On Sequence Spaces and Matrix Transformations in Non-Newtonian Calculus

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Abstract: - This work develops sequence spaces and matrix transformations within a non-Newtonian calculus generated by a strictly increasing function α. The concepts of α-arithmetic, α-metric, α-norm, α-convergence, and α-Cauchy sequences are introduced on the α-real number system $\mathbb{R}_\alpha$. The classical sequence spaces are generalized to spaces ℓ_α^∞, c_α, $c_{\alpha,0}$, and ℓ_α of non-Newtonian bounded, convergent, null, and absolutely summable sequences, respectively. The dual spaces of $c_{\alpha,0}$, and c_α are characterized in terms of α-absolutely summable sequences and α-limit functionals. Necessary and sufficient conditions for α-conservative and α-regular matrices are obtained in a non-Newtonian setting. The α-version of the Silverman-Toeplitz theorem is proven using boundedness, column convergence, and the preservation of the constant sequence (1). Finally, it is shown that various matrix domains associated with α-preserving matrices are closed subspaces of corresponding α-sequence spaces.

Key-Words: - α-arithmetic, α-metrics, α-norms, α-convergence, α-Cauchy sequence, α-matrix, α-conservative, α-regular.

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1 Introduction

Non-Newtonian calculus was first introduced as an alternative to classical calculus [1]. Several studies have developed the theory of sequence spaces within the non-Newtonian framework. For example, it has been established fundamental properties of non-Newtonian sequence spaces, including their vector-space and completeness structures [2]. Subsequent studies have examined sequence spaces over the non-Newtonian complex field and obtained corresponding structural results [3], while matrix transformations between such spaces have also been investigated [4]. In contrast, the present paper develops a framework for sequence spaces and matrix transformations over the non-Newtonian real field generated by an arbitrary strictly increasing function α, with particular attention to α-duals, α-conservative and α-regular matrices, and the closedness of associated matrix domains. Non-Newtonian calculus offers wide-ranging applications in science, engineering, and mathematics. Bigeometric calculus, an important branch of non-Newtonian calculus, has been studied extensively in the literature. Prior work has addressed bigeometric calculus on a multiplicative Lorenz system [5] and examined the elasticity of demand/measure of elasticity in economics via bigeometric calculus [6]. Bigeometric calculus has also been referred to as G-calculus, with investigations into the basic properties of derivatives and integrals in the bigeometric setting alongside applications to numerical analysis [7], [8]. Furthermore, researchers have investigated the solvability of bigeometric differential equations using numerical methods [9], defined Volterra integral equations in the sense of bigeometric calculus, and analyzed their relationship with bigeometric differential equations [10]. In addition, non-Newtonian real series have been examined in detail [11], the foundations of standard topology have been laid on the set of non-Newtonian real numbers [12], and the concept of the Lebesgue measure on real numbers has been extended to the non-Newtonian setting [13]. Further details can be found in [14], [15], [16], [17], [18].

The purpose of this paper is to develop a systematic non-Newtonian analogue of the classical theory of sequence spaces and matrix transformations. More precisely, we establish the α-Banach structure of the basic sequence spaces, characterize their α-duals, and derive necessary and sufficient conditions for α-conservative and α-regular matrix transformations. In particular, the classical Silverman–Toeplitz characterization is

extended to the non-Newtonian setting. We also investigate the closedness of matrix domains generated by α-conservative matrices. Thus, the paper provides a unified framework connecting non-Newtonian sequence spaces, bounded linear functionals, and matrix summability methods. Specifically, the non-Newtonian basis of the concepts and properties in [19] is established, and [20] is also consulted for the functional-analytic concepts and tools required throughout the study.

A generator α is a strictly increasing function from $\mathbb{R}$ to a subset $\mathbb{R}_\alpha$ of $\mathbb{R}$:

$$\alpha: \mathbb{R} \to \mathbb{R}_\alpha,\ \alpha(x) = \dot{x}.$$

Every generator α produces exactly one arithmetic. Conversely, every arithmetic is generated by exactly one generator α. Here, an arithmetic is any mathematical structure that satisfies the axioms of complete ordered fields and has a domain that is a subset of $\mathbb{R}$. Infinite numbers of arithmetic constructs exist.

We define an α-arithmetic using the generator α as follows: For $\dot{x}, \dot{y} \in \mathbb{R}_\alpha$,

α-addition: $\dot{x}\dot{+}\dot{y} = \alpha\left(\alpha^{-1}(\dot{x}) + \alpha^{-1}(\dot{y})\right)$,

α-subtraction: $\dot{x}\dot{-}\dot{y} = \alpha\left(\alpha^{-1}(\dot{x}) - \alpha^{-1}(\dot{y})\right)$,

α-multiplication: $\dot{x} \dot{\times} \dot{y} = \alpha\left(\alpha^{-1}(\dot{x}) \times \alpha^{-1}(\dot{y})\right)$,

α-division: $\dot{x}\dot{/}\dot{y} = \alpha\left(\alpha^{-1}(\dot{x})/\alpha^{-1}(\dot{y})\right)$;
$\alpha^{-1}(\dot{x}) = y \neq 0$,

α-order: $\dot{x} \dot{<} \dot{y} \Leftrightarrow \alpha^{-1}(\dot{x}) < \alpha^{-1}(\dot{y})$,
$x \leq y \Leftrightarrow \alpha^{-1}(\dot{x}) \leq \alpha^{-1}(\dot{y})$.

The set $\mathbb{R}_\alpha = \{\alpha(x) = \dot{x}: x \in \mathbb{R}\}$ is called the set of α-real numbers (or non-Newtonian real numbers). Similarly, the set defined below is called the set of positive α-real numbers (or positive non-Newtonian real numbers):

$$\mathbb{R}_\alpha^+ = \left\{\dot{x} \in \mathbb{R}_\alpha: \dot{x} \dot{>} \dot{0}\right\}.$$

Here, the operators $+, -, \times, /, <$ and $\leq$ are usual operations in classical calculus. The system $(\mathbb{R}_\alpha, \dot{+}, \dot{\times}, \dot{\leq})$ is a complete ordered field. Calculus based on α-arithmetic is called non-Newtonian calculus. In particular, if α is the identity function, then α-arithmetic coincides with the usual arithmetic. On the other hand, if α is an exponential function, then α-arithmetic reduces to the corresponding geometric arithmetic [1], [11], [12]:

$$\begin{aligned}
\dot{x}\dot{+}\dot{y} &= \alpha\left(\alpha^{-1}(\dot{x}) + \alpha^{-1}(\dot{y})\right) \\
&= \alpha(x + y) = e^{\ln \dot{x} + \ln \dot{y}} = \dot{x}.\dot{y}, \\
\dot{x}\dot{-}\dot{y} &= \alpha\left(\alpha^{-1}(\dot{x}) - \alpha^{-1}(\dot{y})\right) = \alpha(x - y) \\
&= e^{\ln \dot{x} - \ln \dot{y}} = \dot{x} \div \dot{y}, \dot{y} \neq \dot{0}, \\
\dot{x} \dot{\times} \dot{y} &= \alpha\left(\alpha^{-1}(\dot{x}) \times \alpha^{-1}(\dot{y})\right) \\
&= \alpha(x \times y) = e^{\ln \dot{x} \times \ln \dot{y}} = \dot{x}^{\ln \dot{y}} = \dot{y}^{\ln \dot{x}}, \\
\dot{x}\dot{/}\dot{y} &= \alpha\left(\alpha^{-1}(\dot{x})/\alpha^{-1}(\dot{y})\right) = \alpha(x/y) \\
&= e^{\ln \dot{x} \div \ln \dot{y}} = \dot{x}^{1/\ln \dot{y}};\ \dot{y} \neq \dot{1}, \\
&\dot{x} \dot{<} \dot{y} \Leftrightarrow \ln \dot{x} < \ln \dot{y} \Leftrightarrow x < y.
\end{aligned}$$

The α-absolute value denoted by $|.|_\alpha$ on $\mathbb{R}_\alpha$ is defined as

$$|\dot{x}|_\alpha = \alpha(|\alpha^{-1}(\dot{x})|) = \alpha(|x|)$$

[11], [12].

Let X be a non-empty set and let $d_\alpha: X \times X \to \mathbb{R}_\alpha$ be a function satisfying the conditions:

(NM1) $d_\alpha(\dot{x}, \dot{y}) = \dot{0} \Leftrightarrow \dot{x} = \dot{y}$,
(NM2) $d_\alpha(\dot{x}, \dot{y}) = d_\alpha(\dot{y}, \dot{x})$,
(NM3) $d_\alpha(\dot{x}, \dot{y}) \dot{\leq} d_\alpha(\dot{x}, \dot{z})\dot{+}\, d_\alpha(\dot{z}, \dot{y})$.

Then d_α is called α-metric (non-Newtonian metric) and the pair $\left(\dot{X}, d_\alpha\right)$ is called an α-metric space (non-Newtonian metric space). If $d_\alpha(\dot{x}, \dot{y}) = |\dot{x}\dot{-}\dot{y}|_\alpha$ is taken, then it is known that $(\mathbb{R}_\alpha, d_\alpha)$ is an α-metric space [11].

Let X be a $\mathbb{R}_\alpha$-vector space and $\|.\|_\alpha: X \times X \to \mathbb{R}_\alpha^+ \cup \{\dot{0}\}$ be a function satisfying the conditions as follows:

(NN1) $\|\dot{x}\|_\alpha = \dot{0} \Leftrightarrow \dot{x} = \dot{0}$,
(NN2) $\left\|\dot{\lambda} \dot{\times} \dot{x}\right\|_\alpha = \left|\dot{\lambda}\right|_\alpha \dot{\times} \|\dot{x}\|_\alpha$ for all $\dot{\lambda} \in \mathbb{R}_\alpha$,
(NN3) $\|\dot{x}\dot{+}\dot{y}\|_\alpha \dot{\leq} \|\dot{x}\|_\alpha \dot{+}\, \|\dot{y}\|_\alpha$ for all $\dot{x}, \dot{y} \in \dot{X}$.

