ and $\sigma \neq 0$, then

$$u_1^\mp(x, y, t) = k_0 \mp \frac{2\sqrt{ac_1^2 + (bc_2 - \rho)c_1 + \eta c_2^2}\sigma}{\sqrt{-\Delta\varepsilon}c_1} \times$$

$$\left(\frac{-\beta}{2\sigma} + \frac{\sqrt{-\Delta}}{2\sigma} \tanh_A \left(\frac{\sqrt{-\Delta}(c_1x + c_2y - \rho t)}{2} \right) \right), \quad (70)$$

$$u_2^\mp(x, y, t) = k_0 \mp \frac{2\sqrt{ac_1^2 + (bc_2 - \rho)c_1 + \eta c_2^2}\sigma}{\sqrt{-\Delta\varepsilon}c_1} \times$$

$$\left(\frac{-\beta}{2\sigma} - \frac{\sqrt{-\Delta}}{2\sigma} \coth_A \left(\frac{\sqrt{-\Delta}(c_1x + c_2y - \rho t)}{2} \right) \right), \quad (71)$$

$$u_{3,4}^\mp(x, y, t) = k_0 \mp \frac{2\sqrt{ac_1^2 + (bc_2 - \rho)c_1 + \eta c_2^2}\sigma}{\sqrt{-\Delta\varepsilon}c_1} \times$$

$$\left(\frac{-\beta}{2\sigma} + \frac{\sqrt{-\Delta}}{2\sigma} \left(\tanh_A \left(\sqrt{-\Delta}(c_1x + c_2y - \rho t) \right) \right. \right. \quad (72)$$

$$\left. \left. \pm \sqrt{pq} \sec_A \left(\sqrt{-\Delta}(c_1x + c_2y - \rho t) \right) \right) \right),$$

$$u_{5,6}^\mp(x, y, t) = k_0 \mp \frac{2\sqrt{ac_1^2 + (bc_2 - \rho)c_1 + \eta c_2^2}\sigma}{\sqrt{-\Delta\varepsilon}c_1} \times$$

$$\left(\frac{-\beta}{2\sigma} + \frac{\sqrt{-\Delta}}{2\sigma} \left(\coth_A \left(\sqrt{-\Delta}(c_1x + c_2y - \rho t) \right) \right. \right. \quad (73)$$

$$\left. \left. \pm \sqrt{pq} \csc_A \left(\sqrt{-\Delta}(c_1x + c_2y - \rho t) \right) \right) \right),$$

$$u_7^\mp(x, y, t) = k_0 \mp \frac{2\sqrt{ac_1^2 + (bc_2 - \rho)c_1 + \eta c_2^2}\sigma}{\sqrt{-\Delta\varepsilon}c_1} \times$$

$$\left(\frac{-\beta}{2\sigma} - \frac{\sqrt{-\Delta}}{4\sigma} \left(\tanh_A \left(\frac{\sqrt{-\Delta}(c_1x + c_2y - \rho t)}{4} \right) \right. \right. \quad (74)$$

$$\left. \left. - \coth_A \left(\frac{\sqrt{-\Delta}(c_1x + c_2y - \rho t)}{4} \right) \right) \right).$$

Case 2: If $\Delta = \beta^2 - 4\alpha\sigma > 0$ and $\sigma \neq 0$, then

$$u_8^\mp(x, y, t) = k_0 \mp \frac{2\sqrt{ac_1^2 + (bc_2 - \rho)c_1 + \eta c_2^2}\sigma}{\sqrt{-\Delta\varepsilon}c_1} \times$$

$$\left(\frac{-\beta}{2\sigma} - \frac{\sqrt{\Delta}}{2\sigma} \tanh_A \left(\frac{\sqrt{\Delta}(c_1x + c_2y - \rho t)}{2} \right) \right), \quad (75)$$

$$u_9^\mp(x, y, t) = k_0 \mp \frac{2\sqrt{ac_1^2 + (bc_2 - \rho)c_1 + \eta c_2^2}\sigma}{\sqrt{-\Delta\varepsilon}c_1} \times$$

$$\left(\frac{-\beta}{2\sigma} - \frac{\sqrt{\Delta}}{2\sigma} \coth_A \left(\frac{\sqrt{\Delta}(c_1x + c_2y - \rho t)}{2} \right) \right), \quad (76)$$

$$u_{10,11}^\mp(x, y, t) = k_0 \mp \frac{2\sqrt{ac_1^2 + (bc_2 - \rho)c_1 + \eta c_2^2}\sigma}{\sqrt{-\Delta\varepsilon}c_1} \times$$

$$\left(\frac{-\beta}{2\sigma} - \frac{\sqrt{\Delta}}{2\sigma} \left(\tanh_A \left(\sqrt{\Delta}(c_1x + c_2y - \rho t) \right) \right. \right. \quad (77)$$

$$\left. \left. \pm i\sqrt{pq} \operatorname{sech}_A \left(\sqrt{\Delta}(c_1x + c_2y - \rho t) \right) \right) \right),$$

$$u_{12,13}^\mp(x, y, t) = k_0 \mp \frac{2\sqrt{ac_1^2 + (bc_2 - \rho)c_1 + \eta c_2^2}\sigma}{\sqrt{-\Delta\varepsilon}c_1} \times$$

$$\left(\frac{-\beta}{2\sigma} - \frac{\sqrt{\Delta}}{2\sigma} \left(\coth_A \left(\sqrt{\Delta}(c_1x + c_2y - \rho t) \right) \right. \right. \quad (78)$$

$$\left. \left. \pm i\sqrt{pq} \operatorname{csch}_A \left(\sqrt{\Delta}(c_1x + c_2y - \rho t) \right) \right) \right),$$

$$u_{14}^\mp(x, y, t) = k_0 \mp \frac{2\sqrt{ac_1^2 + (bc_2 - \rho)c_1 + \eta c_2^2}\sigma}{\sqrt{-\Delta\varepsilon}c_1} \times$$

$$\left(\frac{-\beta}{2\sigma} - \frac{\sqrt{\Delta}}{4\sigma} \left(\tanh_A \left(\frac{\sqrt{\Delta}(c_1x + c_2y - \rho t)}{4} \right) \right. \right. \quad (79)$$

$$\left. \left. + \coth_A \left(\frac{\sqrt{\Delta}(c_1x + c_2y - \rho t)}{4} \right) \right) \right).$$

Case 3: If $\alpha\sigma > 0$ and $\beta = 0$, then

$$u_{15}^\mp(x, y, t) = k_0 \mp \frac{\sqrt{ac_1^2 + (bc_2 - \rho)c_1 + \eta c_2^2}\sigma}{\sqrt{\alpha\sigma\varepsilon}c_1} \times$$

$$\sqrt{\frac{\alpha}{\sigma}} \tanh_A (\alpha\sigma(c_1x + c_2y - \rho t)), \quad (80)$$

$$u_{16}^\mp(x, y, t) = k_0 \mp \frac{\sqrt{ac_1^2 + (bc_2 - \rho)c_1 + \eta c_2^2}\sigma}{\sqrt{\alpha\sigma\varepsilon}c_1} \times$$

$$\left(-\sqrt{\frac{\alpha}{\sigma}} \coth_A (\alpha\sigma(c_1x + c_2y - \rho t)) \right), \quad (81)$$

$$u_{17,18}^\mp(x, y, t) = k_0 \mp \frac{\sqrt{ac_1^2 + (bc_2 - \rho)c_1 + \eta c_2^2}\sigma}{\sqrt{\alpha\sigma\varepsilon}c_1} \times$$

$$\left(\sqrt{\frac{\alpha}{\sigma}} \left(\tanh_A (2\sqrt{\alpha\sigma}(c_1x + c_2y - \rho t)) \right. \right. \quad (82)$$

$$\left. \left. \pm \sqrt{pq} \sec_A (2\sqrt{\alpha\sigma}(c_1x + c_2y - \rho t)) \right) \right),$$

$$u_{19,20}^\mp(x, y, t) = k_0 \mp \frac{\sqrt{ac_1^2 + (bc_2 - \rho)c_1 + \eta c_2^2}\sigma}{\sqrt{\alpha\sigma\varepsilon}c_1} \times$$

$$\left(-\sqrt{\frac{\alpha}{\sigma}} \left(\coth_A (2\sqrt{\alpha\sigma}(c_1x + c_2y - \rho t)) \right. \right. \quad (83)$$

$$\left. \left. \pm \sqrt{pq} \csc_A (2\sqrt{\alpha\sigma}(c_1x + c_2y - \rho t)) \right) \right),$$

$$u_{21}^{\mp}(x, y, t) = k_0 \mp \frac{\sqrt{ac_1^2 + (bc_2 - \rho) c_1 + \eta c_2^2} \sigma}{\sqrt{\alpha \sigma \varepsilon} c_1} \times \left(\frac{1}{2} \sqrt{\frac{\alpha}{\sigma}} \left(\tan_A \left(\frac{\sqrt{\alpha \sigma} (c_1 x + c_2 y - \rho t)}{2} \right) - \cot_A \left(\frac{\sqrt{\alpha \sigma} (c_1 x + c_2 y - \rho t)}{2} \right) \right) \right) \quad (84)$$

