Associated Curves of Timelike Magnetic Curves

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Abstract: A magnetic vector field in three-dimensional space is divergence-free. The trajectories it generates are called magnetic curves. This paper characterises the Mannheim, Bertrand and involute–evolute structures carried by timelike magnetic curves in Minkowski 3-space. For a timelike N-magnetic curve, a single lemma governs all three cases. A Mannheim pair occurs exactly when the magnetic field lies in the rectifying plane of the trajectory and the relation $\kappa = \lambda(\tau^2 - \kappa^2)$ holds with λ constant. The same condition rules out Bertrand pairs. Involutes never occur. The Mannheim structure has codimension one. It is therefore unstable under perturbations of the magnetic field. The trajectory is a slant helix if and only if its Mannheim partner is a general helix, and the two invariants are reciprocal. A closed-form example is given. Timelike B-magnetic curves behave differently. There the governing orthogonality is automatic, and Bertrand pairs do exist. An explicit example is constructed.

Key-Words: Lorentz force equation, Timelike magnetic curves, Mannheim curves, Bertrand curves, Involute-evolute pairs, Slant helices.

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1 Introduction

The study of magnetic curves and vector fields represents a significant area of research in both the fields of physics and mathematics. A key characteristic of any magnetic vector field is that it is divergence-free in three-dimensional space. Magnetic vector fields are generated by electric currents and changing electric fields, and charged particles moving within these fields are influenced by magnetic forces. The paths traced by these particles, known as magnetic trajectories, are termed magnetic curves. The study of magnetic curves is crucial for both theoretical physics and engineering applications, including the analysis of electromagnetism, plasma physics, and magnetic materials. Although the study of magnetic trajectories initially seems to be a problem confined to the realm of physics, recent research highlights that the analysis of magnetic flow within a magnetic field has increasingly adopted a variational and geometrical perspective. Specifically, the formulation of magnetic curves has been advanced through the integration of concepts from differential geometry and variational calculus, extending from fundamental spaces to more complex manifolds.

The study of curves and surfaces is a central focus in differential geometry, both in Euclidean and non-Euclidean spaces. Research has also extensively explored curves associated with a given curve. The Frenet frame is a commonly used tool for describing the geometric characteristics of these curves and for their classification within differential geometry. For example, the definitions of involute–evolute pairs, Bertrand pairs and Mannheim pairs rely on the Frenet frame. Many studies have investigated these associated curves.

The Lorentz force on a Riemannian manifold M exerted by a magnetic field is represented by a skew-symmetric $(1,1)$-type tensor field ϕ satisfying

$$g_M(\phi(U), V) = F(U, V) \tag{1}$$

for any $U, V \in TM$, [1], [2], [3]. Solutions to the Lorentz force equation are analogous to Kirchhoff elastic rods, establishing a connection between two seemingly distinct physical models. The trajectory of a charged particle in a magnetic field F is represented by a smooth curve γ on M which obeys the Lorentz equation $\nabla_{\dot\gamma}\dot\gamma = \phi(\dot\gamma)$, where ϕ is the Lorentz force in (1) and ∇ stands for the Levi-Civita connection. In three dimensions, the magnetic vector field V associated with F is divergence-free, that is, $\operatorname{div} V = 0$. As a charged particle follows a path in a magnetic field, the tangent (velocity), normal and binormal vectors are all influenced by the field; the forces acting in the normal and binormal directions are given by $\phi(N) = V \times N$ and $\phi(B) = V \times B$, respectively, and the trajectories of charged particles are altered accordingly.

It is important to study the relationships between magnetic curves and other special curves such as

Bertrand, Mannheim and involute–evolute curves. In Euclidean 3-space, such relationships were given, [4]. All magnetic curves corresponding to Killing magnetic fields were presented in 3-dimensional Euclidean space, [5]. In addition, magnetic curves were analysed with the aid of the geometric properties of flat para-Kähler manifolds, [6]. Magnetic curves in the three-dimensional Heisenberg group were also investigated, [7]. In higher dimensions, slant normal magnetic curves in S-manifolds were classified, [8], and, more recently, normal magnetic curves in homothetic s-th Sasakian manifolds were characterised, [9]. On the other hand, associated curves have been studied independently of the magnetic setting. Bertrand curves and their characterisation in Lorentzian space were studied, [10]. The concepts of the binormal-direction curve and the binormal-donor curve of a Frenet curve in Euclidean 3-space were introduced, and the connections between the curvature and torsion of these mates were obtained, [11]. Subsequently, new associated curves of a Frenet curve in three- and four-dimensional Euclidean space were explored, [12]. The evolutes and involutes of space curves were investigated, [13]. Further studies on Frenet curves, magnetic curves and associated curves are available in the literature, [1], [3], [14], [15], [16], [17].