Then $\|.\|_\alpha$ is called α-norm (non-Newtonian norm) and the pair $\left(\dot{X}, \|.\|_\alpha\right)$ is called an α-normed space (non-Newtonian normed space) [12], [15].

For a subset $A \subset \mathbb{R}_\alpha$, α-supremum and α-infimum are defined as follows, respectively:

$$\alpha - \sup A = \alpha(\sup\{\alpha^{-1}(\dot{a}): \dot{a} \in A\})$$

and

$$\alpha - \inf A = \alpha(\inf\{\alpha^{-1}(\dot{a}): \dot{a} \in A\}).$$

If $(\dot{x}_k)$ is an α-sequence, then it is easily observed that

$$\begin{aligned}
\alpha - \sup_{k \in \mathbb{N}} |\dot{x}_k|_\alpha &= \alpha - \sup_{k \in \mathbb{N}} \alpha(|\alpha^{-1}(\dot{x}_k)|) \\
&= \alpha - \sup_{k \in \mathbb{N}} \alpha(|x_k|) \\
&= \alpha(\sup_{k \in \mathbb{N}} |x_k|) \\
&= \alpha(\sup_{k \in \mathbb{N}} |\alpha^{-1}(\dot{x}_k)|).
\end{aligned}$$

A function $f: X \to \mathbb{R}_\alpha$ is called an α-linear functional if it satisfies the following conditions: For $\dot{x}, \dot{y} \in X$ and $\dot{\lambda} \in \mathbb{R}_\alpha$

(i) $f(\dot{x}\dot{+}\dot{y}) = f(\dot{x})\dot{+}f(\dot{y})$, (ii) $f\left(\dot{\lambda} \dot{\times} \dot{x}\right) = \dot{\lambda} f(\dot{x})$.

If there is a number $\dot{M} \dot{>} \dot{0}$ such that $|f(\dot{x})|_\alpha \dot{\leq} \dot{M} \dot{\times} \|\dot{x}\|_\alpha$ for all $\dot{x} \in \dot{X}$, then f is said to be the α-bounded linear operator. The set of all α-

bounded operators defined on the space X is denoted by X', and similar to the classical case, it can be easily shown that X' is a vector space and also a normed space with the norm

$$\|f\|'_\alpha = \alpha - \sup\{\, |f(\dot{x})|_\alpha : \|\dot{x}\|_\alpha \dot{\leq} \dot{1}\}$$
$$= \alpha(\sup\{\alpha^{-1}(|f(\dot{x})|) : \alpha^{-1}(\|\dot{x}\|_\alpha) \leq 1\}).$$

Let $(\dot{x}_n)$ be a sequence in $\mathbb{R}_\alpha$ (an α-sequence). If, given $\dot{\varepsilon} \dot{>} \dot{0}$, there exists $n_0 = n_0(\dot{\varepsilon}) \in \mathbb{N}$ such that

$$|\dot{x}_n \dot{-} \dot{x}|_\alpha = \alpha(|\alpha^{-1}(\dot{x}_n) - \alpha^{-1}(\dot{x})|)$$
$$= \alpha(|x_n - x|) \dot{<} \dot{\varepsilon} \Leftrightarrow |x_n - x| < \alpha^{-1}(\dot{\varepsilon}) = \varepsilon$$

for all $n > n_0$, then $(\dot{x}_n)$ is said to be α-convergent (non-Newtonian convergent) to $\dot{x}$ and ${}^{\alpha}\lim_{n\to\infty} \dot{x}_n = \dot{x}$ or $\dot{x}_n \to \dot{x}$ as $n \to \infty$ is written. Also, if, given $\dot{\varepsilon} \dot{>} \dot{0}$, there exists $n_0 = n_0(\dot{\varepsilon}) \in \mathbb{N}$ such that $|\dot{x}_n \dot{-} \dot{x}_m|_\alpha \dot{<} \dot{\varepsilon} \Leftrightarrow |x_n - x_m| < \alpha^{-1}(\dot{\varepsilon}) = \varepsilon$ for all $n, m > n_0$, then $(\dot{x}_n)$ is called α-Cauchy sequence (non-Newtonian Cauchy sequence). If every α-Cauchy sequence in a α-normed space converges, then this space is called α-Banach space. It follows directly from the definition of α-convergence that an α-sequence $(\dot{x}_n)$ is α-convergent to $\dot{x}$ if and only if the sequence $\left(\alpha^{-1}(\dot{x}_n)\right) = (x_n)$ converges to $\alpha^{-1}(\dot{x}) = x$ in the usual real number system. Similarly, $(\dot{x}_n)$ is an α-Cauchy sequence if and only if $\left(\alpha^{-1}(\dot{x}_n)\right) = (x_n)$ is a Cauchy sequence in $\mathbb{R}$ in the usual sense. It is clear that ${}^{\alpha}\lim_{n\to\infty} \dot{x}_n = \alpha(\lim_{n\to\infty} x_n)$ can be written.

Throughout this study, ω and ω_α represents the set of all real sequences and of all α-sequences, respectively. It is can be easily seen that the set ω_α, together with the algebraic operations of addition ($\dot{+}$) and scalar multiplication ($\dot{\times}$) defined on it as follows, forms a vector space over $\mathbb{R}_\alpha$: For $\dot{x} = (\dot{x}_k)$, $\dot{y} = (\dot{y}_k) \in \omega_\alpha$ and $\dot{\alpha} \in \mathbb{R}_\alpha$,

$$\dot{+} : \omega_\alpha \times \omega_\alpha \to \omega_\alpha, \qquad (\dot{x}, \dot{y}) \to \dot{x} \dot{+} \dot{y} = (\dot{x}_k \dot{+} \dot{y}_k)$$

and

$$\dot{\times} : \mathbb{R}_\alpha \times \omega_\alpha \to (\dot{\alpha}, \dot{x}) \to \dot{\alpha} \dot{\times} \dot{x} = (\dot{\alpha} \dot{\times} \dot{x}_k).$$

An α-infinite sum

$$\dot{x}_1 \dot{+} \dot{x}_2 \dot{+} \ldots \dot{+} \dot{x}_k \dot{+} \ldots = {}_{\alpha}\dot{\sum}_{k=1}^{\infty} \dot{x}_k,$$

is called an α-series or a non-Newtonian real series.

The sequence of partial sums of an α-series is a sequence $\left(\dot{S}_m\right)$ with $\dot{S}_m = {}_{\alpha}\dot{\sum}_{k=1}^{m} \dot{x}_k$ for all $m \in \mathbb{N}$. If $\left(\dot{S}_m\right)$ is α-convergent, then the α-series ${}_{\alpha}\dot{\sum}_{k=1}^{\infty} \dot{x}_k$ is said to be α-convergent. If ${}^{\alpha}\lim_{m\to\infty} \dot{S}_m = \dot{S}$ with $\dot{S} \in \mathbb{R}_\alpha$, then ${}_{\alpha}\dot{\sum}_{k=1}^{\infty} \dot{x}_k = \dot{S}$ is written. If ${}^{\alpha}\lim_{m\to\infty} \dot{S}_m$ is not or neither equal to $\dot{-}\infty = \lim_{n\to-\infty} \alpha(n)$ or $\dot{+}\infty = \lim_{n\to+\infty} \alpha(n)$, then the α-series ${}_{\alpha}\dot{\sum}_{k=1}^{\infty} \dot{x}_k$ is said to be α-divergent.

An α-sequence $(\dot{x}_k)$ of α-scalars is briefly denoted by $\dot{x}$. The α-series ${}_{\alpha}\dot{\sum}_{k=1}^{\infty} \dot{x}_k \dot{\times} \dot{y}_k$ is also denoted by $\dot{x} \dot{\times} \dot{y}$ [19].

We define the sets ℓ_α^∞, c_α, $c_{\alpha,0}$ and ℓ_α of α-bounded, α-convergent, α-null and α-absolutely summable sequences over $\mathbb{R}_\alpha$, which correspond to the sets ℓ^∞, c, c_0, and ℓ over $\mathbb{R}$ [20], respectively, as follows:

$$\ell_\alpha^\infty = \{\dot{x} \in \omega_\alpha : \alpha - \sup_{k\in\mathbb{N}} |\dot{x}_k|_\alpha \dot{<} \dot{+}\infty\},$$
$$c_\alpha = \{\dot{x} \in \omega_\alpha : {}^{\alpha}\lim_{k\to\infty} \dot{x}_k \text{ exists}\},$$
$$c_{\alpha,0} = \left\{\dot{x} \in \omega_\alpha : {}^{\alpha}\lim_{k\to\infty} \dot{x}_k = \dot{0}\right\},$$
$$\ell_\alpha = \left\{\dot{x} \in \omega_\alpha : {}_{\alpha}\dot{\sum}_{k=1}^{\infty} |\dot{x}_k|_\alpha \dot{<} \dot{+}\infty\right\}.$$

Let us define the transformation $\Phi(\dot{x}) = \left(\alpha^{-1}(\dot{x}_k)\right)_{k=1}^{\infty}$. Φ is a bijection that maps α-sequences to the corresponding classical real sequences. Furthermore, $\left(\dot{x}^{(n)}\right)$ converges to $\dot{a}$ in the α-sense if and only if $\Phi\left(\dot{x}^{(n)}\right)$ converges to $\Phi(\dot{a})$ in the classical sense. Similarly, $\left(\dot{x}^{(n)}\right)$ is α-Cauchy if and only if $\Phi\left(\dot{x}^{(n)}\right)$ is Cauchy in the classical sense. Thanks to this transformation, classical properties such as completeness, closedness, and boundedness can be transferred to α-spaces.

2 Main Results

Lemma 2.1. The set ℓ_α is an α-vector space over $\mathbb{R}_\alpha$ and

$$\|\dot{x}\|_{\alpha,1} = {}_{\alpha}\dot{\sum}_{k=1}^{\infty} |\dot{x}_k|_\alpha = \alpha\left(\sum_{k=1}^{\infty} |\alpha^{-1}(\dot{x}_k)|\right)$$

is an α-norm on ℓ_α.