Case 4: If $\alpha \sigma < 0$ and $\beta = 0$, then

$$u_{22}^{\mp}(x, y, t) = k_0 \mp \frac{\sqrt{ac_1^2 + (bc_2 - \rho) c_1 + \eta c_2^2} \sigma}{\sqrt{-\alpha \sigma \varepsilon} c_1} \times \left(-\sqrt{\frac{-\alpha}{\sigma}} \tanh_A \left(\sqrt{-\alpha \sigma} (c_1 x + c_2 y - \rho t) \right) \right), \quad (85)$$

$$u_{23}^{\mp}(x, y, t) = k_0 \mp \frac{\sqrt{ac_1^2 + (bc_2 - \rho) c_1 + \eta c_2^2} \sigma}{\sqrt{-\alpha \sigma \varepsilon} c_1} \times \left(-\sqrt{\frac{-\alpha}{\sigma}} \coth_A \left(\sqrt{-\alpha \sigma} (c_1 x + c_2 y - \rho t) \right) \right), \quad (86)$$

$$u_{24,25}^{\mp}(x, y, t) = k_0 \mp \frac{\sqrt{ac_1^2 + (bc_2 - \rho) c_1 + \eta c_2^2} \sigma}{\sqrt{-\alpha \sigma \varepsilon} c_1} \times \left(-\sqrt{\frac{-\alpha}{\sigma}} \left(\tanh_A \left(2\sqrt{-\alpha \sigma} (c_1 x + c_2 y - \rho t) \right) \pm i\sqrt{pq} \operatorname{sech}_A \left(2\sqrt{-\alpha \sigma} (c_1 x + c_2 y - \rho t) \right) \right) \right), \quad (87)$$

$$u_{26,27}^{\mp}(x, y, t) = k_0 \mp \frac{\sqrt{ac_1^2 + (bc_2 - \rho) c_1 + \eta c_2^2} \sigma}{\sqrt{-\alpha \sigma \varepsilon} c_1} \times \left(-\sqrt{\frac{-\alpha}{\sigma}} \left(\coth_A \left(2\sqrt{-\alpha \sigma} (c_1 x + c_2 y - \rho t) \right) \pm i\sqrt{pq} \operatorname{csch}_A \left(2\sqrt{-\alpha \sigma} (c_1 x + c_2 y - \rho t) \right) \right) \right), \quad (88)$$

$$u_{28}^{\mp}(x, y, t) = k_0 \mp \frac{\sqrt{ac_1^2 + (bc_2 - \rho) c_1 + \eta c_2^2} \sigma}{\sqrt{-\alpha \sigma \varepsilon} c_1} \times \left(-\frac{1}{2} \sqrt{\frac{-\alpha}{\sigma}} \left(\tanh_A \left(\frac{\sqrt{-\alpha \sigma} (c_1 x + c_2 y - \rho t)}{2} \right) + \coth_A \left(\frac{\sqrt{-\alpha \sigma} (c_1 x + c_2 y - \rho t)}{2} \right) \right) \right). \quad (89)$$

Case 5: If $\sigma = \alpha$ and $\beta = 0$, then

$$u_{29}^{\mp}(x, y, t) = k_0 \mp \frac{\sqrt{ac_1^2 + (bc_2 - \rho) c_1 + \eta c_2^2} \alpha}{\sqrt{\varepsilon \alpha^2} c_1} \times \tan_A \left(\alpha (c_1 x + c_2 y - \rho t) \right), \quad (90)$$

$$u_{30}^{\mp}(x, y, t) = k_0 \mp \frac{\sqrt{ac_1^2 + (bc_2 - \rho) c_1 + \eta c_2^2} \alpha}{\sqrt{\varepsilon \alpha^2} c_1} \times \left(-\cot_A \left(\alpha (c_1 x + c_2 y - \rho t) \right) \right), \quad (91)$$

$$u_{31,32}^{\mp}(x, y, t) = k_0 \mp \frac{\sqrt{ac_1^2 + (bc_2 - \rho) c_1 + \eta c_2^2} \alpha}{\sqrt{\varepsilon \alpha^2} c_1} \times \left(\tan_A \left(2\alpha (c_1 x + c_2 y - \rho t) \right) \pm \sqrt{pq} \sec_A \left(2\alpha (c_1 x + c_2 y - \rho t) \right) \right), \quad (92)$$

$$u_{33,34}^{\mp}(x, y, t) = k_0 \mp \frac{\sqrt{ac_1^2 + (bc_2 - \rho) c_1 + \eta c_2^2} \alpha}{\sqrt{\varepsilon \alpha^2} c_1} \times \left(-\cot_A \left(2\alpha (c_1 x + c_2 y - \rho t) \right) \pm \sqrt{pq} \csc_A \left(2\alpha (c_1 x + c_2 y - \rho t) \right) \right), \quad (93)$$

$$u_{35}^{\mp}(x, y, t) = k_0 \mp \frac{\sqrt{ac_1^2 + (bc_2 - \rho) c_1 + \eta c_2^2} \alpha}{\sqrt{\varepsilon \alpha^2} c_1} \times \left(\frac{1}{2} \left(\tan_A \left(\frac{\alpha (c_1 x + c_2 y - \rho t)}{2} \right) - \cot_A \left(\frac{\alpha (c_1 x + c_2 y - \rho t)}{2} \right) \right) \right). \quad (94)$$

Case 6: If $\sigma = -\alpha$ and $\beta = 0$, then

$$u_{36}^{\pm}(x, y, t) = k_0 \pm \frac{\sqrt{ac_1^2 + (bc_2 - \rho) c_1 + \eta c_2^2} \alpha}{\sqrt{-\varepsilon \alpha^2} c_1} \times \tanh_A \left(\alpha (c_1 x + c_2 y - \rho t) \right), \quad (95)$$

$$u_{37}^{\pm}(x, y, t) = k_0 \pm \frac{\sqrt{ac_1^2 + (bc_2 - \rho) c_1 + \eta c_2^2} \alpha}{\sqrt{-\varepsilon \alpha^2} c_1} \times \coth_A \left(\alpha (c_1 x + c_2 y - \rho t) \right), \quad (96)$$

$$u_{38,39}^{\pm}(x, y, t) = k_0 \pm \frac{\sqrt{ac_1^2 + (bc_2 - \rho) c_1 + \eta c_2^2} \alpha}{\sqrt{-\varepsilon \alpha^2} c_1} \times \left(\tanh_A \left(2\alpha (c_1 x + c_2 y - \rho t) \right) \pm i\sqrt{pq} \operatorname{sech}_A \left(2\alpha (c_1 x + c_2 y - \rho t) \right) \right), \quad (97)$$

$$u_{40,41}^{\pm}(x, y, t) = k_0 \pm \frac{\sqrt{ac_1^2 + (bc_2 - \rho) c_1 + \eta c_2^2} \alpha}{\sqrt{-\varepsilon \alpha^2} c_1} \times \left(\coth_A \left(2\alpha (c_1 x + c_2 y - \rho t) \right) \pm i\sqrt{pq} \operatorname{csch}_A \left(2\alpha (c_1 x + c_2 y - \rho t) \right) \right), \quad (98)$$

$$u_{42}^{\pm}(x, y, t) = k_0 \pm \frac{\sqrt{ac_1^2 + (bc_2 - \rho) c_1 + \eta c_2^2} \alpha}{\sqrt{-\varepsilon \alpha^2} c_1} \times \left(\frac{1}{2} \left(\tanh_A \left(\frac{\alpha (c_1 x + c_2 y - \rho t)}{2} \right) + \coth_A \left(\frac{\alpha (c_1 x + c_2 y - \rho t)}{2} \right) \right) \right). \quad (99)$$

Case 7: If $\beta^2 = 4\alpha\sigma$, then the parameters A and k_1 are undefined. Therefore, no solution exists for this case and it is not discussed here.