Spacelike magnetic curves and their associated curve pairs have previously been studied in the literature. In particular, associated curves for N-magnetic structures corresponding to the spacelike principal normal of spacelike curves were addressed, [18], [19]. The present study fills this gap by treating timelike N- and B-magnetic curves, which have not yet been examined in detail. Section 1 covers the significance of magnetic curves and associated curves, along with a review of recent literature. Section 2 provides the definitions and basic facts used later. Section 3 develops the theory for timelike N-magnetic curves: after two lemmas that isolate the geometric meaning of the free function ω in the magnetic field, we characterise the Mannheim pairs (Theorem 1), establish the genericity and the slant-helix duality that accompany them (Corollaries 1 and 3), construct a closed-form example, and prove that neither Bertrand pairs nor involutes can occur (Theorems 4 and 5). Section 4 carries the same programme through for timelike B-magnetic curves and shows that the two families behave differently: Bertrand pairs, which are impossible in the N-magnetic setting, do exist here, and an explicit example is exhibited. Section 5 summarises the outcome.

2 Preliminaries

This section recalls some basic formulas and concepts for later use. Let E_1^3 be 3-dimensional Minkowski space with a symmetric, bilinear and non-degenerate metric of signature $(+,+,-)$ and vector product

$$v \times w = \left(\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}, \begin{vmatrix} v_3 & v_1 \\ w_3 & w_1 \end{vmatrix}, - \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \right), \tag{2}$$

where $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3) \in E_1^3$. A vector w in E_1^3 is called spacelike if $\langle w, w \rangle > 0$ or $w = 0$, timelike if $\langle w, w \rangle < 0$, and lightlike or null if $\langle w, w \rangle = 0$ and $w \neq 0$. A parametrized curve $\alpha = \alpha(s)$ in E_1^3 is called a timelike curve if its tangent vector field is timelike, that is, $\langle \alpha'(s), \alpha'(s) \rangle < 0$.

Assume that $\alpha(s)$ is a timelike curve and $\{T,\ N,\ B\}$ is the moving Frenet frame along $\alpha(s)$ in E_1^3, so that $\langle T, T \rangle = -1$ and $\langle N, N \rangle = \langle B, B \rangle = 1$. The equations governing the Frenet derivatives are

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}, \tag{3}$$

where κ and τ are respectively the curvature and the torsion of the timelike curve $\alpha(s)$, [20], [21].

Recall that α is called a *general helix* if the ratio τ/κ is constant, and a *slant helix* if its principal normal makes a constant angle with a fixed direction of E_1^3. The latter class admits the following characterisation.

Proposition 1. *Let $\alpha(s)$ be a timelike space curve with $\kappa^2(s) - \tau^2(s) \neq 0$. Then α is a slant helix if and only if*

$$\frac{\kappa^2}{|\kappa^2 - \tau^2|^{3/2}} \left(\frac{\tau}{\kappa} \right)' \tag{4}$$

is a constant function, [20].

The absolute value in (4) is essential for the present paper: under the standing non-degeneracy assumptions of Sections 3 and 4 one always has $\tau^2 > \kappa^2$, so that $\kappa^2 - \tau^2 < 0$.

3 Associated curves of timelike N-magnetic curves

In this section we investigate the associated curves – Mannheim, Bertrand and involute–evolute pairs – of a timelike N-magnetic curve, and we determine exactly which conditions on the magnetic field turn the magnetic trajectory into each of these curves. The whole discussion is organised around a single structural lemma (Lemma 2 below), from which the Mannheim, Bertrand and involute statements follow as three faces of one computation.

3.1 The magnetic field of a timelike N-magnetic curve

Let β be a timelike curve in E_1^3 with arclength parameter s, curvature $\kappa > 0$ and Frenet frame fields $\{T,\ N,\ B\}$, so that $\langle T, T\rangle = -1$ and $\langle N, N\rangle = \langle B, B\rangle = 1$. With respect to the vector product (2) the frame satisfies

$$T \times N = B, \quad N \times B = -T, \quad B \times T = N. \tag{5}$$

Imposing the Lorentz force equation $\nabla_T N = \phi(N) = V \times N$ together with (5) determines the magnetic field up to a single free function, namely

$$\begin{aligned} V &= -\tau T - \omega N + \kappa B, \\ \phi(T) &= V \times T = \kappa N + \omega B, \end{aligned} \tag{6}$$

where $\omega = \langle \phi(T), B\rangle$ (compare, [3], [22]). The curve β is then an N-magnetic trajectory of V.

The function ω is the only degree of freedom in (6), and it governs the entire section. Its geometric content is the following.

