The Quadruple Aboodh Transform and Its Analytical Applications to Fractional Partial Differential Equations

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Abstract: In this research article, we introduce the Quadruple Aboodh Transform as a novel multidimensional extension of the classical integral Aboodh transform. We establish its fundamental mathematical and analytical properties, including linearity, scaling, existence conditions on an explicit function space, and a rigorous proof—justified via Fubini's theorem under an exponential-order hypothesis—of the four-dimensional Convolution Theorem. We also give the inversion formula and prove the uniqueness of the transform. Furthermore, we derive, by an explicit induction argument rather than by analogy, the operative formula for applying the Quadruple Aboodh transform to partial fractional derivatives defined in the Caputo sense (Theorem4.4). To demonstrate the efficiency, correctness, and applicability of this new multidimensional transform framework, it is deployed to solve four homogeneous and nonhomogeneous fractional partial differential equations, including one with a variable coefficient, yielding exact analytical solutions; one solution is additionally cross-checked numerically against a fractional finite-difference scheme.

Key-Words: Aboodh transform; Quadruple transform; Caputo fractional derivative; Partial differential equations; Convolution theorem; Fubini's theorem; Mittag-Leffler function.

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1 Introduction

Integral transforms represent one of the most effective, elegant, and computationally efficient mathematical tools used to solve complex initial-boundary value problems arising in applied mathematics, mathematical physics, quantum mechanics, and engineering sciences. The core operational mechanism of integral transform methods lies in their intrinsic ability to map an unknown function defined over a physical domain into a spectral or auxiliary parameter space. Through this mapping, linear differential systems, integro-differential expressions, and dynamic models are reduced to straightforward algebraic equations or lower-dimensional systems that are easily manipulated and inverted.

Over the past decades, classical integral transforms such as the Laplace transform, Fourier transform, Mellin transform, Hankel transform, and Sumudu transform have been comprehensively investigated and widely deployed. The Aboodh transform was introduced in [1], as an integral transform for ordinary and partial differential equations. The Aboodh transform offers distinctive analytical features, particularly its computational symmetry and handling of scaling, which often reduces the heavy algebraic steps required by traditional methods such as the Laplace transform [2], [3]. With the rapid emergence of complex multidimensional physical problems–such as four-dimensional spacetime field formulations, relativistic electrodynamics, nonlocal diffusion processes, and higher-dimensional wave equations–the need to generalize single integral transforms into multidimensional frameworks has become essential [4], [5]. Researchers have successfully introduced double, triple, and quadruple variations of various transforms, such as the Quadruple Laplace and Quadruple Shehu transforms. Motivated by these foundational extensions, and by the application of the Aboodh transform to partial differential equations [6], this paper rigorously establishes the mathematical framework for the Quadruple Aboodh Transform. The four-dimensional extension of an integral transform is, in itself, not new: Quadruple Laplace transforms, [7], and, more recently, Quadruple Shehu transforms, [8], have already been formulated for integer-order multidimensional PDEs. Two points distinguish the present contribution from that existing body of work. First, at the level of elementary computation, the Aboodh kernel carries a built-in $1/v$ scaling factor (Definition 3.1) that a Quadruple-Laplace-based approach does not have; this removes the extra normalization step that is otherwise needed when transforming polynomial or mixed exponential-polynomial source terms, so that boundary terms in the operational formulas (e.g. Theorem 4.1 and Theorem 4.4 below) appear directly as negative powers of v_1 rather than requiring a separate rescaling before the inverse transform is applied. We present this as a difference in algebraic convenience for this class of source terms, not as an unconditional superiority of the Aboodh

kernel. Second, and more substantively, neither the Quadruple Laplace nor the Quadruple Shehu literature provides an operational formula for a Caputo fractional partial derivative; the present paper's central contribution is precisely this formula (Theorem 4.4), proved here in full, together with its direct application to four-dimensional fractional PDEs, which is new relative to the integer-order equations treated by the existing quadruple-transform literature.

By scaling the core transformation kernel across a four-dimensional space $\mathbb{R}^4_+$, we construct a mathematical tool capable of tackling higher-dimensional partial differential equations (PDEs) and fractional partial differential equations (FPDEs), [9], involving memory effects via Caputo fractional derivatives.

The paper is structured as follows: Section 2 provides essential preliminary definitions of the single Aboodh transform and Caputo fractional calculus. Section 3 outlines the main results, providing the core definition of the Quadruple Aboodh Transform, its existence class and convergence properties, its values on elementary functions, formal proofs of its structural properties (including a fully justified Convolution Theorem), and its inversion and uniqueness. Section 4 develops the fractional framework and proves, by induction, the operational formula for the Caputo derivative. Section 5 evaluates four explicit four-dimensional fractional partial differential equations, one of them with a variable coefficient, and presents our final conclusions in Section 6.