Case 8: If $\beta = \lambda$, $\alpha = m\lambda$ ($m \neq 0$), and $\sigma = 0$, then $k_1 = 0$. Consequently, $u(x, y, t) = k_0$, which represents a constant (trivial) solution which is ignored here.

Case 9: If $\beta = \sigma = 0$, then $\Delta = \beta^2 - 4\alpha\sigma = 0$. No solution is obtained in this case since the parameters A and k_1 cannot be determined.

Case 10: If $\beta = \alpha = 0$, then $\Delta = \beta^2 - 4\alpha\sigma = 0$. In this case, the parameters A and k_1 are undefined; therefore, no valid solution exists.

Case 11: If $\alpha = 0$ and $\beta \neq 0$, then

$$u_{43}^{\pm}(x, y, t) = k_0 \pm \frac{2\sqrt{ac_1^2 + (bc_2 - \rho)c_1 + \eta c_2^2} \sigma}{\sqrt{-\varepsilon\beta^2} c_1} \times \left(\frac{p\beta}{\sigma (\cosh_A (\beta (c_1 x + c_2 y - \rho t)) - i \sinh_A (\beta (c_1 x + c_2 y - \rho t))) + p} \right). \quad (100)$$

$$u_{44}^{\pm}(x, y, t) = k_0 \pm \frac{2\sqrt{ac_1^2 + (bc_2 - \rho)c_1 + \eta c_2^2} \sigma}{\sqrt{-\varepsilon\beta^2} c_1} \times \left(\frac{\beta (\cosh_A (\beta (c_1 x + c_2 y - \rho t)) + i \sinh_A (\beta (c_1 x + c_2 y - \rho t)))}{\sigma (\cosh_A (\beta (c_1 x + c_2 y - \rho t)) + i \sinh_A (\beta (c_1 x + c_2 y - \rho t))) + q} \right). \quad (101)$$

Case 12: If $\beta = \lambda$, $\sigma = m\lambda$ ($m \neq 0$), and $\alpha = 0$, then

$$u_{45}^{\mp}(x, y, t) = k_0 \mp \frac{2\sqrt{ac_1^2 + (bc_2 - \rho)c_1 + \eta c_2^2} m\lambda}{\sqrt{-\varepsilon\lambda^2} c_1} \times \left(\frac{p \left(e^{\mp \frac{\sqrt{-\varepsilon\lambda^2 (ac_1^2 + bc_1 c_2 + \eta c_2^2 - \rho c_1)}}{\varepsilon \lambda^2 c_1^2}} \right)^{\lambda (c_1 x + c_2 y - \rho t)}}{q - mp \left(e^{\mp \frac{\sqrt{-\varepsilon\lambda^2 (ac_1^2 + bc_1 c_2 + \eta c_2^2 - \rho c_1)}}{\varepsilon \lambda^2 c_1^2}} \right)^{\lambda (c_1 x + c_2 y - \rho t)}} \right). \quad (102)$$

Verification of representative solutions: To verify the correctness of all of the obtained traveling wave solutions, i.e., solutions in Eqs. (55), (57), (61), (63), (70)–(102), they were written in the form $U(\xi)$ with ξ defined in Eq. (3) and were substituted directly into the reduced ordinary differential equation in Eq. (51). The verification process was carried out analytically using the corresponding derivative relations together with symbolic simplification in Maple. In all cases, the resulting residual expressions vanish identically. For example, the solution given in Eq. (70) was substituted into Eq. (51), yielding a zero residual. All remaining solutions were verified analogously. Therefore, under the traveling wave transformation, the reported solutions satisfy Eq. (1) exactly.

4 Graphs of selected exact solutions

In this section, selected exact solutions obtained via the modified rational sine–cosine method and the NEDAM are illustrated in terms of 3D, 2D,

and contour graphs. These graphical representations provide a clear visualization of solution behaviors. Various wave profiles under different parameter values are presented in Figures 1–10. Several of the obtained solutions contain imaginary components, their moduli (absolute values) are plotted to capture the combined influence of both real and imaginary parts on the corresponding wave behaviors.

In Figure 1, the solution in Eq. (55), derived using the modified rational sine–cosine method, is graphically displayed. The 3D and contour graphs are generated with the parameter values $B = 0.25$, $F = 0.55$, $a = 0.4$, $b = 0.25$, $\varepsilon = 0.1$, $c_2 = 0.2$, $\eta = 0.1$, and $y = 1$. The 2D graph is produced using the same parameter values with the different values of $t = -0.5, 0, 0.5, 1$. The physical behavior of this displayed solution corresponds to a singular periodic wave.

In Figure 2, the solution in Eq. (57), obtained via the modified rational sine–cosine method, is graphically illustrated. The 3D and contour graphs are generated using the following values: $A = 1$, $B = 0$, $a = 0.5$, $b = 0.1$, $\varepsilon = 0.1$, $c_1 = 0.5$, $c_2 = 0.15$, $\eta = 0.1$, and $y = 1$. The 2D graph is plotted utilizing the same parameter values with the different values of $t = -0.5, 0, 0.5, 1$. The physical behavior of the solution corresponds to a singular solution.

In Figure 3, the solution in Eq. (70), derived using the NEDAM, is visually presented. The 3D and contour graphs are depicted using the following parameter values: $p = 0.2$, $q = 0.6$, $\alpha = 0.4$, $\sigma = 0.5$, $\beta = 0.3$, $a = 0.3$, $b = 0.4$, $\varepsilon = 0.2$, $c_1 = 0.5$, $c_2 = 0.2$, $\eta = 0.3$, $\rho = 0.2$, $k_0 = 1$, and $y = 1$. Its 2D graph is plotted using the same parameter values as described above by varying $t = -0.5, 0, 0.5, 1$. The physical behavior of this solution corresponds to a periodic wave solution.

In Figure 4, the solution in Eq. (77), obtained via the NEDAM, is graphically displayed. The 3D and contour graphs are generated using the following values: $p = 0.4$, $q = 0.6$, $\alpha = 0.3$, $\sigma = 0.2$, $\beta = 0.6$, $a = 0.6$, $b = 0.25$, $\varepsilon = 0.3$, $c_1 = 0.5$, $c_2 = 0.2$, $\eta = 0.2$, $\rho = 0.5$, $k_0 = 1$, and $y = 1$. The corresponding 2D graph is plotted using the same parameter values with $t = -0.5, 0, 0.5, 1$. This solution physically presents the behavior of a kink soliton wave.

The solution in Eq. (81), derived using the NEDAM, is visually presented in Figure 5. The 3D and contour graphs are depicted utilizing the following parameter values: $p = 0.5$, $q = 0.6$, $\alpha = 1$, $\sigma = 1$, $\beta = 0$, $a = 0.3$, $b = 0.2$, $\varepsilon = 0.1$, $c_1 = 1$, $c_2 = 0.5$, $\eta = 0.2$, $\rho = 0.1$, $k_0 = 1$, and $y = 1$. In addition, the corresponding 2D graph is visualized using the same parameters as described above with

$t = -0.5, 0, 0.5, 1$. The physical behavior of the demonstrated solution is a singular periodic wave.