Lemma 1. *For a timelike N-magnetic curve the following are equivalent: (i) $\omega \equiv 0$; (ii) $\phi(T) = \nabla_T T$, that is, the curve is simultaneously a $T-$magnetic and an N-magnetic trajectory of V; (iii) V lies in the rectifying plane* $\operatorname{span}\{T, B\}$ *of the curve.*

Proof. Immediate from (6), since $\nabla_T T = \kappa N$ and $\langle V, N\rangle = -\omega$.

Throughout this section we work under the non-degeneracy assumption

$$\tau^2 - \omega^2 - \kappa^2 > 0, \tag{7}$$

which allows us to define the normalised magnetic field of β by

$$\begin{aligned} W &= \frac{1}{\delta}(-\tau T - \omega N + \kappa B), \\ \delta &= \sqrt{\tau^2 - \omega^2 - \kappa^2}, \end{aligned} \tag{8}$$

so that W is a unit timelike vector field. Note that (7) forces $\tau^2 > \kappa^2$ and $\tau \neq 0$. If instead $-\tau^2 + \omega^2 + \kappa^2 \geq 0$, then W is spacelike or lightlike, and the timelike Frenet apparatus employed below does not apply. That situation lies outside the scope of the present paper.

Let γ denote the direction curve of W, that is, the curve whose unit tangent field satisfies $T^* = \pm W$, with arclength parameter s^* and Frenet apparatus $\{T^*,\ N^*,\ B^*,\ \kappa^*,\ \tau^*\}$. The correspondence between the two curves is $s \mapsto s^*(s)$, so that

$$\gamma'(s) = T^* \frac{ds^*}{ds}, \tag{9}$$

and in general $ds^*/ds \neq 1$.

The next lemma is the computational core of the section.

Lemma 2. *Along a timelike N-magnetic curve,*

$$\begin{aligned} W' = &-\left[\left(\frac{\tau}{\delta}\right)' + \frac{\omega\kappa}{\delta}\right] T - \left(\frac{\omega}{\delta}\right)' N \\ &+ \left[\left(\frac{\kappa}{\delta}\right)' + \frac{\omega\tau}{\delta}\right] B. \end{aligned} \tag{10}$$

In particular, if $\omega \equiv 0$ then

$$W' = -\left(\frac{\tau}{\delta}\right)' T + \left(\frac{\kappa}{\delta}\right)' B, \tag{11}$$

so that T^*, $T^{*\prime}$ and hence N^* all lie in the rectifying plane $\operatorname{span}\{T, B\}$ *of β. Since the orthogonal complement of that plane is RN, it follows that $B^* \parallel N$.*

Proof. Differentiating (8) and using the Frenet formulas (3) gives (10) after collecting terms; the two contributions $-\tau\kappa/\delta$ and $+\kappa\tau/\delta$ to the N-component cancel. Equivalently, writing $u = \tau T - \kappa B$ one has $u' = \tau' T + \tau\kappa N - \kappa' B - \kappa\tau N = \tau' T - \kappa' B$, which is (11).

Because $\tau^2 - \kappa^2 = \delta^2 > 0$ when $\omega \equiv 0$, we may introduce a hyperbolic parameter $\psi = \psi(s) > 0$ by

$$\frac{\kappa}{\delta} = \sinh\psi, \qquad \frac{\tau}{\delta} = -\cosh\psi. \tag{12}$$

Since $\langle T, T^*\rangle = \langle T, W\rangle = \tau/\delta = -\cosh\psi$, the function ψ is precisely the hyperbolic angle between T and T^*.

3.2 Mannheim pairs

Recall, [17], [23], that a pair of curves (β, γ) is called a Mannheim pair when the principal normal line of β coincides with the binormal line of γ at corresponding points, i.e. $B^* \parallel N$. The following theorem characterises such pairs among the timelike N-magnetic curves and isolates the degenerate case.

Theorem 1. *Let β be a timelike N-magnetic curve satisfying* (7) *and let γ be the direction curve of W. Then (β, γ) is a Mannheim pair if and only if*

$$\omega \equiv 0 \quad \text{and} \quad \kappa = \lambda\left(\tau^2 - \kappa^2\right) \quad \text{for a constant } \lambda \neq 0. \tag{13}$$

In that case $\gamma = \beta + \lambda N$ with λ constant,

$$\frac{ds^*}{ds} = \cosh\psi, \tag{14}$$

$$\kappa = \frac{\sinh^2\psi}{\lambda}, \qquad \tau = -\frac{\sinh\psi\cosh\psi}{\lambda}, \tag{15}$$

$$\kappa^* = \frac{|\psi'|}{\cosh\psi}, \qquad \tau^* = -\frac{\tanh\psi}{\lambda}. \tag{16}$$

Moreover γ is a Frenet curve if and only if ψ is non-constant, equivalently if and only if β is not a general helix; if β is a timelike circular helix then γ degenerates into its axis.