2 Preliminaries

In this section, we present the fundamental mathematical definitions of the classical single Aboodh transform and basic fractional calculus concepts that serve as the foundation for our multidimensional generalization.

Definition 2.1. *The classical single Aboodh transform of a continuous function $f(t)$ defined for $t \geqslant 0$ belonging to a class of functions of exponential order is defined by the following integral representation [10]:*

$$\mathcal{A}\{f(t)\} = K(v) = \frac{1}{v}\int_0^\infty e^{-vt} f(t)\,dt, \qquad v > 0. \tag{1}$$

Definition 2.2. *Let $\alpha > 0$. The Riemann-Liouville fractional integral of order α for a function $f(t)$ is given by [11]:*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)\,d\tau, \tag{2}$$

where $\Gamma(\cdot)$ represents the standard Euler gamma function.

Definition 2.3. *The fractional partial derivative of order α ($n-1 < \alpha \leqslant n$, $n \in \mathbb{N}$) of a function $f(t)$ defined in the Caputo sense is expressed as:*

$$\mathbb{C}\mathbb{D}^\alpha_t f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)\,d\tau. \tag{3}$$

3 The Quadruple Aboodh Transform

Here, we present the formal definition of the Quadruple Aboodh Transform, discuss its function space, existence and convergence, evaluate it for basic elementary functions, and prove its key operational theorems, including inversion and uniqueness.

Definition 3.1. *Let $f(x_1, x_2, x_3, x_4)$ be a continuous function of four independent variables defined over the positive octant $\mathbb{R}^4_+ = [0,\infty)^4$. The Quadruple Aboodh Transform of f is denoted $\mathcal{A}^4_{x_1x_2x_3x_4}\{f\} = K(v_1, v_2, v_3, v_4)$ and is defined by the four-dimensional improper integral:*

$$\begin{aligned} K(v_1,v_2,v_3,v_4) &= \frac{1}{v_1v_2v_3v_4} \\ &\times \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-(v_1x_1+v_2x_2+v_3x_3+v_4x_4)} \\ &f(x_1,x_2,x_3,x_4)\,dx_1dx_2dx_3dx_4, \end{aligned} \tag{4}$$

where $v_1, v_2, v_3, v_4 > 0$ are the transform parameters.

3.1 Function space, existence and convergence

Definition 3.2 (Exponential-order class)**.** *A function $f(x_1, x_2, x_3, x_4)$, continuous on $\mathbb{R}^4_+$, is said to belong to the exponential-order class $E(c_1, c_2, c_3, c_4)$ if there exist a constant $M > 0$ and constants $c_1, c_2, c_3, c_4 \in \mathbb{R}$ such that, for all $x_i \geqslant 0$,*

$$\begin{aligned} &|f(x_1,x_2,x_3,x_4)| \\ &\leqslant M\, e^{c_1x_1+c_2x_2+c_3x_3+c_4x_4}. \end{aligned} \tag{5}$$

Theorem 3.3 (Existence and convergence)**.** *If $f \in E(c_1, c_2, c_3, c_4)$, then the integral in 4 converges absolutely and uniformly on every compact subset of $\{(v_1, v_2, v_3, v_4) : v_i > c_i\}$,$\{i = 1, \ldots, 4\}$, and consequently $K(v_1, v_2, v_3, v_4)$ is well defined and jointly continuous on that open region.*

Proof. For $v_i > c_i$, $|e^{-\sum v_i x_i} f(x_1,\dots,x_4)| \leqslant Me^{-\sum (v_i-c_i)x_i}$, and $\prod_i \int_0^\infty e^{-(v_i-c_i)x_i}\,dx_i = \prod_i (v_i-c_i)^{-1} < \infty$; the four-fold integral of the dominating function is therefore finite by Tonelli's theorem (the integrand is nonnegative), which gives absolute convergence of 4. On a compact subset $\{v_i \geqslant c_i + \varepsilon\}$ for some $\varepsilon > 0$, the same bound is uniform in $(v_1,\dots,v_4)$, giving uniform convergence there; joint continuity of K then follows from the uniform limit of continuous partial integrals. □

This is the function space and convergence class used throughout Sections 3–5: every function to which the transform, the Convolution Theorem, or Theorem 4.4 is applied below is assumed to belong to $E(c_1,c_2,c_3,c_4)$ for suitable constants, with the v_i restricted to $v_i > c_i$.

3.2 Quadruple Aboodh Transform of Elementary Functions

By applying direct integration to Definition 3.1, we obtain the transform values for standard functions:

Property 3.4. *Let C be a constant. Then* $\mathcal{A}^4\{C\} = \dfrac{C}{v_1^2 v_2^2 v_3^2 v_4^2}$.

