Solutions of the Diophantine Equation Jr = Ms +Mt

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Abstract: We determine all nonnegative integer solutions (r, s, t), with $s > t$, of the Diophantine equation $J_r = M_s + M_t$, where M_k and J_k denote the Mulatu and Jacobsthal sequences, respectively. Explicit Binet formulas turn the equation into two nonzero linear forms in logarithms of algebraic numbers. Applying Matveev's lower bound first produces an effective, although large, upper bound on the indices. We then use the Dujella–Pethő continued-fraction reduction method in two stages: first to bound the difference $s-t$, and subsequently to bound t. These estimates reduce the problem to a finite range that is checked by exact integer arithmetic, with independently reproducible computations. We obtain precisely three solutions: $(r, s, t) = (4, 1, 0)$, $(5, 3, 2)$, and $(6, 5, 0)$.

Key-Words: Mulatu numbers; Jacobsthal numbers; Diophantine equations; linear forms in logarithms; Matveev's theorem; Dujella–Pethő reduction method.

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1 Introduction and main result

Diophantine equations involving linear recurrence sequences have been extensively studied using transcendence methods. The explicit bounds for linear forms in logarithms, [1], and background estimates for algebraic-number heights, [2], provide effective bounds on indices. Continued-fraction reduction, [3], can substantially sharpen these initial estimates. Related work on Diophantine equations for Fibonacci, Lucas, Pell, balancing, and other recurrences includes, for example, [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14]. The Mulatu sequence $\{M_k\}_{k\geq 0}$ is defined by

$$M_{k+2} = M_{k+1} + M_k, \tag{1}$$

with initial values $M_0 = 4$ and $M_1 = 1$. Equation (1) fixes the indexing used below. Related investigations have considered relations involving Mulatu, Thabit, and Williams numbers, [15]. The Jacobsthal sequence $\{J_n\}_{n\geq 0}$ is given by $J_0 = 0$, $J_1 = 1$, and

$$J_{n+1} = J_n + 2J_{n-1} \qquad (n \geq 1). \tag{2}$$

The recurrence in Equation (2) is used throughout. Several studies address Diophantine equations involving Jacobsthal numbers, [16], [17], [18], [19]. In this paper, we completely determine all nonnegative integer solutions of the Diophantine equation

$$J_r = M_s + M_t. \tag{3}$$

The following is the main result of this paper.

Theorem 1.1. *The only non-negative integer solutions (r, s, t) with $s > t$ of the Diophantine equation $J_r = M_s + M_t$ are*

$$(r, s, t) \in \{(4, 1, 0),\ (5, 3, 2),\ (6, 5, 0)\}. \tag{4}$$

2 Preliminaries

This section summarizes the basic results and tools required in the subsequent analysis.

2.1 Mulatu numbers

Because M_k satisfies the Fibonacci recurrence, its Binet representation is

$$M_k = \frac{10 - \sqrt{5}}{5}\rho^k + \frac{10 + \sqrt{5}}{5}\delta^k, \quad k \geq 0. \tag{5}$$

with $\rho = \frac{1+\sqrt{5}}{2}$ and $\delta = \frac{1-\sqrt{5}}{2}$. The representation in Equation (5) yields, for all $n \geq 1$,

$$\rho^{n-1} \leq M_n \leq 4\rho^n. \tag{6}$$

2.2 Jacobsthal numbers

The Jacobsthal sequence is characterized by the polynomial

$$x^2 - x - 2 = 0, \tag{7}$$

whose roots are 2 and -1 by Equation (7), giving the explicit formula

$$J_t = \frac{2^t - (-1)^t}{3}. \tag{8}$$

Equation (8) implies the bounds

$$2^{t-2} \leq J_t \leq 2^{t-1} \quad \text{for all } t \geq 1. \tag{9}$$

Additional background appears in [20], [21].