Proof. *Necessity.* Assume $B^* \parallel N$ and write

$$\gamma(s) = \beta(s) + \lambda(s)N(s), \tag{17}$$

where λ is a non-zero differentiable function. Differentiating (17) and using (9) and the Frenet formulas,

$$T^*\frac{ds^*}{ds} = (1+\lambda\kappa)T + \lambda' N - \lambda\tau B. \tag{18}$$

Since $T^* \perp B^*$ and $B^* \parallel N$, taking the inner product of (18) with N gives

$$\lambda' = 0, \tag{19}$$

so, by (19), the Mannheim distance is constant; this is a consequence of the geometric hypothesis and not an additional assumption. Furthermore $T^* = \pm W$ together with $\langle W, N\rangle = -\omega/\delta$ yields

$$\omega \equiv 0. \tag{20}$$

Substituting $T^* = \frac{1}{\delta}(-\tau T + \kappa B)$ into (18) and comparing the T- and B-components,

$$-\tau\frac{ds^*}{ds} = \delta(1+\lambda\kappa), \qquad \kappa\frac{ds^*}{ds} = -\lambda\tau\delta, \tag{21}$$

whence, eliminating ds^*/ds from (21),

$$\kappa = \lambda(\tau^2 - \kappa^2). \tag{22}$$

Sufficiency. Conversely, let (13) hold. By Lemma 2 the condition $B^* \parallel N$ is automatically satisfied. With the substitution (12), relation (22) reads $\kappa = \lambda\delta^2$, hence $\delta = \sinh\psi/\lambda$ and (15) follows. Then (18) becomes

$$\begin{aligned}\gamma' &= \cosh^2\psi\, T + \sinh\psi\cosh\psi\, B \\ &= \cosh\psi\, W.\end{aligned} \tag{23}$$

By (23), (14) holds and $T^* = W$. By (11),

$$\frac{dT^*}{ds^*} = \frac{\psi'}{\cosh\psi}\left(\sinh\psi\, T + \cosh\psi\, B\right), \tag{24}$$

so that, from (24), $\kappa^* = |\psi'|/\cosh\psi$ and $N^* = \operatorname{sgn}(\psi')\left(\sinh\psi\, T + \cosh\psi\, B\right)$. Using (5) one obtains $B^* = T^* \times N^* = -\operatorname{sgn}(\psi')N$, since $T \times B = -N$ and $\cosh^2\psi - \sinh^2\psi = 1$, whence

$$\tau^* = \left\langle \frac{dB^*}{ds^*}, N^* \right\rangle = -\frac{\tanh\psi}{\lambda}. \tag{25}$$

Finally $\kappa^* = 0$ if and only if $\psi' = 0$, i.e. if and only if τ/κ is constant. Combined with (22) this forces κ and τ to be constant, so that β is a timelike circular helix and γ is a straight line.

This result is significant because Mannheim curves can be used to examine the geometric properties of certain physical fields, for instance magnetic field lines; the relationship between the two curves of the pair then contributes to the understanding of the underlying physical process. Lemma 1 gives that interpretation a precise form: by Theorem 1 the Mannheim structure occurs exactly when the magnetic field lies in the rectifying plane of the trajectory, equivalently when the trajectory is simultaneously a $T-$magnetic and an N-magnetic curve.

Corollary 1. *(Genericity.) Condition* (13) *requires $\omega \equiv 0$, which by Lemma 1 is a condition of codimension one on the magnetic field. Hence any generic perturbation $\omega \mapsto \omega + \varepsilon_0$ destroys the Mannheim structure: timelike N-magnetic Mannheim pairs are structurally unstable, whereas the relation* (22) *is a condition on the trajectory alone and is insensitive to the field.*

Corollary 2. *Under* (13) *the partner curve satisfies*

$$\begin{gathered}-\frac{1}{\lambda} < \tau^* < 0, \quad \textit{so } |\tau^*| < \frac{1}{\lambda}, \\ \frac{\kappa^*}{\tau^*} = -\frac{\lambda\,|\psi'|}{\sinh\psi}.\end{gathered} \tag{26}$$

Proof. By (22) and (7) one has $\lambda = \kappa/(\tau^2 - \kappa^2) = \kappa/\delta^2 > 0$, so λ is automatically positive. The assertions in (26) are then immediate from (16) and (25), since $0 < \tanh\psi < 1$.