The solution in Eq. (88), which is derived via the NEDAM, is visually illustrated in Figure 6. The 3D and contour graphs are portrayed using the following parameter values: $p = 0.5, q = 0.6, \alpha = -0.3, \sigma = 0.5, \beta = 0, a = 0.1, b = 0.1, \varepsilon = 0.1, c_1 = 0.15, c_2 = 0.15, \eta = 0.1, \rho = 0.3, k_0 = 1$, and $y = 1$. Its corresponding 2D graph is plotted using the described parameter values with $t = -0.5, 0, 0.5, 1$. From the graphical representations, this solution provides a kink soliton wave.

The solution in Eq. (90), derived using the NEDAM, is graphically presented in Figure 7. The 3D and contour graphs are portrayed employing $p = 0.5, q = 0.6, \alpha = 1, \sigma = 1, \beta = 0, a = 0.2, b = 0.1, \varepsilon = 0.1, c_1 = 1, c_2 = 0.5, \eta = 0.1, \rho = 0.1, k_0 = 1$, and $y = 1$. The 2D graph of the solution is displayed by using the same parameter values mentioned above and varying the values of t with $t = -0.5, 0, 0.5, 1$. The physical behavior of the selected solution represents a singular periodic wave.

In Figure 8, the solution in Eq. (96), obtained via the NEDAM, is graphically illustrated in terms of the 3D, 2D, and contour graphs using $p = 0.5, q = 0.6, \alpha = 0.5, \sigma = -0.5, \beta = 0, a = 0.1, b = 0.1, \varepsilon = 0.1, c_1 = 0.15, c_2 = 0.15, \eta = 0.1, \rho = 0.3, k_0 = 1$, and $y = 1$. However, the values $t = -0.5, 0, 0.5, 1$ are used for its corresponding 2D graph. The physical behavior of this solution shows a singular kink-type soliton.

The solution in Eq. (100), derived using the NEDAM, is graphically presented in Figure 9 with the 3D, 2D, and contour graphs using $p = 0.5, q = 0.6, \alpha = 0, \sigma = 0.5, \beta = 0.1, a = 0.1, b = 0.1, \varepsilon = 0.1, c_1 = 0.15, c_2 = 0.15, \eta = 0.1, \rho = 0.3, k_0 = 1$, and $y = 1$. The corresponding 2D graph is plotted using the above parameter values with $t = -0.5, 0, 0.5, 1$. The physical behavior of the selected solution corresponds to a kink soliton wave.

In Figure 10, the solution in Eq. (102), obtained via the NEDAM, is visually illustrated in terms of the 3D, 2D, and contour graphs using $p = 0.5, q = 0.6, \alpha = 0, m = 5, \lambda = 0.1, \sigma = 0.5, \beta = 0.1, a = 0.1, b = 0.1, \varepsilon = 0.1, c_1 = 0.15, c_2 = 0.15, \eta = 0.1, \rho = 0.3, k_0 = 1$, and $y = 1$. The corresponding 2D graph is plotted using $t = -0.5, 0, 0.5, 1$. This solution physically behaves as a singular kink-type soliton.

Overall, the graphical results demonstrate that different parameter combinations produce markedly different wave profiles, including periodic waves, kink solitons, singular periodic structures, and singular kink-type solutions. In particular, the presented examples indicate that the governing

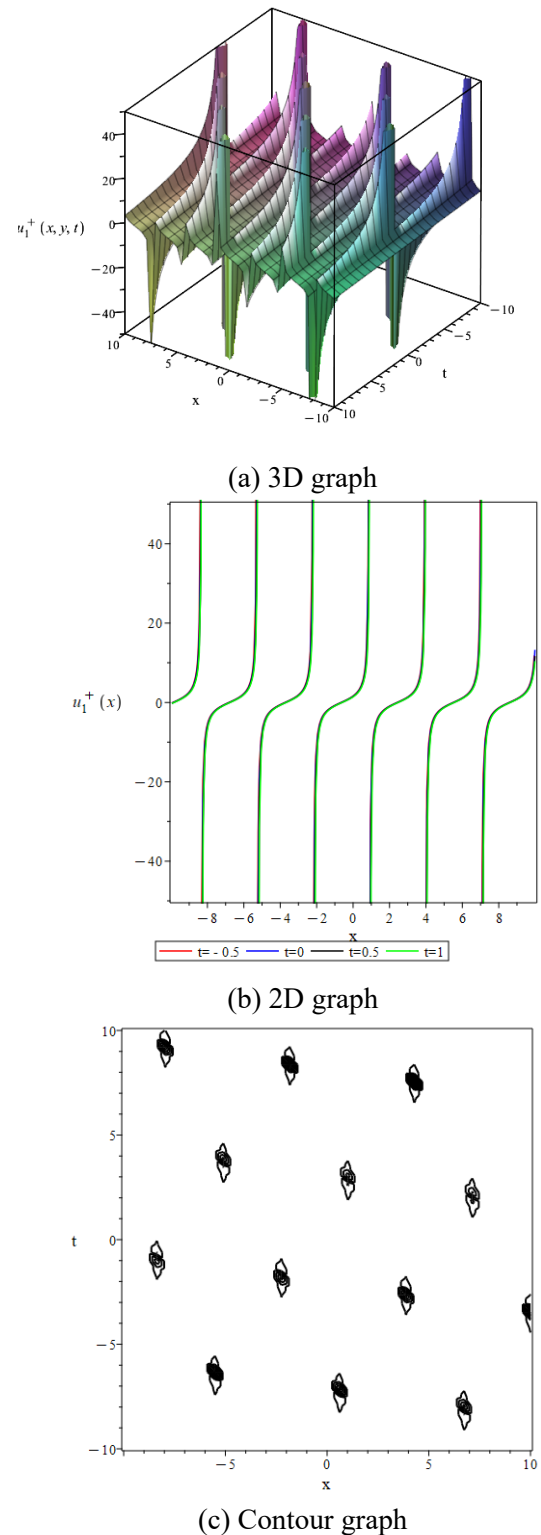
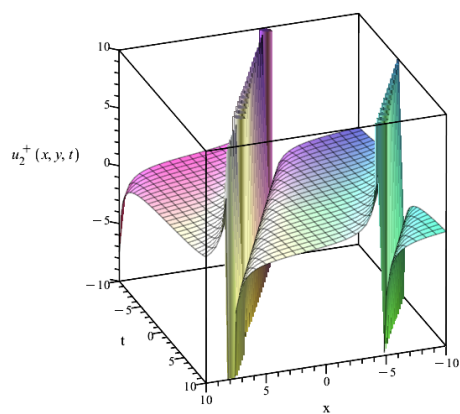
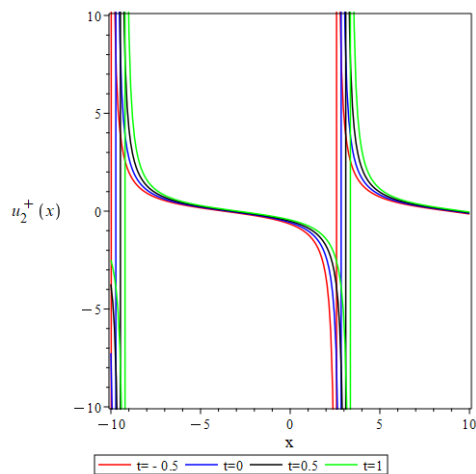


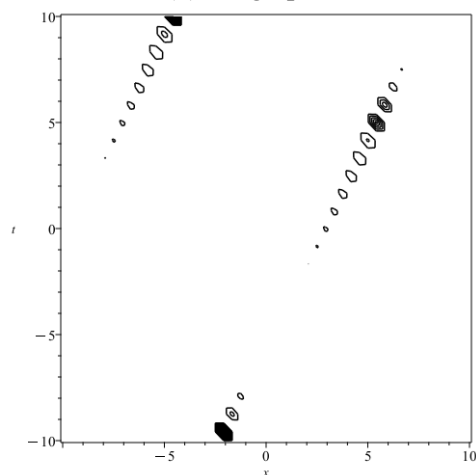
Fig. 1: The 3D, 2D, and contour graphs of solution $u_1^+(x, y, t)$ in Eq. (55) show a singular periodic wave.
Source: created by the authors.