Proof. Let $\dot{x}, \dot{y} \in \ell_\alpha$ and $\dot{\mu} \in \mathbb{R}_\alpha$. From the triangle inequality of the α-absolute value, we obtain

$$|\dot{x}_k \dot{+} \dot{y}_k|_\alpha \dot{\leq} |\dot{x}_k|_\alpha \dot{+} |\dot{y}_k|_\alpha$$

for all $k \in \mathbb{N}$. Performing α-addition yields the inequality

$$\|\dot{x} \dot{+} \dot{y}\|_{\alpha,1} \dot{\leq} \|\dot{x}\|_{\alpha,1} \dot{+} \|\dot{y}\|_{\alpha,1}.$$

Furthermore,

$$\|\dot{\mu} \dot{\times} \dot{y}\|_{\alpha,1} = {}_{\alpha}\dot{\sum}_{k=1}^{\infty} |\dot{\mu} \dot{\times} \dot{x}_k|_\alpha$$
$$= \alpha\left(\sum_{k=1}^{\infty} \alpha^{-1}(|\dot{\mu} \dot{\times} \dot{x}_k|_\alpha)\right)$$
$$= \alpha\left(\sum_{k=1}^{\infty} \alpha^{-1}\{\alpha(|\alpha^{-1}(\dot{\mu} \dot{\times} \dot{x}_k)|)\}\right)$$
$$= \alpha\left(\sum_{k=1}^{\infty} |\alpha^{-1}(\dot{\mu}) \times \alpha^{-1}(\dot{x}_k)|\right)$$
$$= \alpha\left(|\alpha^{-1}(\dot{\mu})| \times \sum_{k=1}^{\infty} |\alpha^{-1}(\dot{x}_k)|\right)$$
$$= |\dot{\mu}|_\alpha \dot{\times} \|\dot{x}\|_{\alpha,1}.$$

Finally, if $\|\dot{x}\|_{\alpha,1} = \dot{0}$, then every $\dot{x}_k = \dot{0}$. Thus, the three α-norm axioms are satisfied.

Theorem 2.2. ℓ_α is an α-Banach space.

Proof. Let $\left(\dot{x}^{(m)}\right)_{m\in\mathbb{N}}$ be an arbitrary α-Cauchy sequence in ℓ_α. For each $m \in \mathbb{N}$, define $x^{(m)} = \left(\alpha^{-1}\left(\dot{x}_k^{(m)}\right)\right)_{k\in\mathbb{N}}$. Then, for each $m, n \in \mathbb{N}$,

$$\left\|\dot{x}^{(m)} \dot{-} \dot{x}^{(n)}\right\|_{\alpha,1} = \alpha\left(\left\|x^{(m)} - x^{(n)}\right\|_1\right).$$

Due to the α-Cauchy property and the fact that α is strictly increasing, $\left(x^{(m)}\right)_{m\in\mathbb{N}}$ is Cauchy in the classical ℓ_1 space. Since ℓ_1 is complete, there exists $x = (x_k) \in \ell_1$ such that $\left\|x^{(m)} - x\right\|_1 \to 0$. Define $\dot{x}_k = \alpha(x_k)$ for each $k \in \mathbb{N}$. Thus, $\dot{x} = (\dot{x}_k) \in \ell_\alpha$ and

$$\left\|\dot{x}^{(m)} \dot{-} \dot{x}\right\|_{\alpha,1} = \alpha\left(\left\|x^{(m)} - x\right\|_1\right) \to \dot{0}.$$

Therefore, ℓ_α is an α -Banach space.

Lemma 2.3. The sets ℓ_α^∞, c_α and $c_{\alpha,0}$ are α-normed spaces with

$$\|\dot{x}\|_{\alpha,\infty} = \alpha\{\sup_{k\in\mathbb{N}}|\alpha^{-1}(\dot{x}_k)|\}.$$

Proof. It is readily verified that ℓ_α^∞, c_α and $c_{\alpha,0}$ are closed under α-addition and α-scalar multiplication. Since $c_\alpha, c_{\alpha,0} \subset \ell_\alpha^\infty$, the above function $\|.\|_{\alpha,\infty}$ is well defined on all three spaces. The axioms (NN1) and (NN2) follow directly from the definition of the α-absolute value and the α-supremum. For (NN3), let $\dot{x}, \dot{y}$ belong to any one of these spaces. Then

$$\begin{aligned}
&\|\dot{x}\dot{+}\dot{y}\|_{\alpha,\infty} = \alpha - \sup_{k\in\mathbb{N}}|\dot{x}_k\dot{+}\dot{y}_k|_\alpha \\
&\leq \alpha - \sup_{k\in\mathbb{N}}(|\dot{x}_k|_\alpha\dot{+}|\dot{y}_k|_\alpha) \\
&\leq \alpha - \sup_{k\in\mathbb{N}}|\dot{x}_k|_\alpha\dot{+}\alpha - \sup_{k\in\mathbb{N}}|\dot{y}_k|_\alpha \\
&= \|\dot{x}\|_{\alpha,\infty}\dot{+}\|\dot{y}\|_{\alpha,\infty}.
\end{aligned}$$

Hence (NN3) holds, and therefore $\|.\|_{\alpha,\infty}$ is an α-norm on each of the three spaces.

Theorem 2.4. ℓ_α^∞, c_α and $c_{\alpha,0}$ are α-Banach spaces.

Proof. Define the mapping

$$\Phi\colon \omega_\alpha \to \omega,\ \Phi(\dot{x}) = \left(\alpha^{-1}(\dot{x}_k)\right)_{k=1}^\infty,$$

where ω is the space of all real sequences. Since α is strictly increasing, Φ is bijective. Moreover,

$$\Phi(\ell_\alpha^\infty) = \ell_\infty,\ \Phi(c_\alpha) = c, \text{ and } \Phi\left(c_{\alpha,0}\right) = c_0.$$

Also, by the definitions of α-convergence and α-Cauchy sequences, a sequence $\left(\dot{x}^{(m)}\right)$ in an α-sequence is α-Cauchy if and only if $\left(\Phi\left(\dot{x}^{(m)}\right)\right)$ is Cauchy in the corresponding classical sequence space.

First, let $\left(\dot{x}^{(m)}\right)$ be an α-Cauchy sequence in ℓ_α^∞. Then $\left(\Phi\left(\dot{x}^{(m)}\right)\right)$ is a Cauchy sequence in ℓ_∞. Since ℓ_∞ is a Banach space, there exists $y \in \ell_\infty$ such that $\Phi\left(\dot{x}^{(m)}\right) \to y$ in ℓ_∞. Since Φ is onto, there exists $\dot{x} \in \ell_\alpha^\infty$ such that $\Phi(\dot{x}) = y$. Hence $\Phi\left(\dot{x}^{(m)}\right) \to \Phi(\dot{x})$, and thus $\dot{x}^{(m)} \to \dot{x}$ in ℓ_α^∞. Therefore, ℓ_α^∞ is an α-Banach space.

Now, let $\left(\dot{x}^{(m)}\right)$ be an α-Cauchy sequence in c_α. Then $\left(\Phi\left(\dot{x}^{(m)}\right)\right)$ is a Cauchy sequence in c. Since c is a closed subspace of the Banach space ℓ_∞, it is also a Banach space. Hence there exists $y \in c$ such that $\Phi\left(\dot{x}^{(m)}\right) \to y$ in c. Since $\Phi(c_\alpha) = c$, there exists $\dot{x} \in c_\alpha$ such that $\Phi(\dot{x}) = y$. Thus $\dot{x}^{(m)} \to \dot{x}$ in c_α. Therefore, c_α is an α-Banach space.

Finally, let $\left(\dot{x}^{(m)}\right)$ be an α-Cauchy sequence in $c_{\alpha,0}$. Then $\left(\Phi\left(\dot{x}^{(m)}\right)\right)$ is a Cauchy sequence in c_0. Since c_0 is a closed subspace of the Banach space ℓ_∞, it is a Banach space. Hence there exists $y \in c_0$ such that $\Phi\left(\dot{x}^{(m)}\right) \to y$ in c_0. Since $\Phi\left(c_{\alpha,0}\right) = c_0$, there exists $\dot{x} \in c_{\alpha,0}$ such that $\Phi(\dot{x}) = y$. Thus $\dot{x}^{(m)} \to \dot{x}$ in $c_{\alpha,0}$. Therefore, $c_{\alpha,0}$ is an α-Banach space.

Consequently, ℓ_α^∞, c_α, and c_α are all α-Banach spaces.

Given a matrix $\dot{A} = (\dot{a}_{nk})$ of α-scalars and an α-sequence $\dot{x}$, we write $\dot{y} = \dot{A} \dot{\times} \dot{x}$ to indicate $\dot{y}_n = \left(\dot{A} \dot{\times} \dot{x}\right)_n = {}_\alpha\dot{\sum}_{k=1}^\infty \dot{a}_{nk} \dot{\times} \dot{x}_k$, assuming that each series here converges.

The set $\omega_{\alpha,\dot{A}} = \left\{\dot{x} \in \omega_\alpha\colon \dot{A} \dot{\times} \dot{x} \text{ is defined}\right\}$ is called the domain of $\dot{A}$. Let ϕ_α represent the set of all finitely non-zero α-sequences. It is clearly true that $\dot{A}$ is α-row-finite(i. e. each row of $\dot{A}$ lies in ϕ_α) if and only if $\omega_{\alpha,\dot{A}} = \omega_\alpha$.

The set $c_{\alpha,\dot{A}} = \left\{\dot{x} \in \omega_\alpha\colon \dot{A} \dot{\times} \dot{x} \in c_\alpha\right\}$ is called the α-convergence domain of $\dot{A}$ and the α-sequences in $c_{\alpha,\dot{A}}$ are called $\dot{A}$-summable. Let ${}^\alpha\lim_{\dot{A}} \dot{x} = {}^\alpha\lim_{n\to\infty}\left(\dot{A} \dot{\times} \dot{x}\right)_n$ for all $\dot{x} \in c_{\alpha,\dot{A}}$, so $\lim_{\dot{A}}\colon c_{\alpha,\dot{A}} \to \mathbb{R}_\alpha$ is defined. If $c_\alpha \subset c_{\alpha,\dot{A}}$, then an α-matrix $\dot{A}$ is called α-conservative. If ${}^\alpha\lim_{\dot{A}} \dot{x} = \dot{m} \dot{\times} {}^\alpha\lim_{n\to\infty} \dot{x}_n$, then an α-matrix $\dot{A}$ is called α-multiplicative $\dot{m}$ and α-regular where $\dot{m} = \dot{1}$.