Corollary 3. *Under* (13)*, the curve β is a slant helix if and only if its Mannheim partner γ is a general helix, and the two invariants are reciprocal:*

$$\left|\frac{\kappa^2}{|\kappa^2 - \tau^2|^{3/2}}\left(\frac{\tau}{\kappa}\right)'\right| \cdot \left|\frac{\tau^*}{\kappa^*}\right| = 1. \tag{27}$$

Proof. By (12) and (15) we have $\tau/\kappa = -\coth\psi$, $(\tau/\kappa)' = \psi'/\sinh^2\psi$, $\kappa^2 - \tau^2 = -\delta^2$ and $\kappa^2 = \delta^2\sinh^2\psi$, so that the quantity inside the first modulus in (27) equals $\psi'/\delta = \lambda\psi'/\sinh\psi$. On the other hand (16) gives $|\tau^*|/\kappa^* = \sinh\psi/(\lambda\,|\psi'|)$, the moduli arising because κ^* is a curvature and therefore non-negative while $\tau^* < 0$ in the present orientation. The product is 1, and by Proposition 1, [20], the left-hand factor is constant precisely when β is a slant helix.

The perturbation statement also has a direct physical and dynamical interpretation. The associated-curve condition represents a special alignment between the magnetic trajectory and the direction of the magnetic field. For a timelike N-magnetic curve, a non-zero normal component ω destroys this alignment and therefore prevents the Mannheim correspondence from persisting under a generic perturbation of the field, as shown in Corollary 1. This structural instability concerns the geometric constraint defining the associated-curve pair and does not bear on the Lyapunov stability of the magnetic trajectory itself, [24].

From a variational and dynamical viewpoint, magnetic curves are solutions of the Lorentz force equation and may be interpreted as trajectories of charged particles in prescribed magnetic fields, [25], [26]. The results obtained here identify constrained subclasses of these solutions for which the moving Frenet frames preserve a specific correspondence. Such structures may provide simplified geometric models for charged-particle trajectories in Lorentzian settings and may form a basis for future investigations in relativistic dynamics and plasma-related problems. A quantitative physical application, however, would require a prescribed electromagnetic field, appropriate initial conditions, and the corresponding field equations.

3.3 Examples

Formulas (15) turn Theorem 1 into an explicit construction: choose a constant $\lambda > 0$ and any positive non-constant $\psi \in C^\infty(I)$, and set

$$\begin{aligned} \omega \equiv 0, \qquad \kappa = \frac{\sinh^2 \psi}{\lambda}, \\ \tau = -\frac{\sinh\psi\cosh\psi}{\lambda}. \end{aligned} \tag{28}$$

By the fundamental theorem of curves in E_1^3, (28) determines a timelike curve uniquely up to a Lorentzian motion, and by Theorem 1 every such curve is a timelike N-magnetic Mannheim curve with partner $\gamma = \beta + \lambda N$. By Corollary 3 the choice

$$\psi' = \frac{\sinh\psi}{\lambda c}, \qquad 0 < c < 1 \text{ constant}, \tag{29}$$

produces exactly the slant helices of this family, with invariant $1/c$; for $c = \sqrt{3}/2$ the resulting quadratures are elementary and yield the following closed-form example.

Example 1. Let $\lambda = 2/\sqrt{3}$ and let $\psi \in (0, \infty)$ be related to the arclength parameter by

$$\begin{aligned} s = \ln\tanh\frac{\psi}{2}, \\ \text{equivalently} \qquad \frac{d\psi}{ds} = \sinh\psi, \end{aligned} \tag{30}$$

so that s ranges over $(-\infty, 0)$. Since $\lambda c = 1$, (30) is the choice (29). Then the curve

$$\beta(\psi) = (\beta_1(\psi),\ \beta_2(\psi),\ \beta_3(\psi))\,, \tag{31}$$

with

$$\begin{aligned} \beta_1(\psi) &= 2\sinh\frac{\psi}{2} - 4\arctan e^{\psi/2}, \\ \beta_2(\psi) &= -\sqrt{3}\,\ln\sinh\psi, \\ \beta_3(\psi) &= 2\ln\tanh\frac{\psi}{4} + 2\cosh\frac{\psi}{2}, \end{aligned} \tag{32}$$

is a unit-speed timelike curve in E_1^3 with

$$\begin{aligned} \kappa = \frac{\sqrt{3}}{2}\sinh^2\psi, \\ \tau = -\frac{\sqrt{3}}{2}\sinh\psi\cosh\psi, \end{aligned} \tag{33}$$

so that $\kappa = \lambda(\tau^2 - \kappa^2)$ with $\lambda = 2/\sqrt{3}$ constant. Taking $\omega \equiv 0$, Theorem 1 shows that the curve (31), with curvature and torsion (33), is a timelike N-magnetic Mannheim curve whose partner is $\gamma = \beta + \lambda N$, at constant distance $\lambda = 2/\sqrt{3}$; the pair, computed from (32), is drawn in Fig. 1. By (16) and (30),

$$\kappa^* = \tanh\psi, \qquad \tau^* = -\frac{\sqrt{3}}{2}\tanh\psi, \tag{34}$$

so $\tau^*/\kappa^* = -\sqrt{3}/2$ is constant and γ is a timelike general helix, while β is a timelike slant helix with invariant $2/\sqrt{3}$, in agreement with Corollary 3. Since $0 < \tanh\psi < 1$, (34) also exhibits the bound $|\tau^*| < \sqrt{3}/2 = 1/\lambda$ of Corollary 2, equality being approached only as $\psi \to \infty$, that is, as $s \to 0^-$.