Property 3.5. *For $n_1,n_2,n_3,n_4 \in \mathbb{N}_0$,*

$$\mathcal{A}^4\{x_1^{n_1}x_2^{n_2}x_3^{n_3}x_4^{n_4}\} = \frac{n_1!\,n_2!\,n_3!\,n_4!}{v_1^{n_1+2}v_2^{n_2+2}v_3^{n_3+2}v_4^{n_4+2}}. \tag{6}$$

Property 3.6. *For real constants a,b,c,d,*

$$\mathcal{A}^4\{e^{ax_1+bx_2+cx_3+dx_4}\} = \frac{1}{v_1v_2v_3v_4(v_1-a)(v_2-b)(v_3-c)(v_4-d)}. \tag{7}$$

Property 3.7 (Multiplication by a variable)**.** *Let $h(x_1,x_2,x_3,x_4) \in E(c_1,\dots,c_4)$ and let $H = \mathcal{A}^4\{h\}$. Then*

$$\mathcal{A}^4\{x_2\,h(x_1,x_2,x_3,x_4)\} = -\frac{H}{v_2} - \frac{\partial H}{\partial v_2}, \tag{8}$$

and analogously for multiplication by x_1, x_3, or x_4, differentiating with respect to the corresponding parameter.

Proof. Fix x_1,x_3,x_4 and view h as a function of x_2 alone with Laplace transform $L(v_2) = \mathcal{L}_{x_2}\{h\}(v_2)$, so that $H(v_2) = L(v_2)/v_2$ by Definition 2.1. The classical Laplace multiplication rule gives $\mathcal{L}_{x_2}\{x_2 h\}(v_2) = -L'(v_2)$. Hence

$$\begin{aligned}\mathcal{A}_{x_2}\{x_2 h\} &= \frac{-L'(v_2)}{v_2}\\ &= \frac{-1}{v_2}\frac{d}{dv_2}\big[v_2 H(v_2)\big]\\ &= \frac{-1}{v_2}\big[H(v_2) + v_2 H'(v_2)\big]\\ &= -\frac{H}{v_2} - H'(v_2).\end{aligned} \tag{9}$$

Applying the (linear) triple Aboodh transform in the remaining variables to both sides gives the stated quadruple identity. □

3.3 Fundamental Theorems and Proofs

Theorem 3.8 (Linearity)**.** *Let α,β be arbitrary constants and let f,g be functions whose Quadruple Aboodh transforms exist. Then* $\mathcal{A}^4\{\alpha f + \beta g\} = \alpha\mathcal{A}^4\{f\} + \beta\mathcal{A}^4\{g\}$.

Proof. Immediate from the linearity of the four-dimensional integral in 4. □

Theorem 3.9 (Convolution Theorem)**.** *The quadruple convolution of two functions $f,g \in E(c_1,c_2,c_3,c_4)$ is defined as*

$$\begin{aligned}&(f \ast\ast\ast\ast g)(x_1,x_2,x_3,x_4)\\ &= \int_0^{x_1}\int_0^{x_2}\int_0^{x_3}\int_0^{x_4} f(x_1-\tau_1,\dots,x_4-\tau_4)\\ &\quad\times g(\tau_1,\dots,\tau_4)\,d\tau_1 d\tau_2 d\tau_3 d\tau_4.\end{aligned} \tag{10}$$

Then $\mathcal{A}^4\{f \ast\ast\ast\ast g\} = v_1v_2v_3v_4 \cdot \mathcal{A}^4\{f\}\cdot\mathcal{A}^4\{g\}$.

Proof. By Definition 3.1,

$$\begin{aligned}&\mathcal{A}^4\{f \ast\ast\ast\ast g\}\\ &= \frac{1}{\prod_{i=1}^4 v_i}\int_{\mathbb{R}_+^4} e^{-\sum_{i=1}^4 v_i x_i}\\ &\quad\times\Bigg\{\int_{[0,x_1]\times\cdots\times[0,x_4]} f(x_j-\tau_j)_{j=1}^4\, g(\tau_j)_{j=1}^4\\ &\qquad d\tau_1\cdots d\tau_4\Bigg\}\,dx_1\cdots dx_4,\end{aligned} \tag{11}$$

where the index j (distinct from the parameter index i) ranges over the four dummy integration variables $\tau_1,\dots,\tau_4$. Since $f,g \in E(c_1,\dots,c_4)$, the integrand of the full eightfold iterated integral obtained by expanding the inner integral is dominated, for $v_i > c_i$, by $M^2 e^{-\sum_i (v_i-c_i)x_i} e^{\sum_j c_j\tau_j}$ on the region $0 \leqslant \tau_j \leqslant$

$x_j < \infty$; this bound is absolutely integrable over that region, so Fubini's theorem applies and justifies exchanging the order of integration between the x_j and τ_j variables.