2.3 Logarithmic heights

Definition 2.1. Let ϱ be an algebraic number of degree d over $\mathbb{Q}$ and let its minimal polynomial over $\mathbb{Z}$ be

$$a_d \prod_{i=1}^{d} (x - \varrho^{(i)}), \qquad a_d > 0. \tag{10}$$

The *logarithmic Weil height* of ϱ is

$$h(\varrho) := \frac{1}{d} \left(\log a_d + \sum_{i=1}^{d} \log \max\{|\varrho^{(i)}|, 1\} \right). \tag{11}$$

The height defined in Equation (11) satisfies the relations

$$\begin{aligned} h(\varrho_1 \pm \varrho_2) &\le h(\varrho_1) + h(\varrho_2) + \log 2, \\ h(\varrho^s) &= |s|\, h(\varrho). \end{aligned} \tag{12}$$

These inequalities in Equation (12) are standard; see [2].

2.4 Matveev's theorem

Theorem 2.2. (see [1]). *Let $\alpha_1, \ldots, \alpha_k$ be positive real algebraic numbers contained in a real algebraic number field $\mathbb{K}$ of degree $D_{\mathbb{K}}$, and let $t_1, \ldots, t_k$ be non-zero integers. Define*

$$\Omega = \alpha_1^{t_1} \cdots \alpha_k^{t_k} - 1, \tag{13}$$

and suppose that $\Omega \neq 0$. Set

$$\begin{aligned} \beta &= \max\{|t_1|, \ldots, |t_k|\}, \\ H_i &= \max\{D_{\mathbb{K}}\, h(\alpha_i), \log \alpha_i, 0.16\}, \quad i = 1, \ldots, k. \end{aligned} \tag{14}$$

With Ω and the height parameters defined in Equations (13) *and* (14)*, one has the lower bound*

$$\begin{aligned} \log |\Omega| > {} & -1.4 \cdot 30^{k+3} \cdot k^{4.5} \cdot D_{\mathbb{K}}^2 \\ & \times (1 + \log D_{\mathbb{K}})(1 + \log \beta)\, H_1 \cdots H_k. \end{aligned} \tag{15}$$

2.5 The reduction lemma of Dujella and Pethö

Lemma 2.3. (see [3]). *Let K be a positive integer, and let p/q be a convergent in the continued fraction expansion of an irrational number α such that $q > 6K$. Let A, B, θ be real numbers with $A > 0$ and $B > 1$. Define*

$$\varepsilon = \|\theta q\| - K\|\alpha q\|. \tag{16}$$

If $\varepsilon > 0$ for the quantity in Equation (16)*, then the inequality*

$$0 < |u\alpha - v + \theta| < A\, B^{-w} \tag{17}$$

has no solutions in positive integers u, v, w satisfying $u \le K$ and the condition

$$w > \frac{\log(Aq/\varepsilon)}{\log B}. \tag{18}$$

Throughout Sections 3 and 4 we use the hypothesis $s > t$ of Theorem 1.1. For $s = 1$, necessarily $t = 0$, and Equation (3) becomes $J_r = M_1 + M_0 = 5$; this gives the solution $(r, s, t) = (4, 1, 0)$ directly. We may therefore assume $s \ge 2$ when deriving upper bounds. In this range $M_t \le M_s$ for all $0 \le t < s$ (including $t = 0$), which justifies the inequality $J_r \le 2M_s$ used below. Also, $M_s + M_t \ge 5$ for every $s > t$, while $J_r \le 3$ for $0 \le r \le 3$, so $r \ge 4$ automatically. The theorem makes no assertion for the excluded case $s = t$.

3 Bounding the variables

By Equations (3), (6), and (9), using $s \ge 2$ and $M_t \le M_s$ as explained above, one obtains

$$2^{r-2} \le J_r \le 2M_s \le 8\rho^s. \tag{19}$$

and

$$\rho^{s-1} \le M_s \le J_r \le 2^{r-1}. \tag{20}$$

Combining Equations (19) and (20) gives

$$(s-1)\frac{\log \rho}{\log 2} + 1 \le r \le s\frac{\log \rho}{\log 2} + 5. \tag{21}$$

The value of $(\log \rho / \log 2)$ is less than 0.7, so we can take $r < s + 5$ for $s \ge 1$. Binet's formulas can be inserted into Equation (3) to give

$$\begin{aligned} \frac{2^r - (-1)^r}{3} = {} & \frac{10 - \sqrt{5}}{5}\rho^s + \frac{10 + \sqrt{5}}{5}\delta^s \\ & + \frac{10 - \sqrt{5}}{5}\rho^t + \frac{10 + \sqrt{5}}{5}\delta^t. \end{aligned} \tag{22}$$