Theorem 2.5. The dual of $c_{\alpha,0}$ is given by ℓ_α. More precisely, for every $\mathcal{g} \in c_{\alpha,0}'$, there exists a unique $\dot{a} \in \ell_\alpha$ such that

$$\mathcal{g}(\dot{x}) = \dot{\sum}_{k=1}^\infty \dot{a}_k \dot{\times} \dot{x}_k .$$

Proof. Let $\mathcal{g} \in c_{\alpha,0}'$. For $x = (x_k) \in c_0$ define $F(x) = \alpha^{-1}\left(\mathcal{g}\left(\Phi^{-1}(x)\right)\right)$. Since $\Phi\colon c_{\alpha,0} \to c_0$ is a bijection, F is well defined. Moreover, because $\mathcal{g}$ is α-linear and α-bounded, F is a bounded linear functional on the classical Banach space c_0. Hence,

by the classical duality $c_0' = \ell_1$, there exists a unique $a = (a_k) \in \ell_1$ such that

$$F(x) = \sum_{k=1}^{\infty} a_k x_k\,,\ x = (x_k) \in c_0.$$

Set $\dot{a}_k = \alpha(a_k), k \in \mathbb{N}$. Since $a \in \ell_1$, it follows that $\dot{a} \in \ell_\alpha$. For $\dot{x} = \big(\alpha(x_k)\big) \in c_{\alpha,0}$, we have

$$\begin{aligned} \mathcal{g}(\dot{x}) &= \alpha\big(F(x)\big) = \alpha\big(\textstyle\sum_{k=1}^{\infty} a_k x_k\big)\big) \\ &= {}_\alpha\textstyle\sum_{k=1}^{\infty} \dot{a}_k \dot{\times} \dot{x}_k. \end{aligned}$$

Thus every $\mathcal{g} \in c_{\alpha,0}'$ has the required representation.

Conversely, let $\dot{a} = (\dot{a}_k) \in \ell_\alpha$, and write $\dot{a}_k = \alpha(a_k)$, where $a = (a_k) \in \ell_1$. Define

$$\mathcal{g}(\dot{x}) = \textstyle\sum_{k=1}^{\infty} \dot{a}_k \dot{\times} \dot{x}_k\,,\ \dot{x} = (\dot{x}_k) \in c_{\alpha,0}.$$

Equivalently,

$$\mathcal{g}(\dot{x}) = \alpha\big(\textstyle\sum_{k=1}^{\infty} a_k x_k\big),\ \dot{x} = \alpha(x_k).$$

Since $a \in \ell_1$ and $x = (x_k) \in c_0 \subset \ell_\infty$, the classical series $\sum_{k=1}^{\infty} a_k x_k$ converges absolutely. Hence $\mathcal{g}$ is well defined. Moreover, the classical estimate

$$\big|\textstyle\sum_{k=1}^{\infty} a_k x_k\big| \le \|a\|_1 \|x\|_\infty$$

shows that $\mathcal{g}$ is α-bounded. The α-linearity of $\mathcal{g}$ follows directly from the α-linearity of the α-series.

Therefore $\mathcal{g} \in c_{\alpha,0}'$.

Finally, the uniqueness of $\dot{a}$ follows from the uniqueness of $a \in \ell_1$ in the classical representation of the dual of c_0. Hence $c_{\alpha,0}' = \ell_\alpha$.

Example 2.6. The α-identity matrix $\dot{I}$, an α-matrix where each component on the main diagonal is $\dot{1}$ and all other components are $\dot{0}$, is α-regular since $\dot{I}\dot{x} = \dot{x}$, also $\omega_{\alpha,\dot{I}} = \omega_\alpha$ and $c_{\alpha,\dot{I}} = c_\alpha$.

Example 2.7. Let $\dot{Q}$ be the α-matrix given by $\big(\dot{Q}\dot{x}\big)_1 = \dot{x}_1$, $\big(\dot{Q}\dot{x}\big)_n = (\dot{x}_{n-1} \dot{+} \dot{x}_n) \dot{/} \dot{2}$ for $n = 2,3,\ldots..$ Since

$$\begin{aligned} {}^\alpha\lim\nolimits_{\dot{Q}}\dot{x} &= {}^\alpha\lim\nolimits_{n\to\infty}\big(\dot{Q}\dot{x}\big)_n \\ &= {}^\alpha\lim\nolimits_{n\to\infty}\alpha\left(\frac{\alpha^{-1}(\dot{x}_{n-1} \dot{+} \dot{x}_n)}{\alpha^{-1}(\dot{2})}\right) \\ &= {}^\alpha\lim\nolimits_{n\to\infty}\alpha\left(\frac{\alpha^{-1}\{\alpha(\alpha^{-1}\{\dot{x}_{n-1}\} + \alpha^{-1}\{\dot{x}_n\})\}}{2}\right) \\ &= {}^\alpha\lim\nolimits_{n\to\infty}\alpha\left(\tfrac{x_{n-1}+x_n}{2}\right) = \\ \alpha&\left(\lim\nolimits_{n\to\infty}\tfrac{x_{n-1}+x_n}{2}\right) \\ &= \alpha(\lim\nolimits_{n\to\infty} x_n) = \alpha(\lim\nolimits_{n\to\infty}\alpha^{-1}\{\dot{x}_n\}) \\ &= {}^\alpha\lim\nolimits_{n\to\infty}\dot{x}_n, \end{aligned}$$

the α-matrix $\dot{Q}$ is α-regular. It is easily seen that the α-matrix $\dot{Q}$ takes the following form:

$$\dot{Q} = \begin{bmatrix} \dot{1} & \dot{0} & \dot{0} & \dot{0} & \dot{0} & \dot{0} & \ldots \\ \dot{0} & \frac{\dot{1}}{\dot{2}} & \frac{\dot{1}}{\dot{2}} & \dot{0} & \dot{0} & \dot{0} & \ldots \\ \dot{0} & \dot{0} & \frac{\dot{1}}{\dot{2}} & \frac{\dot{1}}{\dot{2}} & \dot{0} & \dot{0} & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{bmatrix}_\alpha.$$

Definition 2.8. Let $\dot{X}$ and $\dot{Y}$ be two sequence spaces consisting α-sequences. We denote by $(\dot{X}:\dot{Y})_\alpha$ the class of all α-matrices $\dot{A} = (\dot{a}_{nk})$ of α-scalars such that

$$\dot{A} \dot{\times} \dot{x} = \Big(\big(\dot{A} \dot{\times} \dot{x}\big)_n\Big)_{n\in\mathbb{N}} = \Big({}_\alpha\textstyle\sum_{k=1}^{\infty} \dot{a}_{nk} \dot{\times} \dot{x}_k\Big)_{n\in\mathbb{N}}$$

belongs to $\dot{Y}$ for every $\dot{x} \in \dot{X}$, whenever the corresponding α-series converge. Thus $\dot{A} \in \big(\dot{X}:\dot{Y}\big)_\alpha$ means that $\dot{A}$ defines an α-linear transformation from $\dot{X}$ into $\dot{Y}$.

Obviously, $\dot{A} \in (\omega_{\alpha.\dot{A}}:\omega_\alpha)$ for all α-matrix $\dot{A}$, and $\dot{A} \in (c_\alpha:c_\alpha)$ iff $\dot{A}$ is α-conservative.

Theorem 2.9. For an α-matrix $\dot{A}$, the following statements are equivalent.

(i) $\omega_{\alpha.\dot{A}} \supset \ell_\alpha^\infty$, (ii) $\omega_{\alpha.\dot{A}} \supset c_{\alpha,0}$, (iii) ${}_\alpha\sum_{k=1}^{\infty}|\dot{a}_{nk}|_\alpha \dot{<} \dot{+}\infty$ for each n.

Proof. Let $\dot{A} = (\dot{a}_{nk})$ be an α-matrix. We prove the equivalence of (i), (ii), and (iii).

(iii) ⇒ (i). If (iii) is true and $\dot{x} = (\dot{x}_k) \in \ell_\alpha^\infty$, then

$$\begin{aligned} \Big|\big(\dot{A} \dot{\times} \dot{x}\big)_n\Big|_\alpha &= \Big|{}_\alpha\textstyle\sum_{k=1}^{\infty} \dot{a}_{nk} \dot{\times} \dot{x}_k\Big|_\alpha \\ &\dot{\le} {}_\alpha\textstyle\sum_{k=1}^{\infty}|\dot{a}_{nk}|_\alpha \dot{\times} |\dot{x}_k|_\alpha \\ &\dot{\le} \|\dot{x}\|_{\alpha,\infty} \dot{\times} {}_\alpha\textstyle\sum_{k=1}^{\infty}|\dot{a}_{nk}|_\alpha \dot{<} \dot{+}\infty. \end{aligned}$$

(i) ⇒ (ii). Using the inclusion $c_{\alpha,0} \subset \ell_\alpha^\infty$, if $\omega_{\alpha.\dot{A}} \supset \ell_\alpha^\infty$, then we have $\omega_{\alpha.\dot{A}} \supset c_{\alpha,0}$.