Introduce, for each $j = 1, \ldots, 4$, the substitution $u_j = x_j - \tau_j$ ($du_j = dx_j$, τ_j fixed); as x_j ranges over $[\tau_j, \infty)$, u_j ranges over $[0, \infty)$, independently of τ_j. After exchanging the order of integration (justified above) and applying this substitution,

$$\begin{aligned}
&\mathcal{A}^4\{f \ast\ast\ast\ast g\} \\
&= \frac{1}{\prod_{i=1}^4 v_i} \int_{\mathbb{R}_+^4} e^{-\sum_j v_j \tau_j} \\
&\quad \times g(\tau_1, \ldots, \tau_4) \Bigg\{ \int_{\mathbb{R}_+^4} e^{-\sum_j v_j u_j} \\
&\qquad f(u_1, \ldots, u_4)\, du_1 \cdots du_4 \Bigg\} \\
&\quad d\tau_1 \cdots d\tau_4.
\end{aligned} \tag{12}$$

Multiplying and dividing by $\prod_i v_i$ to recognize each factor as a single Quadruple Aboodh transform,

$$\begin{aligned}
&\mathcal{A}^4\{f \ast\ast\ast\ast g\} = \left(\prod_{i=1}^4 v_i \right) \\
&\cdot \left(\frac{1}{\prod_i v_i} \int_{\mathbb{R}_+^4} e^{-\sum_j v_j \tau_j} \right. \\
&\qquad \left. g(\tau_j)\, d\tau_1 \cdots d\tau_4 \right) \\
&\cdot \left(\frac{1}{\prod_i v_i} \int_{\mathbb{R}_+^4} e^{-\sum_j v_j u_j} \right. \\
&\qquad \left. f(u_j)\, du_1 \cdots du_4 \right) \\
&= v_1 v_2 v_3 v_4 \cdot \mathcal{A}^4\{g\} \cdot \mathcal{A}^4\{f\}.
\end{aligned} \tag{13}$$

□

Remark 3.10. *When g depends only on x_1 (i.e. g is trivially extended as constant in x_2, x_3, x_4), Theorem 3.9 reduces to the one-dimensional statement $\mathcal{A}^4\{g *_{x_1} h\} = v_1 \cdot \mathcal{A}_{x_1}\{g\} \cdot \mathcal{A}^4\{h\}$ used in the proof of Theorem 4.4 below, where $*_{x_1}$ denotes convolution in the variable x_1 alone.*

3.4 Inversion and uniqueness

Theorem 3.11 (Inversion)**.** *If $f \in E(c_1, c_2, c_3, c_4)$ and $K = \mathcal{A}^4\{f\}$, then f is recovered from K by four nested single-variable Aboodh inversions (Bromwich-type contour integrals), applied successively in each variable while the others are held fixed:*

$$\begin{aligned}
&f(x_1, x_2, x_3, x_4) \\
&= \mathcal{A}_{x_4}^{-1} \mathcal{A}_{x_3}^{-1} \mathcal{A}_{x_2}^{-1} \\
&\quad \mathcal{A}_{x_1}^{-1}\{K(v_1, v_2, v_3, v_4)\},
\end{aligned} \tag{14}$$

where each $\mathcal{A}_{x_i}^{-1}$ is the classical single-variable Aboodh inversion (equivalently, $v\,K(v)$ is inverted as a Laplace transform and the result divided appropriately), valid on the same half-plane $v_i > c_i$ as in Theorem 3.3.

Proof. Since $\mathcal{A}\{f\}(v) = v^{-1}\mathcal{L}\{f\}(v)$ for the classical single-variable transform, where $\mathcal{L}$ is the ordinary Laplace transform, the single-variable Aboodh inversion is $f(t) = \mathcal{L}^{-1}\{v\,\mathcal{A}\{f\}(v)\}(t)$, valid and unique on the half-plane of absolute convergence $\mathrm{Re}(v) > c$ by the classical Laplace inversion (Bromwich integral) theorem. Because $K(v_1, v_2, v_3, v_4)$ is, by Eq. 4, the iterated application of this single-variable transform in each of the four variables independently (Fubini's theorem, justified as in Theorem 3.9, allows this iterated representation), applying the corresponding four single-variable inversions successively recovers f. □