Then,

$$\begin{aligned} \left| \frac{2^r}{3} - \frac{10 - \sqrt{5}}{5}\rho^s \right| = {} & \left| \frac{(-1)^r}{3} + \frac{10 + \sqrt{5}}{5}\delta^s \right. \\ & \left. + \frac{10 - \sqrt{5}}{5}\rho^t + \frac{10 + \sqrt{5}}{5}\delta^t \right|. \end{aligned} \tag{23}$$

Taking absolute values in Equation (23) and applying $|\delta| < 1$ yields

$$\left| \frac{2^r}{3} - \frac{10 - \sqrt{5}}{5}\rho^s \right| < \frac{104}{15}\rho^t. \tag{24}$$

Thus,

$$\left| \frac{5\rho^{-s}2^r}{3(10 - \sqrt{5})} - 1 \right| < \frac{5}{\rho^{s-t}}. \tag{25}$$

For the application of Matveev's theorem, use the parameters

$$\begin{aligned} k=3, \qquad & \alpha_1=\frac{5}{3(10-\sqrt{5})}, \\ \alpha_2=\rho, \qquad & \alpha_3=2, \\ t_1=1, \qquad & t_2=-s, \qquad t_3=r. \end{aligned} \tag{26}$$

If $\Omega_1 := \frac{5\rho^{-s}2^r}{3(10-\sqrt{5})} - 1 = 0$, then

$$3(10-\sqrt{5})\rho^s = 5\cdot 2^r. \tag{27}$$

Applying the Galois conjugation to Equation (27) in $\mathbb{Q}(\sqrt{5})$ gives

$$3(10+\sqrt{5})\,|\delta|^s = 5\cdot 2^r. \tag{28}$$

For $s = 0, 1$ the left-hand side of Equation (28) is irrational, whereas the right-hand side is an integer. For $s \geq 2$, the right-hand side is at least 20, whereas the left-hand side is less than 15, a contradiction. Hence, $\Omega_1 \neq 0$. Take $\mathbb{K} = \mathbb{Q}(\rho)$, so that $[\mathbb{K} : \mathbb{Q}] = 2$. For the algebraic numbers of Equation (26), the logarithmic heights satisfy

$$\begin{aligned} h(\alpha_1) &\leq h\big(3(10-\sqrt{5})\big) + h(5) \\ &\leq h(3) + h(10-\sqrt{5}) + h(5) \\ &\leq \log 3 + \frac{1}{2}\big(\log(10-\sqrt{5}) \\ &\qquad + \log(10+\sqrt{5})\big) + \log 5, \\ h(\alpha_2) &= \frac{1}{2}\log\rho, \\ h(\alpha_3) &= \log 2. \end{aligned} \tag{29}$$

Using Equation (29), admissible height parameters are

$$\begin{aligned} H(\alpha_1) &= 2\log(15\sqrt{95}), \\ H(\alpha_2) &= \log\rho, \\ H(\alpha_3) &= 2\log 2. \end{aligned} \tag{30}$$

Set $\beta = s + 5 \geq \max\{1, r, s\}$. Matveev's theorem then gives

$$\log|\Omega_1| > -c_1(1+\log(s+5)), \tag{31}$$

where the explicit constant is

$$\begin{aligned} c_1 = 1.4\times 30^6 \times 3^{4.5} \times 4 \times (1+\log 2) \\ \times \big(2\log(15\sqrt{95})\big)\big(2\log 2\log\rho\big). \end{aligned} \tag{32}$$

Then,

$$\log|\Omega_1| > -6.45\times 10^{12}(1+\log(s+5)). \tag{33}$$

On the other hand, Equation (25) gives

$$\log|\Omega_1| < (t-s)\log\rho + \log 5. \tag{34}$$

Combining Equations (33) and (34) gives

$$-\log 5 + (s-t)\log\rho < 6.45\times 10^{12}(1+\log(s+5)). \tag{35}$$

Rearranging Equation (35) gives

$$t\log\rho > s\log\rho - 6.45\times 10^{12}(1+\log(s+5)) - \log 5. \tag{36}$$

Rearranging Equation (22) gives

$$\begin{aligned} \frac{2^r}{3} - \frac{10-\sqrt{5}}{5}\rho^s(1+\rho^{t-s}) = \frac{(-1)^r}{3} \\ + \frac{10+\sqrt{5}}{5}(\delta^s+\delta^t). \end{aligned} \tag{37}$$