(a) 3D graph

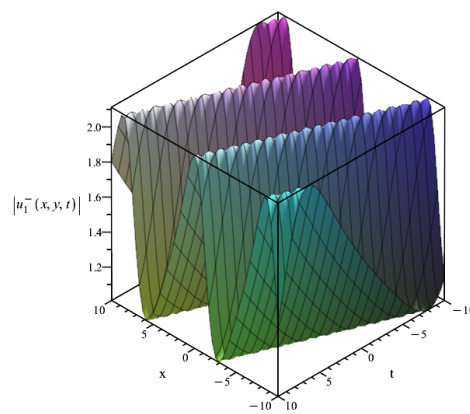


(b) 2D graph

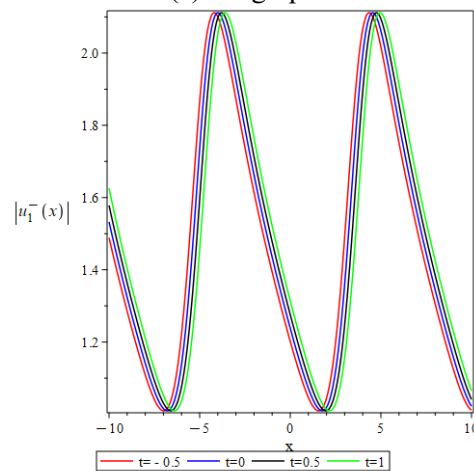


(c) Contour graph

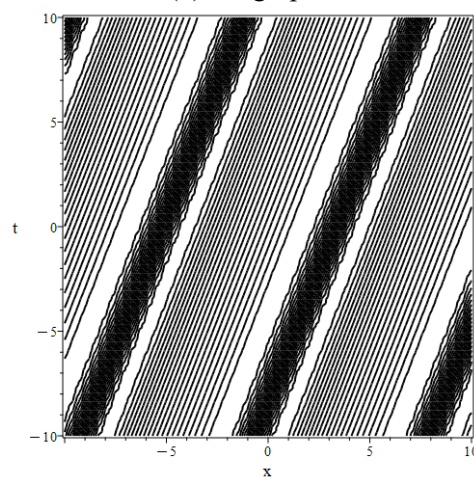
Fig. 2: The 3D, 2D, and contour graphs of the solution $u_2^+(x, y, t)$ in Eq. (57) illustrate a singular solution.
Source: created by the authors.



(a) 3D graph

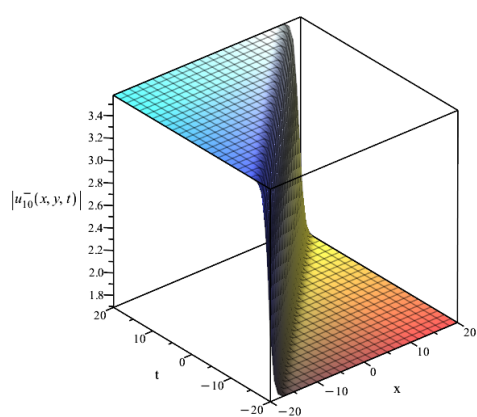


(b) 2D graph

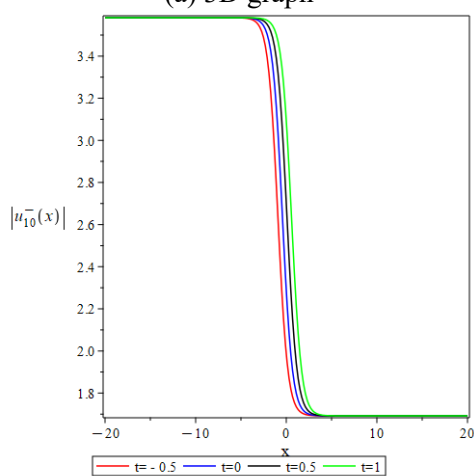


(c) Contour graph

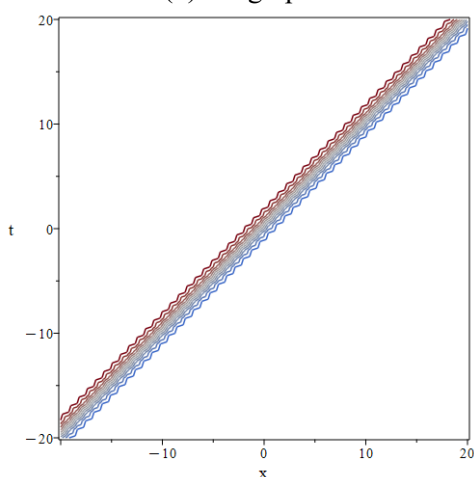
Fig. 3: The 3D, 2D, and contour graphs of solution $|u_1^-(x, y, t)|$ in Eq. (70) show a periodic wave solution.
Source: created by the authors.



(a) 3D graph



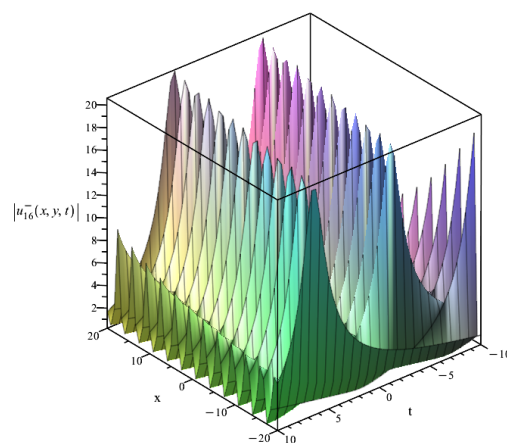
(b) 2D graph



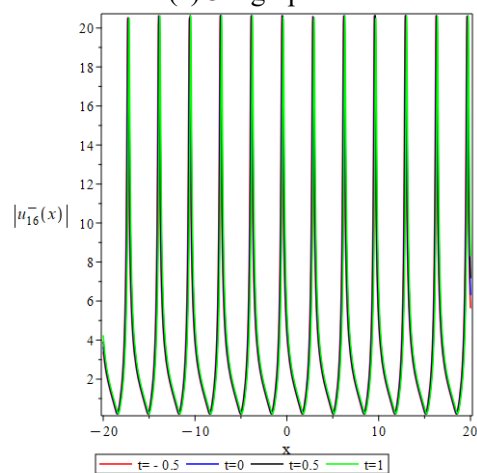
(c) Contour graph

Fig. 4: The 3D, 2D, and contour graphs of solution $|u_{10}^-(x, y, t)|$ in Eq. (77) behave as a kink soliton wave.

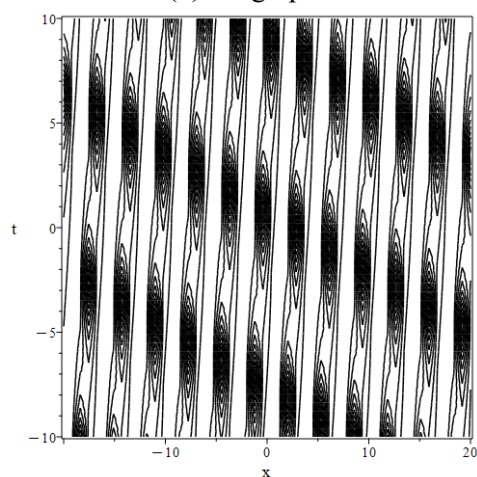
Source: created by the authors.