(ii) ⇒ (iii). Assume (ii), that is, $\omega_{\alpha.\dot{A}} \supset c_{\alpha,0}$. Then, we can write $\omega_{\alpha.\dot{A}} \supset c_{\alpha,0} \Leftrightarrow \omega_A \supset c_0$ with $A = \big(\alpha^{-1}(\dot{a}_{nk})\big)$. Indeed, if $\omega_{\alpha.\dot{A}} \supset c_{\alpha,0}$, then, given $x = (x_k) \in c_0$, $\dot{x} = \alpha(x_k) = (\dot{x}_k) \in c_{\alpha,0}$ and hence $\dot{x} \in \omega_{\alpha.\dot{A}}$, and therefore

$$\begin{aligned} \dot{A} \dot{\times} \dot{x} &= \Big({}_\alpha\textstyle\sum_{k=1}^{\infty} \dot{a}_{nk} \dot{\times} \dot{x}_k\Big)_{n\in\mathbb{N}} \\ &= \big(\alpha(\textstyle\sum_{k=1}^{\infty} \alpha^{-1}(\dot{a}_{nk}) \times x_k)\big)_{n\in\mathbb{N}} \in \omega_\alpha, \end{aligned}$$

and thus $Ax = (\sum_{k=1}^{\infty} \alpha^{-1}(\dot{a}_{nk}) \times x_k)_{n\in\mathbb{N}} \in \omega$, and hence $\omega_A \supset c_0$. The converse can be proven similarly. By Theorem 1.3.2 in [19],

$$\textstyle\sum_{k=1}^{\infty}|\alpha^{-1}(\dot{a}_{nk})| < \infty$$

for each n. Then, for each n, we have

$$\begin{aligned} {}_\alpha\textstyle\sum_{k=1}^{\infty}|\dot{a}_{nk}|_\alpha &= {}_\alpha\textstyle\sum_{k=1}^{\infty}\alpha(|\alpha^{-1}(\dot{a}_{nk})|) \\ &= \alpha(\textstyle\sum_{k=1}^{\infty}|\alpha^{-1}(\dot{a}_{nk})|) \dot{<} \dot{+}\infty. \end{aligned}$$

This completes the proof of (ii) ⇒ (iii).

Theorem 2.10. For an α-matrix $\dot{A} = (\dot{a}_{nk})$, the following statements are equivalent:

(i) $\dot{A} \in (\ell_\alpha^\infty : \ell_\alpha^\infty)$, (ii) $\dot{A} \in (c_{\alpha,0} : \ell_\alpha^\infty)$, (iii) $\|\dot{A}\|_\alpha = \alpha - \sup_n {}_\alpha\sum_{k=1}^\infty |\dot{a}_{nk}|_\alpha \dot{<} \dot{+\infty}$.

Proof. We prove the three implications separately.

(iii) ⇒ (i). Assume that (iii) holds. Let $\dot{x} = (\dot{x}_k) \in \ell_\alpha^\infty$. For every $n \in \mathbb{N}$, we have

$$\begin{aligned}\left|(\dot{A} \dot{\times} \dot{x})_n\right|_\alpha &= \left|{}_\alpha\sum_{k=1}^\infty \dot{a}_{nk} \dot{\times} \dot{x}_k\right|_\alpha \\ &\dot{\leq} {}_\alpha\sum_{k=1}^\infty |\dot{a}_{nk}|_\alpha \dot{\times} |\dot{x}_k|_\alpha \\ &\dot{\leq} \|x\|_{\alpha,\infty} \dot{\times} {}_\alpha\sum_{k=1}^\infty |\dot{a}_{nk}|_\alpha.\end{aligned}$$

Taking the α-supremum over n and using (iii), we obtain

$$\begin{aligned}\|\dot{A} \dot{\times} \dot{x}\|_{\alpha,\infty} &= \alpha - \sup_{n\in\mathbb{N}} \left|(\dot{A} \dot{\times} \dot{x})_n\right|_\alpha \\ \dot{\leq}\ \alpha - \sup_{n\in\mathbb{N}} {}_\alpha\sum_{k=1}^\infty |\dot{a}_{nk}|_\alpha \dot{\times} \|x\|_{\alpha,\infty} &\dot{<} \dot{+\infty}.\end{aligned}$$

Hence $\dot{A} \dot{\times} \dot{x} \in \ell_\alpha^\infty$ for every $\dot{x} \in \ell_\alpha^\infty$. Therefore $\dot{A} \in (\ell_\alpha^\infty : \ell_\alpha^\infty)$.

(i) ⇒ (ii). Assume that (i) holds. Then $\dot{A} \dot{\times} \dot{x} \in \ell_\alpha^\infty$ for every $\dot{x} \in \ell_\alpha^\infty$. Since $c_{\alpha,0} \subset \ell_\alpha^\infty$, it follows immediately that $\dot{A} \dot{\times} \dot{x} \in \ell_\alpha^\infty$ for every $\dot{x} \in c_{\alpha,0}$. Thus $\dot{A} \in (c_{\alpha,0} : \ell_\alpha^\infty)$.

(ii) ⇒ (iii). Assume that $\dot{A} = (\dot{a}_{nk}) \in (c_{\alpha,0}, \ell_\alpha^\infty)$. Let $A = (\alpha^{-1}(\dot{a}_{nk}))$. A is an infinite matrix with real entries. We show that $A \in (c_0 : \ell^\infty)$ holds. Given an arbitrary $x = (x_k) \in c_0$, $\dot{x} = (\alpha(x_k)) = (\dot{x}_k) \in c_{\alpha,0}$ and thus $\dot{A} \dot{\times} \dot{x} \in \ell_\alpha^\infty$. If we use that

$$\begin{aligned}\dot{A} \dot{\times} \dot{x} &= \left({}_\alpha\sum_{k=1}^\infty \dot{a}_{nk} \dot{\times} \dot{x}_k\right)_{n\in\mathbb{N}} \\ &= \left(\alpha\left(\sum_{k=1}^\infty \alpha^{-1}(\dot{a}_{nk}) \times x_k\right)\right)_{n\in\mathbb{N}},\end{aligned}$$

then $\dot{A} \in (c_{\alpha,0}, \ell_\alpha^\infty)$ implies that $Ax \in \ell^\infty$, and hence $A \in (c_0, \ell^\infty)$. By Theorem 1.3.3 in [19], $\|A\| = \sup_n \sum_{k=1}^\infty |\alpha^{-1}(\dot{a}_{nk})| < +\infty$. Hence

$$\begin{aligned}\|\dot{A}\|_\alpha &= \alpha - \sup_n {}_\alpha\sum_{k=1}^\infty |\dot{a}_{nk}|_\alpha \\ &= \alpha\left(\sup_n \alpha^{-1}\left({}_\alpha\sum_{k=1}^\infty |\dot{a}_{nk}|_\alpha\right)\right) \\ &= \alpha\left(\sup_n \alpha^{-1}\left(\alpha\left(\sum_{k=1}^\infty \alpha^{-1}(|\dot{a}_{nk}|_\alpha)\right)\right)\right) \\ &= \alpha\left(\sup_n \sum_{k=1}^\infty \alpha^{-1}(|\dot{a}_{nk}|_\alpha)\right) \\ &= \alpha\left(\sup_n \sum_{k=1}^\infty |\alpha^{-1}(\dot{a}_{nk})|\right) \dot{<} \dot{+\infty}.\end{aligned}$$

This completes the proof of (ii) ⇒ (iii). Thus (i), (ii), and (iii) are equivalent.

If $\phi_\alpha = \left\{\dot{A} = (\dot{a}_{nk}) : \|\dot{A}\|_\alpha \dot{<} \dot{+\infty}\right\}$ is taken, then then clearly $\phi_\alpha = (\ell_\alpha^\infty : \ell_\alpha^\infty) = (c_{\alpha,0} : \ell_\alpha^\infty)$.

Let $A = (a_{nk})$ be an infinite matrix with real entries and let $\|A\| = \sup_n \sum_{k=1}^\infty |a_{nk}|$. Also, let

$$\phi = \{A = (a_{nk}) : \|A\| < \infty\}.$$

For a matrix A, the equivalence of the following three statements is known from Theorem 1.3.3 in [19] that $\|A\|$ is α-operator norm of the operator $T(x) = A \times x$ from c_0 or ℓ^∞ to ℓ^∞.

It is easily seen that $\|\dot{A}\|_\alpha = \alpha(\|A\|)$, and hence $\dot{A} = (\dot{a}_{nk}) \in \phi_\alpha$ if and only if $A = (a_{nk}) \in \phi$ with $a_{nk} = (\alpha^{-1}(\dot{a}_{nk}))$ for alln, k. Also, $\|\dot{A}\|_\alpha$ is α-operator norm of the operator $\dot{T}(\dot{x}) = \dot{A} \dot{\times} \dot{x}$ from $c_{\alpha,0}$ or ℓ_α^∞ to ℓ_α^∞.

For an α-matrix $\dot{A} = (\dot{a}_{nk})$, if α-limits exist, then we write $\dot{a}_k = {}^\alpha\lim_{n\to\infty} \dot{a}_{nk}$ for each $k \in \mathbb{N}$. For each $k = 1,2,\ldots$, $\dot{\delta}^k$ is the α-sequence whose only non-zero term is the α-number $\dot{1}$ in the kth input. Depending on the context, we take 1 as either an integer or a sequence where all terms are $\dot{1}$. $\dot{a}_k = {}^\alpha\lim_{\dot{A}} \dot{\delta}^k$It is easily seen that $\dot{a}_k = {}^\alpha\lim_{\dot{A}} \dot{\delta}^k$, and this is important data for us.

Lemma 2.11. The spaces c_α and $c_{\alpha,0}$ is a closed subspace of ℓ_α^∞.

Proof. First, we show that c_α is a linear subspace of ℓ_α^∞. Let $\dot{x} = (\dot{x}_k)$ and $\dot{y} = (\dot{y}_k)$ be arbitrary elements of c_α. Then, there exist $\dot{a}, \dot{b} \in \mathbb{R}_\alpha$ such that

$$\dot{a} = {}^\alpha\lim_{k\to\infty} \dot{x}_k,\ \dot{b} = {}^\alpha\lim_{k\to\infty} \dot{y}_k.$$

For any $\dot{\mu}, \dot{\nu} \in \mathbb{R}_\alpha$, the algebraic properties of α-limit imply that

$${}^\alpha\lim_{k\to\infty} (\dot{\mu} \dot{\times} \dot{x}_k \dot{+} \dot{\nu} \dot{\times} \dot{y}_k) = \dot{\mu} \dot{\times} \dot{a} \dot{+} \dot{\nu} \dot{\times} \dot{b}.$$

Hence $\dot{\mu} \dot{\times} \dot{a} \dot{+} \dot{\nu} \dot{\times} \dot{b} \in c_\alpha$. Moreover, since every α-convergent sequence is α-bounded,

$c_\alpha \subset \ell_\alpha^\infty$,

and consequently c_α is a subspace of ℓ_α^∞.