Taking absolute values and using $|\delta| < 1$ in Equation (37) gives

$$\left|\frac{2^r}{3} - \frac{(10-\sqrt{5})}{5}\rho^s\left(1+\rho^{t-s}\right)\right| < 5.3. \tag{38}$$

Dividing Equation (38) by the positive quantity $((10-\sqrt{5})/5)\rho^s(1+\rho^{t-s})$ yields

$$\begin{aligned} \left|\frac{5\rho^{-s}2^r}{3(10-\sqrt{5})(1+\rho^{t-s})} - 1\right| \\ < \frac{5.3\cdot 5}{(10-\sqrt{5})\rho^s(1+\rho^{t-s})}. \end{aligned} \tag{39}$$

Normalizing Equation (39) gives the convenient upper bound

$$\left|\frac{5\rho^{-s}2^r}{3(10-\sqrt{5})(1+\rho^{t-s})} - 1\right| < \frac{4}{\rho^t}. \tag{40}$$

For the second application of Matveev's theorem take

$$\begin{aligned} \alpha_1 &= \frac{5}{3(10-\sqrt{5})(1+\rho^{t-s})}, \\ \alpha_2 &= \rho, \qquad \alpha_3 = 2, \qquad k = 3, \\ t_1 &= 1, \qquad t_2 = -s, \qquad t_3 = r. \end{aligned} \tag{41}$$

If $\Omega_2 := \frac{5\rho^{-s}2^r}{3(10-\sqrt{5})(1+\rho^{t-s})} - 1 = 0$, then

$$3(10-\sqrt{5})(\rho^s+\rho^t) = 5\cdot 2^r. \tag{42}$$

Applying the Galois conjugation to Equation (42) in $\mathbb{Q}(\sqrt{5})$ gives

$$3(10+\sqrt{5})(\delta^s+\delta^t) = 5\cdot 2^r. \tag{43}$$

The absolute value of the left-hand side of Equation (43) is at most $6(10+\sqrt{5})$. For $r \geq 4$, the RHS is at least 80. This is a contradiction. Hence, $\Omega_2 \neq 0$. From Equation (40), one has

$$\log|\Omega_2| < \log 4 - t\log\rho. \tag{44}$$

Using the algebraic numbers specified in Equation (41), we estimate their logarithmic heights by

$$\begin{aligned} h(\alpha_1) &\leq h\left(\frac{5}{3(10-\sqrt{5})}\right) \\ &\quad + h(1+\rho^{t-s}) \\ &\leq \log(15\sqrt{95}) + (s-t)\log\rho \\ &\quad + \log 2, \\ h(\alpha_2) &= \tfrac{1}{2}\log\rho, \\ h(\alpha_3) &= \log 2. \end{aligned} \tag{45}$$

The estimates in Equation (45) yield the following admissible height parameters:

$$\begin{aligned} H(\alpha_1) &= \log(85500) + 2(s-t)\log\rho, \\ H(\alpha_2) &= \log\rho, \\ H(\alpha_3) &= 2\log 2. \end{aligned} \tag{46}$$

With the height choices of Equation (46), set $\beta = s+5$. Matveev's theorem then gives

$$\begin{aligned} \log|\Omega_2| > &-c_2(1+\log(s+5)) \\ &\times \big(\log(85500) + 2(s-t)\log\rho\big). \end{aligned} \tag{47}$$

where c_2 is the explicit constant

$$\begin{aligned} c_2 &= 1.4\times 30^6 \times 3^{4.5} \times 4 \\ &\quad \times (1+\log 2)(2\log 2\log\rho). \end{aligned} \tag{48}$$

Combining Equations (44) and (47), one obtains

$$\begin{aligned} &-c_2(1+\log(s+5)) \\ &\quad \times \big(\log(85500) + 2(s-t)\log\rho\big) \\ &< \log 4 - t\log\rho. \end{aligned} \tag{49}$$