The next example shows that the non-helical hypothesis in Theorem 1 cannot be removed.

Example 2. Let $a, w > 0$ and consider the timelike circular helix

$$\beta(s) = \left(a\cos ws,\ a\sin ws,\ \sqrt{1 + a^2w^2}\,s\right), \tag{35}$$

which is unit speed and timelike, with

$$\begin{aligned} \kappa = aw^2, \qquad \tau = -w\sqrt{1 + a^2w^2}, \\ \tau^2 - \kappa^2 = w^2. \end{aligned} \tag{36}$$

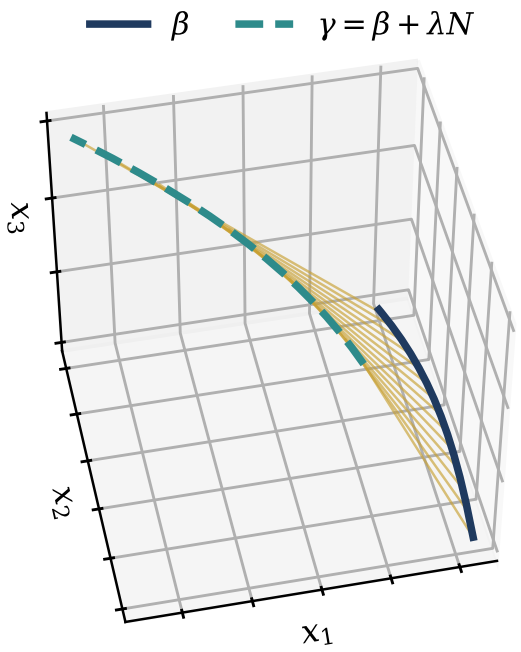


Figure 1: The timelike N-magnetic curve β of Example 1 (solid) and its Mannheim partner $\gamma = \beta + \lambda N$ (dashed), for $\psi \in [0.55, 1.75]$. The connecting segments join corresponding points and show the constant distance $\lambda = 2/\sqrt{3}$ guaranteed by Theorem 1.

Source: created by the authors.

Taking $\omega \equiv 0$, (36) shows that relation (22) holds for the helix (35) with $\lambda = \kappa/(\tau^2 - \kappa^2) = a$. Here ψ is constant, so $\kappa^* = 0$ by (16) and indeed

$$\gamma = \beta + aN = \left(0,\ 0,\ \sqrt{1 + a^2 w^2}\, s\right) \tag{37}$$

is the axis of the helix, a timelike straight line. The line (37) is exactly the degenerate case isolated in Theorem 1, and it is consistent with the Theorem 3 below.

3.4 The partner curve, straight lines and slant helices

Throughout this subsection, $\epsilon := \langle B^*, N\rangle \in \{\pm 1\}$ denotes the relative orientation of the two frames, which is well defined whenever $B^* \parallel N$. By the proof of Theorem 1 one has $\epsilon = -\operatorname{sgn}(\psi')$ in the situation of that theorem.

Theorem 2. *Let β be a timelike Mannheim curve and (β, γ) the timelike Mannheim curve pair. Then*

$$\frac{\tau}{\delta} = \tau^* \left(\frac{\epsilon}{\kappa} - \mu\right) \sinh\theta, \tag{38}$$

where θ is the hyperbolic angle between T and T^*.

Proof. Since β is the Mannheim partner curve of γ, we have

$$\beta(s^*) = \gamma(s^*) + \mu(s^*) B^*(s^*), \tag{39}$$

where μ is a non-zero differentiable function. Differentiating (39) with respect to s^*, using the Frenet frame of γ and $B^* = \epsilon N$,

$$\begin{aligned}\frac{d\beta}{ds}\frac{ds}{ds^*} = &-\frac{\tau}{\delta} T + \left(\epsilon\mu' - \frac{\omega}{\delta}\right) N \\ &+ \frac{\kappa}{\delta} B + \mu\tau^* N^*.\end{aligned} \tag{40}$$