It remains to prove that c_α is closed in ℓ_α^∞. Let $(\dot{x}^{(n)})$ be an α-sequence in c_α such that $\dot{x}^{(n)} \to \dot{x}$ in ℓ_α^∞ as $n \to \infty$. Since every $\dot{x}^{(n)}$ belongs to c_α, there exists $\dot{a}^{(n)} \in \mathbb{R}_\alpha$ satisfying

$${}^\alpha\lim_{k\to\infty} \dot{x}_k^{(n)} = \dot{a}^{(n)}$$

for every $n \in \mathbb{N}$. We first show that $(\dot{a}^{(n)})_{n\in\mathbb{N}}$ is an α-Cauchy sequence. Let an arbitrary $\dot{\varepsilon} \dot{>} \dot{0}$ be given. Since $\dot{x}^{(n)} \to \dot{x}$ in ℓ_α^∞, there exists $n_0 \in \mathbb{N}$ such that, for all $n, m \geq n_0$,

$$\|\dot{x}^{(n)} \dot{-} \dot{x}^{(m)}\|_{\alpha,\infty} \dot{<} \frac{\dot{\varepsilon}}{\dot{3}}$$

with $n, m \geq n_0$. Hence, for every $k \in \mathbb{N}$,

$$\left|\dot{x}_k^{(n)} \dot{-} \dot{x}_k^{(m)}\right|_\alpha \dot{<} \frac{\dot{\varepsilon}}{\dot{3}}$$

Taking the α-limit as $k \to \infty$, we obtain

$$\left|\dot{a}^{(n)} \dot{-} \dot{a}^{(m)}\right|_\alpha \dot{\leq} \frac{\dot{\varepsilon}}{\dot{3}} \dot{<} \dot{\varepsilon}.$$

Thus $\left(\dot{a}^{(n)}\right)_{n\in\mathbb{N}}$ is an α-Cauchy sequence in $\mathbb{R}_\alpha$. Because $\mathbb{R}_\alpha$ is α-complete, there exists $\dot{a} \in \mathbb{R}_\alpha$ such that $\dot{a}^{(n)} \to \dot{a}$ as $n \to \infty$. We now prove that

$$^{\alpha}\lim_{k\to\infty} \dot{x}_k = \dot{a}.$$

Choose $n_1 \in \mathbb{N}$ sufficiently large so that

$$\left\|\dot{x} \dot{-} \dot{x}^{(n_1)}\right\|_{\alpha,\infty} \dot{<} \frac{\dot{\varepsilon}}{\dot{3}}$$

and

$$\left|\dot{a} \dot{-} \dot{a}^{(n_1)}\right|_\alpha \dot{<} \frac{\dot{\varepsilon}}{\dot{3}}.$$

Because $\dot{x}^{(n_1)} \in c_\alpha$,

$$^{\alpha}\lim_{k\to\infty} \dot{x}_k^{(n_1)} = \dot{a}^{(n_1)}.$$

Therefore, there exists $k_0 \in \mathbb{N}$ such that, for all $k \geq k_0$,

$$\left|\dot{x}_k^{(n_1)} \dot{-} \dot{a}^{(n_1)}\right|_\alpha \dot{<} \frac{\dot{\varepsilon}}{\dot{3}}.$$

Using the α-triangle inequality, we obtain, for $k \geq k_0$,

$$\begin{aligned}|\dot{x}_k \dot{-} \dot{a}|_\alpha &\dot{\leq} \left|\dot{x}_k \dot{-} \dot{x}_k^{(n_1)}\right|_\alpha \dot{+} \left|\dot{x}_k^{(n_1)} \dot{-} \dot{a}^{(n_1)}\right|_\alpha \\ &\dot{+} \left|\dot{a}^{(n_1)} \dot{-} \dot{a}\right|_\alpha \dot{<} \frac{\dot{\varepsilon}}{\dot{3}} \dot{+} \frac{\dot{\varepsilon}}{\dot{3}} \dot{+} \frac{\dot{\varepsilon}}{\dot{3}} = \dot{\varepsilon}.\end{aligned}$$

Hence $^{\alpha}\lim_{k\to\infty} \dot{x}_k = \dot{a}$, and thus $\dot{x} \in c_\alpha$. We have shown that the α-limit of an α-convergent sequence in c_α also belongs to c_α. Consequently, c_α is closed in ℓ_α^∞.

That $c_{\alpha,0}$ is a closed subspace of ℓ_α^∞ can be proven in a similar way.

Theorem 2.12. An α-matrix $\dot{A} = (\dot{a}_{nk})$ is α-conservative if and only if (i) $\dot{A} \in \phi_\alpha$, (ii) $\dot{a}_k$ exists for all k, and (iii) $\dot{1} \in c_{\alpha,\dot{A}}$.

Moreover, the conditions (i) and (ii) are necessary and sufficient that $c_{\alpha,0} \subset c_{\alpha,\dot{A}}$.

Proof. Assume that $\dot{A} = (\dot{a}_{nk})$ is α-conservative, that is, $c_\alpha \subset c_{\alpha,\dot{A}}$. Then $\dot{A} \in (c_\alpha : c_\alpha)$ and since $c_{\alpha,0} \subset c_\alpha$, we also have $\dot{A} \in (c_{\alpha,0} : c_\alpha)$. Because $c_{\alpha,0} \subset \ell_\alpha^\infty$, Theorem 2.10 yields condition (i). For each fixed $k \in \mathbb{N}$, $\dot{\delta}^k \in c_{\alpha,0}$. Hence $\dot{A}\dot{\delta}^k \in c_\alpha$. Since $\left(\dot{A}\dot{\delta}^k\right)_n = \dot{a}_{nk}$ for all n, the α-limit $^{\alpha}\lim_{n\to\infty} \dot{a}_{nk}$ exists, proving (ii). Finally, the constant sequence $\dot{1} = (\dot{1}, \dot{1}, \ldots)$ belongs to c_α. Therefore $\dot{A} \dot{\times} \dot{1} \in c_\alpha$, and thus (iii) holds.

Conversely, assume that (i)-(iii) hold. We first prove that $\dot{A} \in (c_{\alpha,0} : c_\alpha)$. Let $\dot{x} = (\dot{x}_k) \in c_{\alpha,0}$. By (i), put

$$\dot{M} = \alpha - \sup_{n\in\mathbb{N}} {}_{\alpha}\dot{\sum}_{k=1}^{\infty} |\dot{a}_{nk}|_\alpha \dot{<} +\dot{\infty}.$$

Thus ${}_{\alpha}\dot{\sum}_{k=1}^{\infty} |\dot{a}_{nk}|_\alpha \dot{\leq} \dot{M}$ for every n. If $\dot{M} = \dot{0}$, every row of $\dot{A}$ is α-zero and therefore $\dot{A}\dot{x} = \dot{0} \in c_\alpha$. Suppose $\dot{M} \dot{>} \dot{0}$. Let $\dot{\varepsilon} \dot{>} \dot{0}$. Since $\dot{x}_k \to \dot{0}$, choose $m \in \mathbb{N}$ such that $|\dot{x}_k|_\alpha \dot{<} \dot{\varepsilon}/(\dot{3} \dot{\times} \dot{M})$ for every $k > m$. For $n, p \in \mathbb{N}$, split the difference as follows:

$$\begin{aligned}\left(\dot{A}\dot{x}\right)_n \dot{-} \left(\dot{A}\dot{x}\right)_p &= {}_{\alpha}\dot{\sum}_{k=1}^{m} \left(\dot{a}_{nk} \dot{-} \dot{a}_{pk}\right) \dot{\times} \dot{x}_k \\ &\dot{+} {}_{\alpha}\dot{\sum}_{k=m+1}^{\infty} \left(\dot{a}_{nk} \dot{-} \dot{a}_{pk}\right) \dot{\times} \dot{x}_k.\end{aligned}$$

By (ii), for every fixed $k \leq m$, $\dot{a}_{nk} \to \dot{a}_k$ and $\dot{a}_{pk} \to \dot{a}_k$ as $n, p \to \infty$. Hence the finite sum tends to α-zero. Therefore there exists $N \in \mathbb{N}$ such that, for $n, p \geq N$, its α-absolute value is less than $\dot{\varepsilon}/\dot{3}$. For the tail, the α-triangle inequality gives

$$\begin{aligned}&\left| {}_{\alpha}\dot{\sum}_{k=m+1}^{\infty} \left(\dot{a}_{nk} \dot{-} \dot{a}_{pk}\right) \dot{\times} \dot{x}_k \right|_\alpha \\ &\dot{\leq} \left({}_{\alpha}\dot{\sum}_{k=m+1}^{\infty} |\dot{a}_{nk}|_\alpha \dot{+} \dot{\sum}_{k=m+1}^{\infty} \left|\dot{a}_{pk}\right|_\alpha \right) \\ &\dot{\times} \dot{\varepsilon}/(\dot{3} \dot{\times} \dot{M}) \dot{<} (\dot{2} \dot{\times} \dot{\varepsilon})/\dot{3}.\end{aligned}$$

Consequently, $\left|\left(\dot{A}\dot{x}\right)_n \dot{-} \left(\dot{A}\dot{x}\right)_p\right|_\alpha \dot{<} \dot{\varepsilon}$ for all $n, p \geq N$. Thus $\left(\left(\dot{A}\dot{x}\right)_n\right)$ is α-Cauchy in $\mathbb{R}_\alpha$. Since $\mathbb{R}_\alpha$ is α-complete, there exists $L \in \mathbb{R}_\alpha$ such that $\left(\dot{A} \dot{\times} \dot{x}\right)_n \to L$ as $n \to \infty$. Therefore $\dot{A} \dot{\times} \dot{x} \in c_\alpha$, and hence $\dot{A} \in (c_{\alpha,0} : c_\alpha)$. Now let $\dot{x} \in c_\alpha$ and let $^{\alpha}\lim_{k\to\infty} \dot{x}_k = l$. Then $\dot{x} \dot{-} l \dot{\times} \dot{1} \in c_{\alpha,0}$. By the first part, $\dot{A} \dot{\times} (\dot{x} \dot{-} l \dot{\times} \dot{1}) \in c_\alpha$. By (iii), $\dot{A} \dot{\times} \dot{1} \in c_\alpha$. Hence $\dot{A} \dot{\times} \dot{x} = \dot{A} \dot{\times} (\dot{x} \dot{-} l \dot{\times} \dot{1}) \dot{+} l \dot{\times} (\dot{A} \dot{\times} \dot{1}) \in c_\alpha$. Therefore $\dot{A}$ is α-conservative.