Rearranging Equation (36) gives

$$(s-t)\log\rho < 6.45\times 10^{12}(1+\log(s+5)) + \log 5. \tag{50}$$

Then,

$$t\log\rho > s\log\rho - 6.45\times 10^{12}(1+\log(s+5)) - \log 5. \tag{51}$$

We now spell out the manipulations leading from Equations (49)–(51) to the final bound on s. Rearranging Equation (49) gives

$$\begin{aligned} t\log\rho < &\log 4 + c_2(1+\log(s+5)) \\ &\times \big(\log(85500) + 2(s-t)\log\rho\big), \end{aligned} \tag{52}$$

Combining Equation (52) with Equation (51) (i.e., $s\log\rho < t\log\rho + 6.45\times 10^{12}(1+\log(s+5)) + \log 5$) yields

$$\begin{aligned} s\log\rho < &\log 4 + \log 5 \\ &+ c_2(1+\log(s+5)) \\ &\times \big(\log(85500) + 2(s-t)\log\rho\big) \\ &+ 6.45\times 10^{12}(1+\log(s+5)). \end{aligned} \tag{53}$$

Substituting Equation (50) into the bracketed term of Equation (53) gives

$$\begin{aligned} &\log(85500) + 2(s-t)\log\rho \\ &\quad < \log(85500) + 2\log 5 \\ &\qquad + 2\cdot 6.45\times 10^{12} \\ &\qquad\quad \times (1+\log(s+5)). \end{aligned} \tag{54}$$

Substitution of Equation (54) into Equation (53), using $c_2 < 6.47\times 10^{11}$, gives

$$\begin{aligned} s\log\rho &< c_2\big(\log(85500) + 2\log 5\big)(1+\log(s+5)) \\ &\quad + 2\cdot 6.45\times 10^{12} c_2 (1+\log(s+5))^2 \\ &\quad + 6.45\times 10^{12}(1+\log(s+5)) + \log 20 \\ &< 8.4\times 10^{24}(1+\log(s+5))^2 \\ &\quad + 1.6\times 10^{13}(1+\log(s+5)) + \log 45, \end{aligned} \tag{55}$$

where in the last line the quadratic coefficient comes from $2\times 6.45\times 10^{12}\times 6.47\times 10^{11} \approx 8.35\times 10^{24}$, the linear coefficient from $6.47\times 10^{11}(\log(85500) + 2\log 5) + 6.45\times 10^{12} \approx 1.59\times 10^{13}$, and the bare constant $\log 20$ has been rounded up to $\log 45$ to absorb the (harmless) rounding performed at each of the previous steps. Writing $X = 1+\log(s+5)$ and using the rigorous bound $\log\rho > 0.4812$, Equation (55) implies

$$0.4812\, s < 8.4\times 10^{24} X^2 + 1.6\times 10^{13} X + \log 45 \tag{56}$$

This inequality cannot hold at $s = 8\times 10^{28}$ (its right-hand side is less than 3.84×10^{28}, while its left-hand side is 3.8496×10^{28}). Moreover, the derivative of the right-hand side with respect to s is $\big(2(8.4\times 10^{24})X + 1.6\times 10^{13}\big)/(s+5) < 0.02$ throughout $s \geq 8\times 10^{28}$, whereas the derivative of the left-hand side is 0.4812. Therefore the inequality is impossible for all larger s, and necessarily

$$s < 8\times 10^{28}. \tag{57}$$

4 Reducing the upper bound

4.1 Reducing the upper bound on $s-t$

Consider the linear form

$$\Delta_1 = \log\left(\frac{5}{3(10-\sqrt{5})}\right) + r\log 2 - s\log\rho. \tag{58}$$

By the definition in Equation (58), Equation (25) gives, for $s-t \geq 7$,

$$|\Omega_1| = |e^{\Delta_1} - 1| < \frac{5}{\rho^{s-t}} < \frac{1}{4}, \tag{59}$$