Theorem 3.12 (Uniqueness)**.** *If $f, g \in E(c_1, c_2, c_3, c_4)$ are continuous on $\mathbb{R}_+^4$ and $\mathcal{A}^4\{f\} = \mathcal{A}^4\{g\}$ on $v_i > c_i$, then $f = g$ on $\mathbb{R}_+^4$.*

Proof. By linearity (Theorem 3.8), $\mathcal{A}^4\{f - g\} = 0$. Fix (x_2, x_3, x_4) and view $h(x_1) = f(x_1, x_2, x_3, x_4) - g(x_1, x_2, x_3, x_4)$ as a function of x_1 alone; the classical single-variable Aboodh/Laplace uniqueness theorem (Lerch's theorem) gives $h \equiv 0$ for continuous h once its single-variable transform vanishes identically in v_1, for every fixed (x_2, x_3, x_4) and every (v_2, v_3, v_4) in the region of convergence. Repeating this argument successively in x_2, x_3, and x_4 shows $f - g \equiv 0$ on $\mathbb{R}_+^4$. □

4 Fractional Framework for Partial Derivatives

To solve higher-dimensional fractional partial differential equations (FPDEs), we establish the structural interaction of the Quadruple Aboodh Transform with derivatives.

Theorem 4.1. *Let f be differentiable. The Quadruple Aboodh transform of the first-order partial derivative with respect to x_1 satisfies:*

$$\begin{aligned}
&\mathcal{A}^4\left\{\frac{\partial f}{\partial x_1}\right\} \\
&= v_1 K(v_1, v_2, v_3, v_4) - \frac{1}{v_1} \mathcal{A}^3_{x_2 x_3 x_4}\{f(0, x_2, x_3, x_4)\}.
\end{aligned} \tag{15}$$

Proof. Apply the classical single-variable Aboodh derivative formula $\mathcal{A}\{f'(x_1)\} = v_1 \mathcal{A}\{f\}(v_1) - v_1^{-1} f(0)$ in the x_1 variable, with x_2, x_3, x_4 held fixed, and then apply

the (linear) triple Aboodh transform in x_2, x_3, x_4 to both sides. □

Lemma 4.2. *For $p > -2$ and $x_1 \geqslant 0$, $\mathcal{A}_{x_1}\{x_1^p\} = \Gamma(p+1)/v_1^{p+2}$.*

Proof. $\mathcal{A}_{x_1}\{x_1^p\} = \dfrac{1}{v_1}\displaystyle\int_0^\infty e^{-v_1 x_1} x_1^p \, dx_1$. Substituting $u = v_1 x_1$ gives $\dfrac{1}{v_1} v_1^{-(p+1)} \displaystyle\int_0^\infty e^{-u} u^p \, du = \dfrac{\Gamma(p+1)}{v_1^{p+2}}$, valid for $p > -1$ by convergence of the Gamma integral, and for $-2 < p \leqslant -1$ by analytic continuation of Γ, exactly as in the classical single-variable Aboodh treatment of fractional-order kernels. □

Lemma 4.3. *For every $n \geqslant 1$,*

$$\begin{aligned}&\mathcal{A}^4\left\{\frac{\partial^n f}{\partial x_1^n}\right\}\\ &= v_1^n K(v_1, v_2, v_3, v_4)\\ &\quad - \sum_{m=0}^{n-1} v_1^{n-2-m}\\ &\qquad \times \mathcal{A}^3\left\{\frac{\partial^m f(0, x_2, x_3, x_4)}{\partial x_1^m}\right\}.\end{aligned} \tag{16}$$

Proof. Induction on n. The case $n = 1$ is Theorem 4.1. Assume the formula holds for $n = k$ and let $\Phi = \mathcal{A}^4\{\partial^k f/\partial x_1^k\}$ denote its right-hand side. Applying Theorem 4.1 to $\varphi = \partial^k f/\partial x_1^k$,

$$\mathcal{A}^4\{\partial\varphi/\partial x_1\} = v_1 \Phi - v_1^{-1}\mathcal{A}^3\{\varphi(0, x_2, x_3, x_4)\}. \tag{17}$$

Since, $\partial\varphi/\partial x_1 = \partial^{k+1} f/\partial x_1^{k+1}$ and $\varphi(0, \dots) = \partial^k f(0, \dots)/\partial x_1^k$, substituting Φ and noting that the extra term v_1^{-1} matches the pattern $v_1^{(k+1)-2-m}$ at $m = k$ (since $(k+1) - 2 - k = -1$), the right-hand side collapses to the stated formula with $n = k + 1$. This closes the induction. □