The α-dual of c_α is characterized as follows.

Theorem 2.13. A α-bounded and α-linear functional f on c_α, that is, $f \in c_\alpha'$ has a representation

$$f(\dot{x}) = {}_{\alpha}\dot{\sum}_{k=1}^{\infty} \dot{a}_k \dot{\times} \dot{x}_k + \dot{\chi} \dot{\times} {}^{\alpha}\lim_{k\to\infty} \dot{x}_k,$$

where $\dot{a} = (\dot{a}_k) \in \ell_\alpha$, $\dot{\chi} = f(\dot{1}) \dot{-} {}_{\alpha}\dot{\sum}_{k=1}^{\infty} \dot{a}_k$, and $\dot{x} = (\dot{x}_k) \in c_\alpha$. Conversely, every expression of this form defines a bounded and α-linear functional on c_α.

Proof. Let $f \in c_\alpha'$. For each $k \in \mathbb{N}$, put $\dot{a}_k = f\left(\dot{\delta}^k\right)$, where $\dot{\delta} = \left(\dot{\delta}^k\right)$ is the sequence whose k-th term is $\dot{1}$ and whose other terms are $\dot{0}$. Since $c_{\alpha,0}$ is a subspace of c_α, the restriction of f to $c_{\alpha,0}$ is a α-bounded and α-linear functional on $c_{\alpha,0}$. By Theorem 2.5, its coefficient sequence $\dot{a} = (\dot{a}_k)$ belongs to ℓ_α and, for every $\dot{y} = (\dot{y}_k) \in c_{\alpha,0}$,

$$f(\dot{y}) = {}_{\alpha}\dot{\sum}_{k=1}^{\infty} \dot{a}_k \dot{\times} \dot{y}_k.$$

Now take an arbitrary $\dot{x} = (\dot{x}_k) \in c_\alpha$ and write ${}^{\alpha}\lim_{k\to\infty}\dot{x}_k = l$. Let $\dot{1} = (\dot{1}, \dot{1}, \dots)$. Then $\dot{x}\dot{-}\dot{l}\dot{\times}\dot{1}$ belongs to $c_{\alpha,0}$. Applying the preceding identity and α-linearity gives

$$f\left(\dot{x}\dot{-}\dot{l}\dot{\times}\dot{1}\right) = {}_{\alpha}\dot{\sum}_{k=1}^{\infty} \dot{a}_k \dot{\times} \left(\dot{x}_k\dot{-}\dot{l}\right).$$

Therefore,

$$f(\dot{x}) = {}_{\alpha}\dot{\sum}_{k=1}^{\infty} \dot{a}_k \dot{\times} \dot{x}_k \dot{+} \dot{l} \dot{\times} \left(f(\dot{1})\dot{-}\ {}_{\alpha}\dot{\sum}_{k=1}^{\infty} \dot{a}_k\right).$$

Setting $\dot{\chi} = \dot{\chi}(f) = f\left(\dot{1}\right)\dot{-}\ {}_{\alpha}\dot{\sum}_{k=1}^{\infty} \dot{a}_k$ proves the required representation.

Conversely, let $\dot{a} = (\dot{a}_k) \in \ell_\alpha$ and $\dot{\rho} \in \mathbb{R}_\alpha$ and define, for $\dot{x} = (\dot{x}_k) \in c_\alpha$,

$$f(\dot{x}) = {}_{\alpha}\dot{\sum}_{k=1}^{\infty} \dot{a}_k \dot{\times} \dot{x}_k \dot{+} \dot{\rho} \dot{\times} \alpha\dot{-}\lim_{k\to\infty}\dot{x}_k.$$

The α-linearity of f follows from the linearity of the α-series and of the α-limit. Moreover, since every α-convergent sequence is α-bounded,

$$|f(\dot{x})|_\alpha \dot{\leq} \left(\|\dot{a}\|_{\alpha,1}\dot{+}|\dot{\rho}|_\alpha\right) \dot{\times} \|\dot{x}\|_{\alpha,\infty}$$

where $\|\dot{a}\|_{\alpha,1} = {}_{\alpha}\dot{\sum}_{k=1}^{\infty}|\dot{a}_k|_\alpha$ and $\|\dot{x}\|_{\alpha,\infty} = \alpha - \sup_{k\in\mathbb{N}}|\dot{x}_k|_\alpha$. Thus f is bounded on c_α. Finally, evaluating at the constant sequence $\dot{1}$ yields

$$\dot{\rho} = f\left(\dot{1}\right)\dot{-}\ {}_{\alpha}\dot{\sum}_{k=1}^{\infty} \dot{a}_k = \dot{\chi}.$$

This completes the proof.

Theorem 2.14. If $\dot{A} = (\dot{a}_{nk}) \in \phi_\alpha$ and each $\dot{a}_k$ exists, then $a = (\dot{a}_k) \in \ell_\alpha$.

Proof. Clearly ${}_{\alpha}\dot{\sum}_{k=1}^{m}|\dot{a}_k|_\alpha \dot{\leq} \left\|\dot{A}\right\|_\alpha$ for each m. Hence $a = \dot{a}_k \in \ell_\alpha$.

Theorem 2.15. If $\dot{A} = (\dot{a}_{nk})$ is α-conservative and $\dot{x} = (\dot{x}_k) \in c_\alpha$, then

$${}^{\alpha}\lim_{\dot{A}}\dot{x} = \dot{\chi} \dot{\times} {}^{\alpha}\lim_{k\to\infty}\dot{a}_k\dot{+}\dot{a}\dot{\times}\dot{x},$$

where $\dot{\chi} = \dot{\chi}(\dot{A}) = {}^{\alpha}\lim_{\dot{A}}\dot{1}\dot{-}\ {}_{\alpha}\dot{\sum}_{k=1}^{\infty} \dot{a}_k$.

Proof. Let $\dot{A} = (\dot{a}_{nk})$ be α-conservative, that is, $c_\alpha \subset c_{\alpha,A}$. We show that $A = \left(\alpha^{-1}(\dot{a}_{nk})\right)$ is a conservative matrix, that is, $c \subset c_A$. Let an arbitrary $x = (x_k) \in c$ be given. Then $\dot{x} = \left(\alpha(x_k)\right) = (\dot{x}_k) \in c_\alpha$. Since $c_\alpha \subset c_{\alpha,A}$, we have

$$\dot{A} \dot{\times} \dot{x} = \left(\alpha(\textstyle\sum_{k=1}^{\infty} \alpha^{-1}(\dot{a}_{nk}) \times x_k)\right)_{n\geq 1} \in c_\alpha,$$

and therefore $Ax = (\sum_{k=1}^{\infty} \alpha^{-1}(\dot{a}_{nk}) \times x_k)_{n\geq 1} \in c$. Thus $x \in c_A$. This shows that A is conservative. Consequently, by Theorem 1.3.8 in [19], we obtain

$$\lim_A x = \chi.\lim_{k\to\infty} a_k + ax,$$

where $\chi = \chi(A) = \lim_A 1 - \sum_{k=1}^{\infty} a_k$, $a_k = \lim_{n\to\infty}\alpha^{-1}(\dot{a}_{nk})$ and $a = (a_k)$. On the other hand, since

$$\begin{aligned}
\dot{\chi} = \dot{\chi}(\dot{A}) &= {}^{\alpha}\lim_{\dot{A}}\dot{1}\dot{-}\ {}_{\alpha}\dot{\sum}_{k=1}^{\infty} \dot{a}_k \\
&= {}^{\alpha}\lim_{n\to\infty}\dot{a}_{nk}\dot{-}\alpha(\textstyle\sum_{k=1}^{\infty}\alpha^{-1}(\dot{a}_k)) \\
&= \alpha\left(\lim_{n\to\infty}\alpha^{-1}(\dot{a}_k)\right)\dot{-}\alpha(\textstyle\sum_{k=1}^{\infty}\alpha^{-1}(\dot{a}_k)) \\
&= \alpha(\lim_{n\to\infty}\alpha^{-1}(\dot{a}_k) - \textstyle\sum_{k=1}^{\infty}\alpha^{-1}(\dot{a}_k)) \\
&= \alpha(\lim_A 1 - \textstyle\sum_{k=1}^{\infty} a_k),
\end{aligned}$$

we have

$$\begin{aligned}
{}^{\alpha}\lim_{\dot{A}}\dot{x} &= {}^{\alpha}\lim_{n\to\infty}\left(\dot{A}\dot{\times}\dot{x}\right)_n \\
&= {}^{\alpha}\lim_{n\to\infty}\alpha(\textstyle\sum_{k=1}^{\infty}\alpha^{-1}(\dot{a}_{nk})\times x_k) \\
&= \alpha(\lim_{n\to\infty}\textstyle\sum_{k=1}^{\infty}\alpha^{-1}(\dot{a}_{nk})\times x_k) \\
&= \alpha(\lim_{n\to\infty}(Ax)_n) = \alpha(\lim_A x) \\
&= \alpha(\chi.\lim_{k\to\infty}a_k + ax) \\
&= \alpha\{(\lim_A 1 - \textstyle\sum_{k=1}^{\infty} a_k)\lim_{k\to\infty}a_k + ax\} \\
&= \alpha\{\alpha^{-1}(\dot{\chi})\times\alpha^{-1}\left(\alpha(\lim_{k\to\infty}a_k)\right) + \\
&\qquad \alpha^{-1}\left(\alpha(ax)\right)\} \\
&= \alpha\{\alpha^{-1}(\dot{\chi})\times\alpha^{-1}\left(\alpha(\lim_{k\to\infty}a_k)\right) \\
&\qquad + \alpha^{-1}\left(\alpha(ax)\right)\} \\
&= \alpha\left\{\alpha^{-1}\left(\alpha\left(\alpha^{-1}(\dot{\chi})\times\alpha^{-1}\left(\alpha(\lim_{k\to\infty}a_k)\right)\right)\right)\right. \\
&\qquad \left. + \alpha^{-1}\left(\alpha(ax)\right)\right\} \\
&= \alpha\left(\alpha^{-1}(\dot{\chi})\times\alpha^{-1}\left(\alpha(\lim_{k\to\infty}a_k)\right)\right)\dot{+}\alpha(ax) \\
&= \dot{\chi}\dot{\times}\alpha(\lim_{k\to\infty}a_k)\dot{+}\alpha(ax) \\
&= \dot{\chi}\dot{\times}\ {}^{\alpha}\lim_{k\to\infty}\dot{a}_k\dot{+}\dot{a}\dot{\times}\dot{x}.
\end{aligned}$$

Theorem 2.16. An α-matrix $\dot{A} = (\dot{a}_{nk})$ is α-regular iff (i) $\left\|\dot{A}\right\|_\alpha \dot{<} \dot{+}\infty$, (ii) $\dot{a}_k = {}^{\alpha}\lim_{n\to\infty}\dot{a}_{nk} = \dot{0}$ for each $k \in \mathbb{N}$ and (iii) ${}^{\alpha}\lim_{\dot{A}}\dot{1} = \dot{1}$.