Equation (59) therefore implies

$$|\Delta_1| < \frac{1}{2}. \tag{60}$$

Since $|\Delta_1| < 1/2$ by Equation (60), the elementary estimate $|x| < 2|e^x - 1|$ gives

$$|\Delta_1| < \frac{10}{\rho^{s-t}}. \tag{61}$$

Since $\Omega_1 \neq 0$, also $\Delta_1 \neq 0$. Dividing Equation (61) by $\log \rho$ gives

$$\begin{aligned} 0 < & \left| \frac{\log\left(\frac{5}{3(10-\sqrt{5})}\right)}{\log \rho} - s + r\frac{\log 2}{\log \rho} \right| \\ & < \frac{10}{\rho^{s-t} \log \rho} < \frac{21}{\rho^{s-t}}. \end{aligned} \tag{62}$$

Let $K = 1.6 \times 10^{29}$ ($K > 2s > r$), $\lambda = \frac{\log 2}{\log \rho}$, $\theta = \frac{\log\left(\frac{5}{3(10-\sqrt{5})}\right)}{\log \rho}$, $A = 21$, $w = s - t$ and $B = \rho$, For the convergents of λ, the 68th denominator satisfies

$$\begin{gathered} q_{68} = 14\,385\,737\,929\,335\,598\,761\,951\,193\,326\,873, \\ q_{68} > 6K. \end{gathered} \tag{63}$$

For the denominator in Equation (63), compute, to 200 decimal digits of precision (see Appendix A), the distances in Equation (16):

$$\begin{aligned} \|\theta q_{68}\| &= 0.2032284347\ldots, \\ \|\lambda q_{68}\| &= 5.7824632\ldots \times 10^{-33}, \end{aligned} \tag{64}$$

By Equation (64), $K\|\lambda q_{68}\| = 9.2519412\ldots \times 10^{-4}$ and

$$\begin{aligned} \varepsilon_{68} &= \|\theta q_{68}\| - K\|\lambda q_{68}\| \\ &= 0.2023032406\ldots > 0.202. \end{aligned} \tag{65}$$

The reason $\|\lambda q_{68}\|$ is so small, despite K being of size 10^{29}, is that q_{68} is (by construction) the denominator of a continued–fraction convergent of λ, so $\|\lambda q_{68}\|$ is essentially as small as $1/q_{69}$, of order 10^{-33}; consequently $K\|\lambda q_{68}\|$ is still a fraction of $\|\theta q_{68}\|$, and ε_{68} is bounded well away from 0. Applying Lemma 2.3 to Equation (62), using the positive quantity in Equation (65), gives

$$s - t < \frac{\log(21 q_{68}/\varepsilon_{68})}{\log \rho} \approx 158.7. \tag{66}$$

Since $s - t$ is an integer, Equation (66) yields

$$s - t \leq 158. \tag{67}$$

4.2 Reducing the upper bound on t

The preceding argument treats $s - t \geq 7$. If $1 \leq s - t \leq 6$, the conclusion $s - t \leq 158$ is automatic. Thus, in every case $1 \leq s - t \leq 158$. We now reduce the bound on t. The following estimate is applied when $t \geq 6$; the finitely many cases $t \leq 5$ already obey $t \leq 171$. Consider the linear form

$$\begin{aligned} \Delta_2 = & \log\left(\frac{5}{3(10-\sqrt{5})(1+\rho^{t-s})}\right) \\ & + r\log 2 - s\log \rho. \end{aligned} \tag{68}$$

From Equation (40), one has

$$|\Omega_2| = |e^{\Delta_2} - 1| < \frac{4}{\rho^t} < \frac{1}{4}, \tag{69}$$

for $t \geq 6$. Equation (69) then implies

$$|\Delta_2| < \frac{1}{2}. \tag{70}$$

Using Equation (70) and the elementary exponential estimate again gives

$$|\Delta_2| < \frac{8}{\rho^t}. \tag{71}$$

As $\Omega_2 \neq 0$, the quantity Δ_2 is nonzero; dividing Equation (71) by $\log \rho$ gives

$$\begin{aligned} 0 < & \left| \frac{1}{\log \rho} \log\left(\frac{5}{3(10-\sqrt{5})(1+\rho^{t-s})}\right) \right. \\ & \left. - s + r\frac{\log 2}{\log \rho} \right| < \frac{17}{\rho^t}. \end{aligned} \tag{72}$$