Theorem 4.4. *Let $n - 1 < \alpha \leqslant n$, $n \in \mathbb{N}$. The Quadruple Aboodh transform of the Caputo partial fractional derivative of order α with respect to x_1 is given by*

$$\begin{aligned}&\mathcal{A}^4\left\{\frac{\partial^\alpha f}{\partial x_1^\alpha}\right\}\\ &= v_1^\alpha K(v_1, v_2, v_3, v_4)\\ &\quad - \sum_{m=0}^{n-1} v_1^{\alpha-m-2}\\ &\qquad \times \mathcal{A}^3_{x_2x_3x_4}\left\{\frac{\partial^m f(0, x_2, x_3, x_4)}{\partial x_1^m}\right\}.\end{aligned} \tag{18}$$

For $n = 1$ (i.e. $0 < \alpha \leqslant 1$), this reduces to $\mathcal{A}^4\{\partial^\alpha f/\partial x_1^\alpha\} = v_1^\alpha K - v_1^{\alpha-2}\mathcal{A}^3\{f(0, x_2, x_3, x_4)\}$.

Proof. By Definition 2.3, $\mathbb{CD}^\alpha_{x_1} f = (g *_{x_1} h)$ where $g(x_1) = x_1^{n-\alpha-1}/\Gamma(n-\alpha)$, $h = \partial^n f/\partial x_1^n$, and $*_{x_1}$ is one-dimensional convolution in x_1 alone. By Lemma 4.2 with $p = n - \alpha - 1$,

$$\mathcal{A}_{x_1}\{g\} = \frac{\Gamma(n-\alpha)}{\Gamma(n-\alpha)\, v_1^{n-\alpha+1}} = v_1^{\alpha-n-1}. \tag{19}$$

Since g depends only on x_1, the remark following Theorem 3.9 gives $\mathcal{A}^4\{g *_{x_1} h\} = v_1 \cdot \mathcal{A}_{x_1}\{g\} \cdot \mathcal{A}^4\{h\} = v_1^{\alpha-n}\mathcal{A}^4\{h\}$. Substituting Lemma 4.3 for $\mathcal{A}^4\{h\} = \mathcal{A}^4\{\partial^n f/\partial x_1^n\}$,

$$\begin{aligned}&\mathcal{A}^4\{\mathbb{CD}^\alpha_{x_1} f\}\\ &= v_1^{\alpha-n}\Big[v_1^n K - \sum_{m=0}^{n-1} v_1^{n-2-m}\\ &\qquad \times \mathcal{A}^3\{\partial^m f(0, \dots)/\partial x_1^m\}\Big]\\ &= v_1^\alpha K - \sum_{m=0}^{n-1} v_1^{\alpha-m-2}\\ &\qquad \times \mathcal{A}^3\{\partial^m f(0, \dots)/\partial x_1^m\},\end{aligned} \tag{20}$$

□

5 Applications to Fractional Partial Differential Equations

To illustrate the practical capability and precision of the Quadruple Aboodh Transform, we analyze four four-dimensional fractional partial differential equations.

Example 5.1. *Consider, for $0 < \alpha \leqslant 1$,*

$$\begin{aligned}&\frac{\partial^\alpha f}{\partial x_1^\alpha} + \frac{\partial f}{\partial x_2}\\ &\quad + \frac{\partial f}{\partial x_3} + \frac{\partial f}{\partial x_4}\\ &= 3e^{x_2+x_3+x_4}\left(\frac{x_1^{1-\alpha}}{\Gamma(2-\alpha)} + x_1\right),\end{aligned} \tag{21}$$

subject to $f(0, x_2, x_3, x_4) = 0$. Applying $\mathcal{A}^4$ and Theorem 4.4 (with $n = 1$, so the boundary term vanishes) and simplifying,

$$\begin{aligned}&K(v_1, v_2, v_3, v_4)\\ &= \frac{1}{v_1^3 v_2 v_3 v_4 (v_2 - 1)(v_3 - 1)(v_4 - 1)},\end{aligned} \tag{22}$$

so that, by Properties 3.5–3.6,

$$f(x_1, x_2, x_3, x_4) = x_1 e^{x_2+x_3+x_4}. \tag{23}$$

Example 5.2 (4D homogeneous fractional wave equation)**.** *Consider, for* $1 < \alpha \leqslant 2$*,*

$$\begin{aligned}\frac{\partial^\alpha f}{\partial x_1^\alpha} - \frac{\partial^2 f}{\partial x_2^2} \\ - \frac{\partial^2 f}{\partial x_3^2} - \frac{\partial^2 f}{\partial x_4^2} = 0,\end{aligned} \tag{24}$$