Taking the inner product of (40) with N and using $N^* \perp N$ (Lemma 2) gives

$$\omega = \epsilon\, \delta \mu', \tag{41}$$

so that (40) reduces to

$$\frac{d\beta}{ds}\frac{ds}{ds^*} = -\frac{\tau}{\delta} T + \frac{\kappa}{\delta} B + \mu\tau^* N^*. \tag{42}$$

Since T and T^* are unit timelike vectors we may write

$$T = T^* \cosh\theta + N^* \sinh\theta. \tag{43}$$

Differentiating (43) with respect to s^* and using the Frenet formulas of γ,

$$\begin{aligned}\frac{dT}{ds}\frac{ds}{ds^*} = &(\theta' + \kappa^*) \sinh\theta\, T^* \\ &+ (\theta' + \kappa^*) \cosh\theta\, N^* \\ &- \tau^* \sinh\theta\, B^*.\end{aligned} \tag{44}$$

Comparing (44) with $dT/ds = \kappa N$ and using $N = \epsilon B^*$ we obtain

$$\theta' = -\kappa^* \qquad \text{and} \qquad \kappa \frac{ds}{ds^*} = -\epsilon\, \tau^* \sinh\theta. \tag{45}$$

Combining (42), (43), (44) and (45) gives (38).

Remark 1. By Theorem 1 the hypothesis of Theorem 2 forces $\omega \equiv 0$, hence $\mu' = 0$ by (41) and $\mu = -\epsilon\lambda = \lambda \operatorname{sgn}(\psi')$; with $\theta = -\operatorname{sgn}(\psi')\psi$ both sides of (38) equal $-\cosh\psi$, so (38) is consistent with the explicit data (15)–(16) of Theorem 1. Solving (41) for ω under (7) gives, in general,

$$\omega = \epsilon\, \mu' \sqrt{\frac{\tau^2 - \kappa^2}{1 + (\mu')^2}}. \tag{46}$$

On Mannheim pairs, (46) degenerates to the trivial statement $0 = 0$.

Theorem 3. *γ is a straight line if and only if*

$$\begin{aligned}&\left(\frac{\tau}{\delta}\right)' = -\frac{\omega\kappa}{\delta}, \qquad \left(\frac{\omega}{\delta}\right)' = 0, \\ &\left(\frac{\kappa}{\delta}\right)' = -\frac{\omega\tau}{\delta}.\end{aligned} \tag{47}$$

If moreover ω does not vanish, (47) *is equivalent to*

$$\frac{\tau' + \omega\kappa}{\tau} = \frac{\omega'}{\omega} = \frac{\kappa' + \omega\tau}{\kappa}. \tag{48}$$

Proof. The curve γ is a straight line precisely when $T^{*\prime} = 0$, i.e. when $W' = 0$. Setting the three components of (10) equal to zero gives (47). Since $\delta^2 = \tau^2 - \omega^2 - \kappa^2$ gives $\delta\delta' = \tau\tau' - \omega\omega' - \kappa\kappa'$, carrying out the differentiations in (47) yields

$$\begin{aligned} (\tau' + \omega\kappa)\delta^2 &= \tau(\tau\tau' - \omega\omega' - \kappa\kappa'), \\ \omega'\delta^2 &= \omega(\tau\tau' - \omega\omega' - \kappa\kappa'), \\ (\kappa' + \omega\tau)\delta^2 &= \kappa(\tau\tau' - \omega\omega' - \kappa\kappa'), \end{aligned} \tag{49}$$

and dividing the three equations in (49) by τ, ω and κ respectively gives (48).

Remark 2. For $\omega \equiv 0$ system (47) reduces to τ/δ and κ/δ constant, i.e. to $\psi' = 0$. This is exactly the degenerate case of Theorem 1, realised by the circular helix of Example 2. The quotient ω'/ω in (48) is of course meaningless in this case, which is why (47) rather than (48) is the correct general statement.

Corollary 4. *Let γ be a straight line and let $\omega \neq 0$ be constant. Then the N-magnetic curve β is a slant helix.*

Proof. By the second equation of (47), ω/δ is constant; since ω is a non-zero constant, δ is constant, and therefore so is $\tau^2 - \kappa^2 = \delta^2 + \omega^2$. From (48),

$$\begin{aligned} &\frac{\tau' + \omega\kappa}{\tau} = \frac{\kappa' + \omega\tau}{\kappa} \\ &\quad\Longrightarrow\ \kappa\tau' - \tau\kappa' = \omega(\tau^2 - \kappa^2), \end{aligned} \tag{50}$$

and solving (50) for ω gives

$$\omega = \frac{\kappa^2}{\tau^2 - \kappa^2}\left(\frac{\tau}{\kappa}\right)'. \tag{51}$$

Since ω and $\tau^2 - \kappa^2$ are constant, (51) shows that so is

$$\frac{\kappa^2}{|\kappa^2 - \tau^2|^{3/2}}\left(\frac{\tau}{\kappa}\right)', \tag{52}$$

and β is a slant helix by Proposition 1, [20].