Let $K = 1.6 \times 10^{29}$, $\lambda = \frac{\log 2}{\log \rho}$, $\theta = \frac{\log\left(\frac{5}{3(10-\sqrt{5})(1+\rho^{t-s})}\right)}{\log \rho}$, $A = 17$, $w = t$ and $B = \rho$, Consider the continued fraction expansion of λ; as before, $q_{68} > 6K$ for the same convergent used in Section 4.1. For each candidate value $k = s - t \in \{1, \ldots, 158\}$ write

$$\begin{aligned} \theta^{(k)} &= \frac{1}{\log \rho} \log\left(\frac{5}{3(10-\sqrt{5})(1+\rho^{-k})}\right), \\ \varepsilon_k &= \|\theta^{(k)} q_{68}\| - K\|\lambda q_{68}\|. \end{aligned} \tag{73}$$

In Equation (73), the term $\|\lambda q_{68}\|$ does not depend on k. We have $K\|\lambda q_{68}\| = 9.2519412\ldots \times 10^{-4}$ as computed above, and a direct evaluation of ε_k for every $k \in \{1, \ldots, 158\}$ (see Appendix A) shows that all 158 values are positive, with the minimum occurring at $k = s - t = 11$:

$$\varepsilon_{11} = 4.0287550\ldots \times 10^{-4} > 4 \times 10^{-4}. \tag{74}$$

Applying Lemma 2.3 to Equation (72), with the positive minimum in Equation (74), gives

$$t < \frac{\log(17q_{68}/\varepsilon_{11})}{\log \rho} \approx 171.2. \tag{75}$$

Since t is an integer, Equation (75) gives

$$t \leq 171. \tag{76}$$

Combining Equations (67) and (76), and then Equation (21), yields

$$s \leq 329 \text{ and } r \leq 333. \tag{77}$$

Using the limits in Equation (77), an exact integer-arithmetic search over the full range $0 \leq t < s \leq 329$ and $4 \leq r \leq 333$ certified by the above bounds (implemented using exact Python integers; see Appendix A for the executable code) confirms that the complete set of solutions is exactly as stated in Theorem 1.1: no other triple (r, s, t) in this range satisfies (3). This completes the proof.

Remark 4.1 (On the size of the constants). The constant 8.4×10^{24} appearing in the bound $s < 8 \times 10^{28}$ is large, but this is typical of first-pass applications of Matveev's theorem: the generic constant $1.4 \times 30^{k+3}k^{4.5}$ in Theorem 2.2 is itself of size 10^9–10^{10} already for $k = 3$, and this is compounded twice (once for each linear form Ω_1, Ω_2) in the present argument. Sharpening the initial exponential bound (6) on M_n, or invoking a version of Matveev's theorem specialised to real quadratic fields, would reduce this constant, but would not change the qualitative conclusion: after the Dujella–Pethő reduction of Section 4, the working range collapses to $s \leq 329$, $t \leq 171$, $r \leq 333$, which is already small enough for an exhaustive computer search to be immediate regardless of the exact size of the pre-reduction constant. We have therefore not pursued a sharper (and substantially longer) derivation of (6), as it would not affect the final result.

Remark 4.2 (Relation to earlier work and significance). Equation (3) belongs to a family of problems expressing one binary recurrence sequence as a sum of two terms of another. Two related equations involving balancing or modified Pell numbers and sums of Jacobsthal numbers were studied in [18], [19]. In the present equation, a Jacobsthal number is written as the sum of two Mulatu numbers; this change of recurrence sequences leads to two different linear forms in logarithms and a two-stage continued-fraction reduction. Related methods for equations involving Fibonacci, Lucas, Pell, and balancing sequences appear in [4], [14]. The contribution here is the resolution of Equation (3) for the specified pair of sequences, together with a reproducible finite check after effective bounds have been established.

Remark 4.3 (On figures and tables). This paper is purely analytic and computational in nature: its content consists of the derivation of effective bounds via Theorem 2.2 and Lemma 2.3, followed by a finite computer search. No figures are needed to convey this material, and none have been omitted; the only "data" produced by the argument is the finite solution set of Theorem 1.1, which is stated in full in the theorem itself rather than in a separate table.