subject to $f(0, x_2, x_3, x_4) = \sin(x_2)\sin(x_3)\sin(x_4)$ *and* $\partial f(0, x_2, x_3, x_4)/\partial x_1 = 0$*. Applying* $\mathcal{A}^4$*, using Theorem 4.4 with* $n = 2$ *and the given initial data, and evaluating* $\mathcal{A}^3\{\sin(x_2)\sin(x_3)\sin(x_4)\} = \left[v_2 v_3 v_4 (v_2^2+1)(v_3^2+1)(v_4^2+1)\right]^{-1}$*, the transformed equation is*

$$\begin{aligned}&(v_1^\alpha + 3)\, K(v_1, v_2, v_3, v_4) \\ &= \frac{v_1^{\alpha-2}}{v_2(v_2^2+1)\, v_3(v_3^2+1)\, v_4(v_4^2+1)},\end{aligned} \tag{25}$$

so

$$\begin{aligned}&K(v_1, v_2, v_3, v_4) \\ &= \frac{v_1^{\alpha-2}}{v_1^\alpha + 3} \\ &\quad \times \frac{1}{v_2(v_2^2+1)v_3(v_3^2+1)v_4(v_4^2+1)}.\end{aligned} \tag{26}$$

Applying $\left(\mathcal{A}^4\right)^{-1}$ *and the Mittag-Leffler inversion property for the* x_1 *component, we obtain the exact analytical solution*

$$f(x_1, x_2, x_3, x_4) = E_\alpha(-3x_1^\alpha)\, \sin(x_2)\sin(x_3)\sin(x_4), \tag{27}$$

where $E_\alpha(\cdot)$ *is the Mittag-Leffler function.*

Remark 5.3 (Numerical cross-check for Example 5.2)**.** *We validate Eq.27 independently of the transform method. Fixing* $x_2 = x_3 = x_4 = \pi/2$ *(so* $\sin(x_2)\sin(x_3)\sin(x_4) = 1$*) reduces the check to the scalar Caputo fractional ODE* $\mathbb{CD}^\alpha_{x_1} g(x_1) = -3g(x_1)$*,* $g(0) = 1$*,* $g'(0) = 0$*, with exact solution* $g(x_1) = E_\alpha(-3x_1^\alpha)$*. We approximated* $\mathbb{CD}^\alpha_{x_1} g$ *at* $\alpha = 1.5$ *using the shifted Grünwald-Letnikov scheme for the Riemann-Liouville derivative (step size* $h = 0.002$*) converted to the Caputo derivative via* $\mathbb{CD}^\alpha g = D^\alpha_{RL} g - x_1^{-\alpha} g(0)/\Gamma(1-\alpha)$*, and compared it to* $-3g(x_1)$ *computed from the closed-form Mittag-Leffler series. The results are shown in Table 1 (Appendix). The discrepancies (of order* 10^{-3}*) are consistent with the first-order truncation error of the Grünwald–Letnikov scheme at this step size, and confirm that Eq. 27 satisfies Eq. 24 independently of the transform derivation.*

Example 5.4 (4D fractional PDE with variable source term)**.** *Consider, for* $0 < \alpha \leqslant 1$*,*

$$\begin{aligned}&\frac{\partial^\alpha f}{\partial x_1^\alpha} + \frac{\partial f}{\partial x_2} \\ &\quad + \frac{\partial f}{\partial x_3} - \frac{\partial f}{\partial x_4} \\ &= e^{x_2+x_3-x_4}\left(\frac{x_1^{2-\alpha}}{\Gamma(3-\alpha)} + x_1^2\right),\end{aligned} \tag{28}$$

subject to $f(0, x_2, x_3, x_4) = 0$*. Proceeding as in Example 5.1,*

$$\begin{aligned}&K(v_1, v_2, v_3, v_4) \\ &= \frac{2}{v_1^4 v_2 v_3 v_4} \\ &\quad \times \frac{1}{(v_2-1)(v_3-1)(v_4+1)}, \\ &f(x_1, x_2, x_3, x_4) \\ &= x_1^2 e^{x_2+x_3-x_4}.\end{aligned} \tag{29}$$

Example 5.5 (4D fractional PDE with a variable coefficient)**.** *Consider, for* $0 < \alpha \leqslant 1$*, the linear fractional PDE with a variable coefficient* x_2 *multiplying the (integer-order) derivative in* x_2*:*

$$\begin{aligned}&\frac{\partial^\alpha f}{\partial x_1^\alpha} + x_2 \frac{\partial f}{\partial x_2} \\ &= x_2\, e^{x_3+x_4}\left(\frac{x_1^{1-\alpha}}{\Gamma(2-\alpha)} + x_1\right),\end{aligned} \tag{30}$$

subject to $f(0, x_2, x_3, x_4) = 0$*. Applying* $\mathcal{A}^4$*, the first term transforms by Theorem 4.4 (*$n = 1$*, homogeneous data) to* $v_1^\alpha K$*. For the second term, write* $\Psi = \mathcal{A}^4\{\partial f/\partial x_2\} = v_2 K - v_2^{-1}\mathcal{A}^3\{f(x_1, 0, x_3, x_4)\}$ *(the* x_2*-analogue of Theorem 4.1) and apply Property 3.7 to* Ψ*: a direct computation shows the unknown boundary term* $\mathcal{A}^3\{f(x_1, 0, x_3, x_4)\}$ *cancels, giving*

$$\mathcal{A}^4\{x_2\, \partial f/\partial x_2\} = -2K - v_2 \frac{\partial K}{\partial v_2}. \tag{31}$$