(a) 3D graph



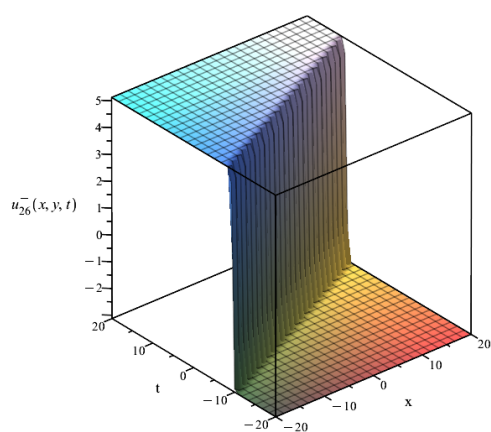
(b) 2D graph



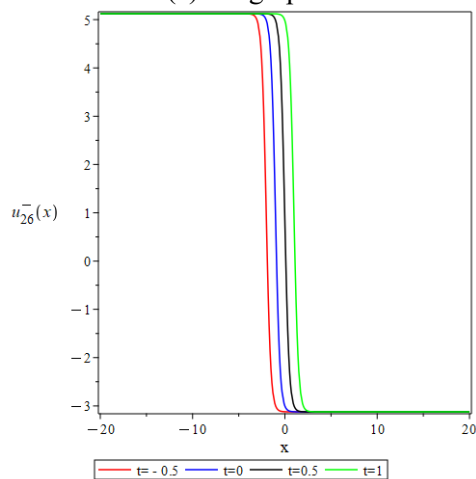
(c) Contour graph

Fig. 5: The 3D, 2D, and contour graphs of solution $|u_{16}^-(x, y, t)|$ in Eq. (81) show a singular periodic wave.

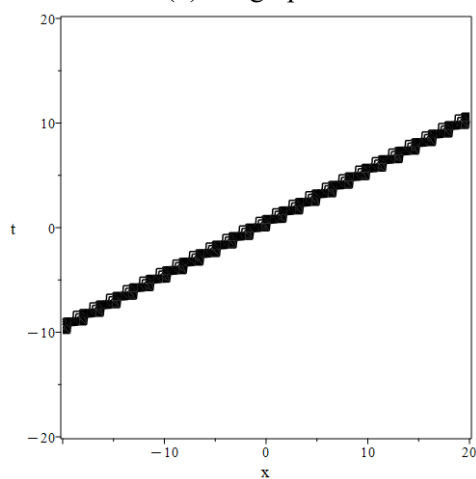
Source: created by the authors.



(a) 3D graph

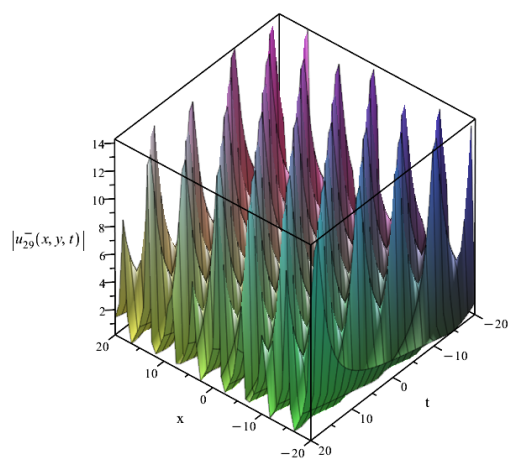


(b) 2D graph

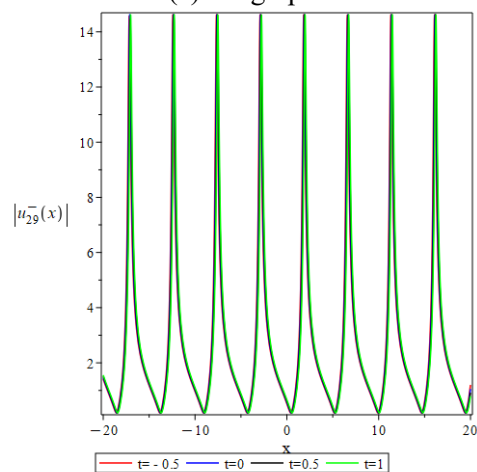


(c) Contour graph

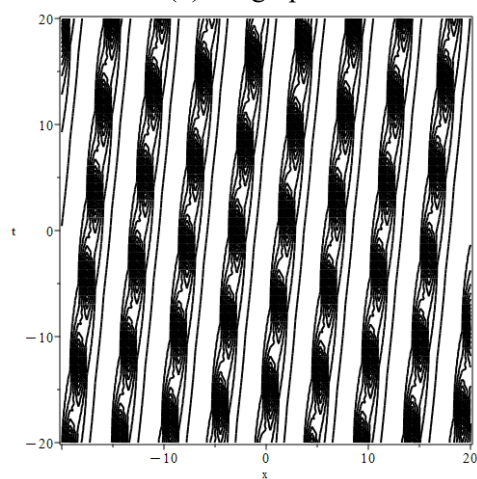
Fig. 6: The 3D, 2D, and contour graphs of solution $u_{26}^-(x, y, t)$ in Eq. (88) display a kink soliton wave.
Source: created by the authors.



(a) 3D graph

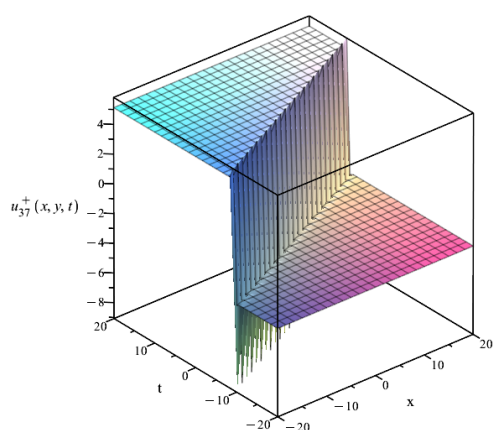


(b) 2D graph

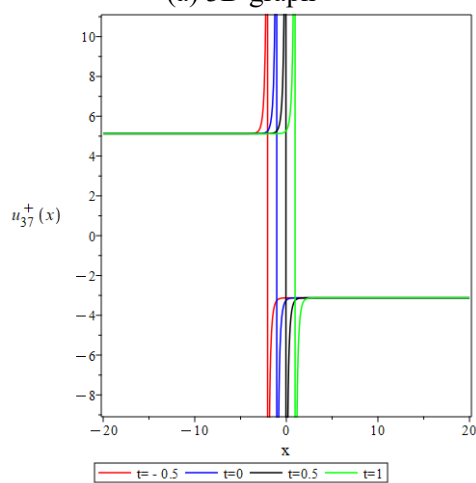


(c) Contour graph

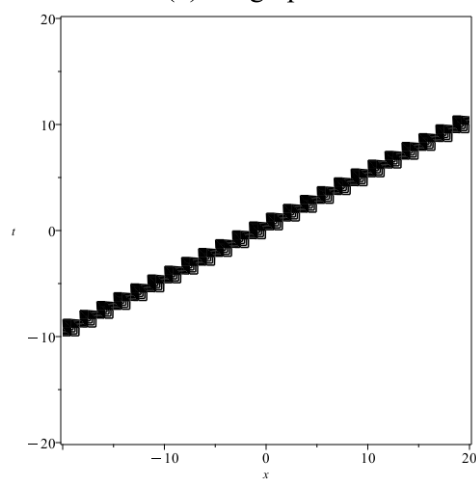
Fig. 7: The 3D, 2D, and contour graphs of the solution $|u_{29}^-(x, y, t)|$ in Eq. (90) behave as a singular periodic wave.
Source: created by the authors.



(a) 3D graph



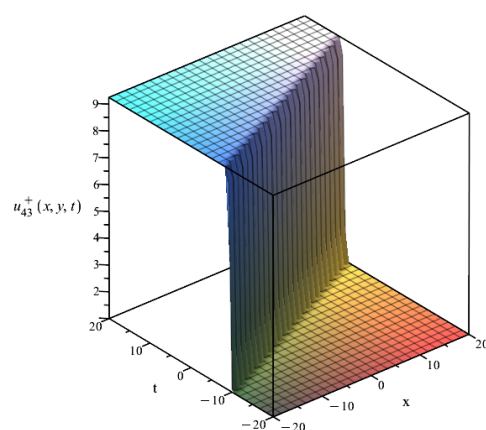
(b) 2D graph



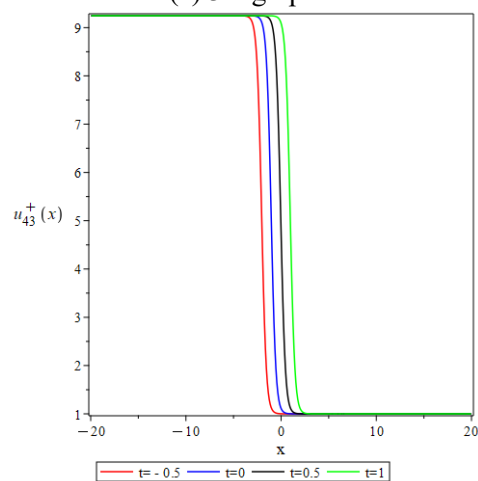
(c) Contour graph

Fig. 8: The 3D, 2D, and contour graphs of the solution $u_{37}^+(x, y, t)$ in Eq. (96) provide a singular kink-type solution.