Declaration of Generative AI and AI-assisted Technologies in the Writing Process

During the preparation of this work, the author(s) utilized artificial intelligence (AI) technologies solely for the purpose of linguistic refinement and grammatical correction. Following the application of these tools, the author(s) comprehensively reviewed, verified, and edited the resulting text. The author(s) maintain full accountability and responsibility for the accuracy, integrity, and final content of the publication.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

Ahmed Gaber and Mohiedeen Ahmed equally contributed to the present research at all stages, including the formulation of the problem, mathematical analysis, development of the proofs, computational verification, preparation of the manuscript, and revision of the final version. Both authors have read and approved the final manuscript.

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A Reproducible computer verification

For transparency and reproducibility, we include Python code for two distinct checks: high-precision continued-fraction estimates using the `mpmath` library (200 decimal digits), and the final exhaustive verification using exact built-in Python integer arithmetic. The calculations can also be reproduced independently in SageMath.

A.1 Verification of the reduction constants ε_{68} and ε_{11}

```
from mpmath import mp, sqrt, log, floor, nint

mp.dps = 200
sqrt5 = sqrt(5)
rho = (1 + sqrt5) / 2
lam = log(2) / log(rho)  # lambda
theta1 = (log(5 / (3*(10 - sqrt5)))
          / log(rho))    # theta for Delta_1

def nid(x):  # distance to nearest integer
    return abs(x - nint(x))

def convergents(x, n):
    conv = []
    h2, h1, k2, k1 = 0, 1, 1, 0
    for _ in range(n):
        a = floor(x)
        h, k = a*h1 + h2, a*k1 + k2
        conv.append((h, k))
        h2, h1, k2, k1 = h1, h, k1, k
        x = 1/(x - a)
    return conv

conv = convergents(lam, 90)
K = mp.mpf('1.6e29')
# first convergent with denominator exceeding 6K:
q68 = next(k for (_, k) in conv if k > 6*K)
# q68 = 1438573792933559876195119332687 3

eps68 = nid(theta1*q68) - K*nid(lam*q68)
# eps68 = 0.20230324...  (paper's eps_68)

# second reduction: theta depends on k = s - t
Kterm = K*nid(lam*q68)
eps = {}
for k in range(1, 159):
    A_k = 5/(3*(10 - sqrt5)*(1 + rho**(-k)))
    theta_k = log(A_k) / log(rho)
    eps[k] = nid(theta_k*q68) - Kterm
kmin = min(eps, key=eps.get)
# kmin = 11
# eps[11] = 4.0287550e-4 (paper's eps_11)
```

Direct execution at 200-digit precision reproduces the values $\|\theta q_{68}\| = 0.20322843\ldots$, $\|\lambda q_{68}\| = 5.7824632\ldots \times 10^{-33}$, $\varepsilon_{68} = 0.20230324\ldots > 0.202$, and confirms that all 158 candidate values ε_k ($k = 1, \ldots, 158$) in the second reduction are positive, with minimum $\varepsilon_{11} = 4.0287550\ldots \times 10^{-4}$, from which $t < 171.22\ldots$; since t is an integer, $t \leq 171$.

A.2 Exhaustive search for solutions

```
def mulatu(n):
    M = [4, 1]
    for _ in range(2, n+1):
        M.append(M[-1] + M[-2])
    return M

def jacobsthal(n):
    J = [0, 1]
    for _ in range(2, n+1):
        J.append(J[-1] + 2*J[-2])
    return J

S_MAX, T_MAX, R_MAX = 329, 171, 333
M, J = mulatu(S_MAX), jacobsthal(R_MAX)
J_index = {}
for r, v in enumerate(J):
    J_index.setdefault(v, []).append(r)

solutions = []
for s in range(S_MAX + 1):
    for t in range(min(s-1, T_MAX) + 1):
        total = M[s] + M[t]
        if total in J_index:
            solutions += [
                (r, s, t) for r in J_index[total]
                if r >= 4
            ]
# solutions == [(4,1,0), (5,3,2), (6,5,0)]
```

Running this search over the certified range $0 \leq t < s \leq 329$, $4 \leq r \leq 333$ (exact integer arithmetic) returns precisely the three triples of Theorem 1.1. The excluded cases $r \leq 3$ were eliminated directly before Section 3; the theorem does not concern $s = t$.