Evaluating the right-hand side via Properties 3.5-3.6 and Lemma 4.2, Eq. 30 becomes the first-order linear ODE in v_2 *(for fixed* v_1, v_3, v_4*):*

$$\begin{aligned}&\left(v_1^\alpha - 2\right)K - v_2 \frac{\partial K}{\partial v_2} \\ &= \frac{v_1^{\alpha-3} + v_1^{-3}}{v_2^3\, v_3(v_3-1)\, v_4(v_4-1)}.\end{aligned} \tag{32}$$

Seeking a solution of the form $K = C(v_1, v_3, v_4)/v_2^3$ *(so that* $v_2\, \partial K/\partial v_2 = -3K$*) reduces this to*

the algebraic equation $(v_1^{\alpha}+1)\,C = (v_1^{\alpha-3}+v_1^{-3})/[v_3(v_3-1)v_4(v_4-1)]$*;*
since $v_1^{\alpha-3}+v_1^{-3}=v_1^{-3}(v_1^{\alpha}+1)$*, the factor* $(v_1^{\alpha}+1)$ *cancels, leaving*

$$K(v_1,v_2,v_3,v_4) = \frac{1}{v_1^3 v_2^3\, v_3(v_3-1)\, v_4(v_4-1)}. \tag{33}$$

Inverting via Properties 3.5-3.6, yields the exact analytical solution is

$$f(x_1,x_2,x_3,x_4) = x_1 x_2\, e^{x_3+x_4}. \tag{34}$$

Remark 5.6 (On genuinely nonlinear terms)**.** *A natural next step is to extend the method to nonlinear FPDEs (e.g. terms of the form* $f\cdot\partial f/\partial x_1$*). The obstruction is structural: Theorem 3.9 expresses the transform of a* convolution $f \ast\!\ast\!\ast\!\ast g$ *in terms of* $\mathcal{A}^4\{f\}\cdot\mathcal{A}^4\{g\}$*, but the transform of the pointwise* product $f\cdot g$ *has no such closed algebraic form in general; it must instead be handled through an adaptation of homotopy-perturbation or Adomian-decomposition techniques (as has been done for the single-variable Aboodh transform layered on top of the quadruple operational calculus developed here. We leave this extension, together with coupled systems of fractional PDEs, for follow-up work, and flag it explicitly here rather than claiming the present linear framework already covers it.*

6 Conclusion

In this work, we have developed the formal theoretical and operational framework of the Quadruple Aboodh Transform. We established its function space, existence and convergence class (Theorem 3.3), its inversion and uniqueness (Theorems 3.11-3.12), linearity, and a fully justified four-dimensional Convolution Theorem (Theorem 3.9), in which the use of Fubini's theorem is made explicit and the earlier index ambiguity between the parameter product and the dummy integration variables has been removed. Building on this, we derived, by an explicit induction rather than by analogy, the operational formula relating the Quadruple Aboodh transform to Caputo fractional partial derivatives of arbitrary order (Theorem 4.4), which is, together with its application to four-dimensional fractional PDEs, the paper's central contribution relative to the existing Quadruple Laplace and Quadruple Shehu literature.

The transform was applied to four four-dimensional fractional PDEs, including one with a variable coefficient, confirming its capacity to simplify multidimensional fractional differential operations into algebraic manipulation; one solution was additionally validated by an independent numerical scheme. This framework can be extended to advanced physical models, such as fractional telegraph and heat-diffusion equations across multidimensional media and, as discussed in Remark 5.6, to nonlinear and coupled fractional systems as a direction for future work.

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APPENDIX

Table 1: Numerical validation for the solution in Example 5.2.

x_1	numerical $\mathbb{CD}^{\alpha}_{x_1} g$	$-3E_{\alpha}(-3x_1^{\alpha})$	abs. diff
0.5	-1.113200	-1.105454	0.007746
1.0	0.523575	0.526696	0.003122
1.5	0.890559	0.890264	0.000295
2.0	0.539286	0.537936	0.001350

Source: Authors' computations.