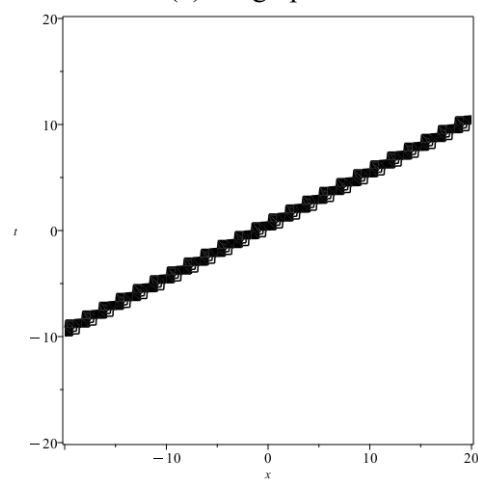
Source: created by the authors.



(a) 3D graph



(b) 2D graph



(c) Contour graph

Fig. 9: The 3D, 2D, and contour graphs of solution $u_{43}^+(x, y, t)$ in Eq. (100) provide a kink soliton wave.

Source: created by the authors.

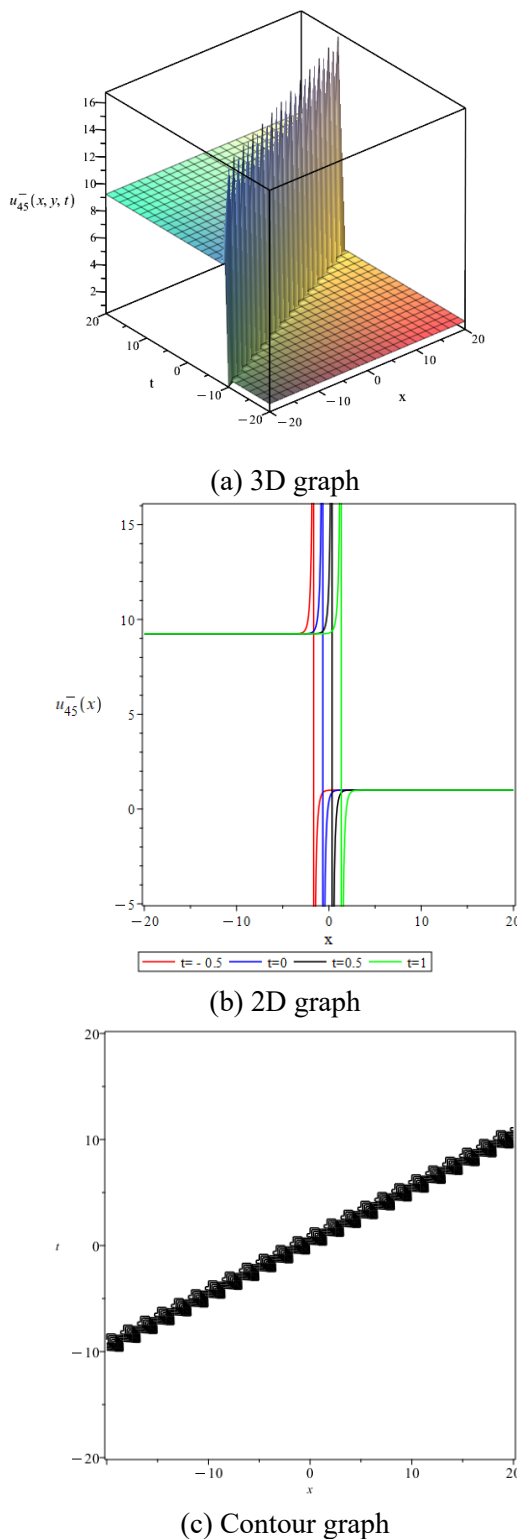


Fig. 10: The 3D, 2D, and contour graphs of solution $u_{45}^-(x, y, t)$ in Eq. (102) show a singular kink-type solution.

Source: created by the authors.

coefficients significantly influence the amplitudes, shapes, localization, and propagation characteristics of the obtained traveling waves. For the traveling wave transformation (3), variations in ρ are associated with phase translation, while c_1 and c_2 affect the wave numbers, spatial orientation, and propagation direction. Using the NEDAM, the generalized parameters p , q , and A modify the shapes and deformation of the wave profiles. Moreover, the parameter k_0 determines the background level of the solutions, whereas k_1 controls the amplitude scaling and overall intensity of the wave structures. In addition, the auxiliary parameters α , β , and σ play important roles in determining whether the resulting solutions are periodic, localized, hyperbolic-type, or singular. The model coefficients, including ε , a , b , and η , are also reflected in the displayed wave structures through dispersive and coupling effects. These comparisons provide a systematic qualitative understanding of parameter-dependent physical behaviors of the selected solutions.

5 Stability analysis

To study the modulation behavior of Eq. (1), we use the standard linear stability analysis method, [29], [30]. The perturbed solution of the $(2+1)$ -dimensional eKPE is assumed as

$$u(x, y, t) = s + \delta \phi(x, y, t), \quad 0 < \delta \leq 1, \quad (103)$$

where s is the steady-state solution of Eq. (1), δ is a small perturbation parameter, and $\phi(x, y, t)$ represents the perturbation function. Substituting Eq. (103) into Eq. (1), we get

$$6\delta^2\varepsilon\phi_x\phi_{xx} + a\delta\phi_{xx} + b\delta\phi_{xy} + \delta\varepsilon\phi_{xxxx} + \eta\delta\phi_{yy} + \delta\phi_{xt} = 0. \quad (104)$$

On linearizing Eq. (104) in terms of δ , we obtain

$$a\delta\phi_{xx} + b\delta\phi_{xy} + \delta\varepsilon\phi_{xxxx} + \eta\delta\phi_{yy} + \delta\phi_{xt} = 0. \quad (105)$$

We consider the solution of Eq. (105) in the form

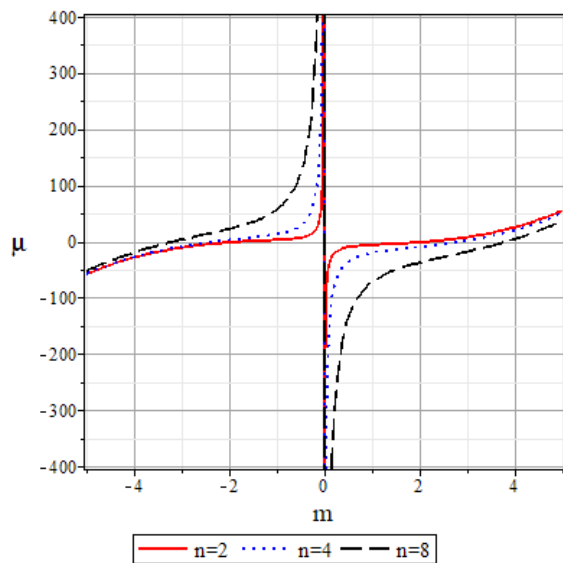
$$\phi(x, y, t) = e^{i(mx+ny+\mu t)}, \quad (106)$$

where μ is the frequency of perturbation and m, n are the normalized wave numbers. Substituting Eq. (106) into Eq. (105), it follows that