Proof. Let $\dot{A}$ be α-regular. Obviously, $\dot{A}$ is α-conservative and by Theorem 2.12, $\dot{A} \in \phi_\alpha$, that is $\left\|\dot{A}\right\|_\alpha \dot{<} \dot{+}\infty$. Since $\dot{\delta}^k \in c_{\alpha,0}$, that is, ${}^{\alpha}\lim_{n\to\infty}\dot{\delta}_n^k = \dot{0}$, by the hypothesis, ${}^{\alpha}\lim_{\dot{A}}\dot{\delta}^k = \dot{0}$. On the other hand $\left(\dot{A}\dot{\times}\dot{\delta}^k\right)_n = \dot{a}_{nk}$ and hence $\dot{a}_k = {}^{\alpha}\lim_{n\to\infty}\dot{a}_{nk} = {}^{\alpha}\lim_{\dot{A}}\dot{\delta}^k = \dot{0}$. Finally, since $\dot{A}$ is α-regular, the α-limit of $\dot{A}\dot{\times}\dot{x}$ must coincide with the α-limit of $\dot{x}$. Hence condition (iii) follows.

Conversely, suppose that the conditions (i), (ii) and (iii) hold. By Theorem 2.12, $\dot{A}$ is conservative. Now, let $\dot{x} = (\dot{x}_k) \in c_\alpha$ with ${}^{\alpha}\lim_{k\to\infty}\dot{x}_k = \dot{l}$. Then, by Theorem 2.15,

$${}^{\alpha}\lim_{\dot{A}}\dot{x} = \dot{\chi}\dot{\times}\dot{l}\dot{+}\dot{a}\dot{\times}\dot{x},$$

where $\dot{\chi} = lim_{\dot{A}}\dot{1}\dot{-}\ {}_{\alpha}\dot{\sum}_{k=1}^{\infty}\dot{a}_k$. In this case, by (ii) we get $\dot{\chi} = lim_{\dot{A}}\dot{1} = \dot{1}$. Also, $\dot{a}\dot{\times}\dot{x} = \dot{0}$ and thus ${}^{\alpha}\lim_{\dot{A}}\dot{x} = \dot{l}$. Hence ${}^{\alpha}\lim_{\dot{A}}\dot{x}$= ${}^{\alpha}\lim_{k\to\infty}\dot{x}_k$ for each $\dot{x} \in c_\alpha$, therefore $\dot{A}$ is α-regular.

Theorem 2.17. If $\dot{A} \in \phi_\alpha$, then $c_{\alpha,\dot{A}} \cap \ell_\alpha^\infty$ and $c_{\alpha,\dot{A}}^0 \cap \ell_\alpha^\infty$ with $c_{\alpha,\dot{A}}^0 = \left\{\dot{x}: \dot{A}\dot{x} \in c_{\alpha,0}\right\}$ are closed subspaces of ℓ_α^∞.

Proof. Let $\dot{A} \in \phi_\alpha$. Then, by Theorem 2.10, $\dot{A} \in (\ell_\alpha^\infty, \ell_\alpha^\infty)$. Thus $\mathcal{T}: \ell_\alpha^\infty \to \ell_\alpha^\infty$, $\mathcal{T}(\dot{x}) = \dot{A}\dot{x}$ is a α-bounded and α-linear operator. Clearly $c_{\alpha,\dot{A}} \cap \ell_\alpha^\infty = \mathcal{T}^{-1}(c_\alpha)$ and $c_{\alpha,\dot{A}}^0 \cap \ell_\alpha^\infty = \mathcal{T}^{-1}(c_{\alpha,0})$. Since $\mathcal{T}$ is continuous, and by Lemma 2.11, c_α and $c_{\alpha,0}$ are closed spaces of ℓ_α^∞, $\mathcal{T}^{-1}(c_\alpha)$ and $\mathcal{T}^{-1}(c_{\alpha,0})$ are also closed subsets of ℓ_α^∞. On the other hand, since c_α and $c_{\alpha,0}$ are vector spaces, and also $\mathcal{T}^{-1}$ is linear, $c_{\alpha,\dot{A}} \cap \ell_\alpha^\infty$ and $c_{\alpha,\dot{A}}^0 \cap \ell_\alpha^\infty$ are closed subspaces of ℓ_α^∞.

2.1 Discussion of Last Section

The results obtained in this section provide a non-Newtonian counterpart of several fundamental facts from the classical theory of sequence spaces and matrix transformations. First, Theorems 2.2 and 2.4 establish the completeness of the basic α-sequence spaces under their natural α-norms. These results ensure that the spaces considered in the subsequent matrix-transformation theory have the required Banach-space structure.

The duality results in Theorem 2.5 and Theorem 2.13 describe bounded α-linear functionals in terms of α-absolutely summable sequences and α-limit functionals. Consequently, the action of a bounded functional on the corresponding α-sequence spaces can be analyzed through the inverse generator α^{-1}, which connects the non-Newtonian formulation with the classical theory.

The matrix-transformation results provide a corresponding extension of classical matrix summability theory. Theorem 2.12 gives necessary and sufficient conditions for an α-matrix to be α-conservative, while Theorem 2.16 characterizes α-regular matrices. In particular, the latter result yields the non-Newtonian version of the classical Silverman–Toeplitz characterization.

Finally, Theorem 2.17 shows that the matrix domains generated by the matrices under consideration are closed subspaces of the relevant α-sequence space. Thus, the results of this section establish a connection between the structural properties of α-sequence spaces, their duals, and matrix transformations. This provides a unified framework for studying summability methods in the non-Newtonian setting.

2.2 Comparison with the Classical Theory and Statement of Contribution

Let Φ denote the coordinatewise map induced by the generator α, defined by $\Phi((\dot{x}_k)) = (\alpha^{-1}(\dot{x}_k))$ whenever an α-sequence is represented in the α-real number system. Under the assumptions stated in Section 1, Φ converts α-addition and α-scalar multiplication into the corresponding classical operations and preserves the relevant notions of convergence and Cauchy convergence. For the sequence spaces considered here, the corresponding α-spaces are therefore transported to their classical counterparts by Φ. In particular, completeness and closedness results obtained solely from this correspondence are structural transfers of classical results rather than independent new phenomena.

The same distinction applies to the Silverman–Toeplitz theorem. In classical form, a matrix A is regular on convergent sequences precisely when its rows are uniformly bounded in ℓ^1, each column tends to zero, and the row sums tend to one. The α-version in Theorem 2.16 expresses these conditions using α-arithmetic, α-limits, and the α-norm. Applying Φ coordinatewise translates these conditions into the classical ones, provided the matrix action and all series in question are well defined. Thus the theorem supplies an explicit formulation and proof in the α-framework; under the stated generator correspondence, its validity follows from the classical Silverman–Toeplitz characterization. The contribution of the present paper is the systematic formulation of these correspondences together with the stated α-dual, α-conservative-matrix, and matrix-domain results in one framework, rather than a claim that the generator correspondence by itself creates a new classical summability principle.

The present work builds on earlier research on non-Newtonian real series and topology, particularly the works cited as [11] and [12], and on the established literature on non-Newtonian sequence spaces cited in [2], [3], [4], [15], and [18]. Those references provide background and foundational results; the results claimed here are the sequence-space duality and matrix-transformation characterizations stated and proved in Section 2, including Theorems 2.5, 2.9–2.10, and 2.12–2.17, insofar as these specific statements and proofs are not already contained in the cited works.

The author declares that the present manuscript is intended to present the results identified above and to acknowledge the earlier publications on which its background relies.

3 Conclusion

In this paper, we developed a non-Newtonian framework for sequence spaces and matrix transformations generated by an arbitrary strictly increasing function α. The classical spaces of

bounded, convergent, null, and absolutely summable sequences were transferred to the α-setting, and their α-normed and α-Banach structures were established.

We also characterized the relevant bounded α-linear functionals and investigated the matrix transformations acting between α-sequence spaces. Necessary and sufficient conditions for α-conservative matrices were obtained, and an α-version of the Silverman–Toeplitz theorem was established for α-regular matrices. These results show that several fundamental principles of classical summability theory remain valid after being formulated in the non-Newtonian framework.

Furthermore, the closedness of the corresponding matrix domains was established. Together with the duality and completeness results, this provides a structural basis for further investigations of summability methods, matrix transformations, and operator-theoretic questions in non-Newtonian sequence spaces.

The approach developed here also suggests several directions for further study, including compact matrix transformations, multiplier spaces, α-duals of other sequence spaces, and matrix domains generated by more general non-Newtonian transformations.

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Declaration of Generative AI and AI-assisted Technologies in the Writing Process

During the preparation of this work, the author(s) utilized artificial intelligence (AI) technologies solely for the purpose of linguistic refinement and grammatical correction. Following the application of these tools, the author(s) comprehensively reviewed, verified, and edited the resulting text. The author(s) maintain full accountability and responsibility for the accuracy, integrity, and final content of the publication.

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