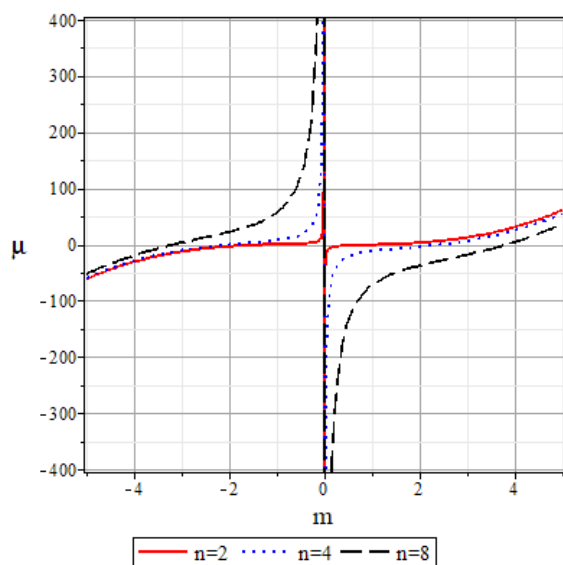
$$-\varepsilon m^4 + am^2 + bmn + \eta n^2 + m\mu = 0. \quad (107)$$

From Eq. (107), the dispersion relation is obtained as

$$\mu = \frac{\varepsilon m^4 - am^2 - bmn - \eta n^2}{m}. \quad (108)$$



(a) $m - \mu$ relation with one parameter set



(b) $m - \mu$ relation with three parameter sets

Fig. 11: Variation of the perturbation frequency μ with respect to the wave number m : (a) $\{\varepsilon = 0.5, a = b = \eta = 1\}$, (b) $\{\varepsilon = 0.5, a = b = \eta = 0.2, n = 2\}$, $\{\varepsilon = 0.5, a = b = \eta = 0.6, n = 4\}$, and $\{\varepsilon = 0.5, a = b = \eta = 1, n = 8\}$.

Source: created by the authors.

For the specific parameter set $\{\varepsilon = 0.5, a = b = \eta = 1\}$, the relationship between the perturbation frequency μ and the wave number m is depicted in Figure 11(a) for varying values of n ($n = 2, 4$, and 8). Furthermore, varying the parameter sets exerts only a minor influence on the $m - \mu$ relation, as illustrated in Figure 11(b) for cases where $\{\varepsilon = 0.5, a = b = \eta = 0.2\}$ at $n = 2$, $\{\varepsilon = 0.5, a = b = \eta = 0.6\}$ at $n = 4$, and $\{\varepsilon = 0.5, a = b = \eta = 1\}$ at

$n = 8$. It is observed that the perturbation frequency μ depends strongly on both the wave number m and the power index n , with different values of n yielding distinctly different dispersion profiles. Notably, the discontinuity near $m = 0$ originates from the singular denominator in Eq. (108), which causes μ to become undefined in this localized region.

From the linear stability viewpoint, the regions where $\mu < 0$ indicate decaying perturbations and therefore correspond to a stable regime, whereas the regions where $\mu > 0$ represent growing perturbations and hence an unstable regime. The critical case $\mu = 0$ corresponds to a neutral or marginally stable state separating the stable and unstable regions.

Moreover, increasing the parameter value of n significantly modifies the wave-frequency profile, confirming that the coefficients ε , a , b , and η play important roles in determining the propagation characteristics of the obtained traveling wave solutions. These results provide a qualitative linear stability description of the obtained traveling wave solutions.

6 Conclusions

In this research article, we presented interesting exact traveling wave solutions of the (2+1)-dimensional extended Kadomtsev-Petviashvili equation (eKPE) in Eq. (1) which were obtained via the modified rational sine-cosine method and the new extended direct algebraic method (NEDAM). The modified rational sine-cosine method produced the exact solutions with simple mathematical forms, primarily expressed in terms of trigonometric and rational functions. The selected graphical solutions illustrate a singular periodic wave and a singular solution in Figure 1, Figure 2, respectively. From the reported results, it indicates that this method is effective for constructing exact traveling wave solutions with definite and simple wave structures.

Furthermore, the NEDAM generated a broader class of solutions for the proposed problem including trigonometric, hyperbolic, rational, exponential, and complex-valued functions. The physical behaviors of the selected solutions include

- a periodic wave solution as shown in Figure 3,
- a kink soliton wave as displayed in Figure 4, Figure 6, and Figure 9,
- a singular kink-type solution as illustrated in Figure 8 and Figure 10,
- a singular periodic wave as shown in Figure 5 and Figure 7.

These outcomes demonstrated that the NEDAM is an effective and reliable methodology and could be successfully applied to a range of NPDEs. A comparative discussion between the results previously reported for the KPE/eKPE and our current findings is provided in Table 1 (Appendix).

It should be clearly noted that the obtained solutions are derived under the assumed solution frameworks of the modified rational sine-cosine method and the NEDAM. Therefore, the presented results are not claimed to represent all possible exact solutions of the extended Kadomtsev–Petviashvili equation (1). Rather, they provide broad and representative families of exact traveling wave solutions. Compared with previously reported results of the eKPE, several obtained solution families are expressed in generalized deformed trigonometric and hyperbolic forms involving the parameters p , q , and A . Some solutions reduce to known classical forms under suitable parameter choices, for example, $A = e$ and $p = q = 1$, whereas others, particularly the complex-valued, singular kink-type, and generalized deformed structures, appear to provide previously unreported exact traveling wave structures for the eKPE. These observations indicate that the proposed methods are effective techniques for constructing a wide variety of exact solutions rather than methods intended to classify all possible solutions.

The stability characteristics of the obtained traveling wave solutions depend strongly on their structural forms and parameter regimes. In general, localized hyperbolic-type solutions such as $\tanh$, $\coth$, and sech -related profiles are commonly associated with solitary-wave behaviors and are expected to exhibit greater robustness under small perturbations. In contrast, periodic trigonometric-type solutions may be more sensitive to parameter variations and phase disturbances due to their oscillatory nature. Singular solutions containing denominator zeros may possess limited physical applicability unless suitable parameter restrictions are imposed. Since the present work primarily focuses on the derivation of exact analytical traveling wave solutions together with a modulation-based linear stability analysis, convergence issues associated with iterative numerical approximations do not arise. However, a more rigorous spectral or nonlinear perturbation stability investigation is beyond the present scope and may be considered in future work.

Future studies may focus on more rigorous spectral or nonlinear stability analyses of the obtained soliton solutions under different parameter conditions. In addition, the present approaches can be extended to fractional-order versions of the eKPE and applied to other nonlinear evolution equations. Moreover, future investigations may also explore potential applications of the derived solutions in nonlinear wave phenomena arising in fluid dynamics, plasma physics, and optical systems.

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Declaration of Generative AI and AI-assisted Technologies in the Writing Process

During the preparation of this work, the author(s) utilized artificial intelligence (AI) technologies solely for the purpose of linguistic refinement and grammatical correction. Following the application of these tools, the author(s) comprehensively reviewed, verified, and edited the resulting text. The author(s) maintain full accountability and responsibility for the accuracy, integrity, and final content of the publication.

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Conflicts of Interest

The authors declare that they have no conflicts of interest.

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APPENDIX

Table 1. Comparison of previous studies and the present work on KP/eKP-type equations.

Reference	Method	Main reported solutions	Relation to the present work
[28]	Hirota bilinear method	Lump solutions with analyticity and localization conditions for the eKP equation	Focused mainly on rationally localized lump waves, whereas the present work derives broader traveling wave families such as periodic, kink-type, singular, exponential, and complex-valued solutions.
[26]	Painlevé analysis, Hirota method	Multi-soliton waves, shock waves, lump structures	Mainly emphasized integrability-related structures, while the present study applies two algebraic approaches to obtain additional exact traveling wave profiles.
[27]	Exact solution methods with stability discussion	Exact solutions together with sensitivity/stability properties	Complementary to the present paper focusing on deriving large classes of explicit closed-form traveling wave solutions.
Present work	Modified rational sine–cosine method and NEDAM	Singular solutions, periodic waves, singular periodic waves, kink soliton waves, singular kink-type waves, trigonometric and hyperbolic forms, rational solutions, exponential solutions, and complex-valued solutions	Generating diverse exact solutions. Several generalized forms reduce to classical trigonometric and hyperbolic solutions under suitable parameter choices.

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