Remark 3. Under (7) one always has $\tau^2 > \kappa^2$, so the quantity $\kappa^2 - \tau^2$ appearing in Proposition 1 is negative and the absolute value in (52) is essential.

3.5 Bertrand pairs and involutes

Recall, [10], that (β, γ) is a Bertrand pair when the principal normal lines of the two curves coincide at corresponding points, i.e. $N^* \parallel N$.

Theorem 4. *There is no timelike N-magnetic curve β such that (β, γ) is a Bertrand curve pair.*

Proof. Assume that (β, γ) is a Bertrand pair and write

$$\gamma(s) = \beta(s) + \xi(s)N(s), \tag{53}$$

with ξ a non-zero differentiable function. Differentiating (53),

$$T^*\frac{ds^*}{ds} = (1 + \xi\kappa)T + \xi' N - \xi\tau B. \tag{54}$$

Since $N^* \parallel N$ and $T^* \perp N^*$, we have $\langle T^*, N\rangle = 0$; taking the inner product of (54) with N therefore gives $\xi' = 0$. On the other hand $T^* = \pm W$ and $\langle W, N\rangle = -\omega/\delta$ give

$$\omega \equiv 0. \tag{55}$$

Note that (55) is a condition on the magnetic field, not yet a contradiction: by Lemma 1 it merely states that V lies in the rectifying plane of β. The obstruction appears one order later. Indeed, by Lemma 2, (55) places T^*, $T^{*\prime}$ and hence N^* inside $\mathrm{span}\{T, B\}$, whereas the Bertrand hypothesis requires $N^* \parallel N$ and $N \perp \mathrm{span}\{T, B\}$. This is impossible unless $T^{*\prime} = 0$, in which case γ is a straight line and N^* is not defined. Hence no such pair exists.

Remark 4. Theorems 1 and 4 rest on the same condition $\omega \equiv 0$, see (20) and (55), which by Lemma 2 confines the entire Frenet frame of γ to the rectifying plane of β. This is precisely what the Mannheim condition $B^* \parallel N$ requires and precisely what the Bertrand condition $N^* \parallel N$ forbids. The two theorems are therefore two readings of a single computation, which also explains why the Mannheim case admits solutions while the Bertrand case does not. Section 4 shows that this obstruction is specific to the N-magnetic setting.

Theorem 5. *There is no timelike N-magnetic curve β such that γ is an involute of β.*

Proof. Assume that γ is an involute of β, [13], so that the tangent vectors of the two curves are orthogonal at corresponding points, $\langle T^*, T\rangle = 0$. Writing

$$\gamma(s) = \beta(s) + \vartheta(s)T(s) \tag{56}$$

with ϑ a non-zero differentiable function and differentiating (56),

$$T^*\frac{ds^*}{ds} = (1 + \vartheta')T + \vartheta\kappa N. \tag{57}$$

Taking the inner product of (57) with T and using $\langle T, T\rangle = -1$ gives

$$1 + \vartheta' = 0, \tag{58}$$

so that $\vartheta(s) = c - s$ for a constant c and $\gamma' = \vartheta\kappa N$. Hence $T^* \parallel N$. Comparing with $T^* = \pm\frac{1}{\delta}(-\tau T -$

$\omega N + \kappa B)$ forces $\tau = \kappa = 0$, which contradicts $\kappa > 0$.

Remark 5. The inner product in (57) has to be taken with T and not with N: the involute condition is $\langle T^*, T\rangle = 0$, and it is the timelike character of T that produces the sign in (58).

4 Associated curves of timelike B-magnetic curves

We now repeat the analysis of Section 3 for timelike B-magnetic curves. The computations run in parallel, but the conclusions do not: the obstruction that rules out Bertrand pairs in the N-magnetic setting disappears here, and Bertrand pairs do occur.

4.1 The magnetic field of a timelike B-magnetic curve

Let β_1 be a timelike curve in E_1^3 with arclength parameter s, curvature $\kappa > 0$ and Frenet frame fields $\{T,\ N,\ B\}$. Imposing $\nabla_T B = \phi(B) = V_1 \times B$ and using (5) determines the magnetic field up to one free function,

$$V_1 = -\tau T + \omega_1 B, \quad \phi(T) = V_1 \times T = \omega_1 N, \tag{59}$$

where $\omega_1 = \langle \phi(T), N\rangle$. The curve β_1 is then a B-magnetic trajectory of V_1.

The essential structural difference with Section 3 is already visible in (59): the field V_1 has *no* N-component, and therefore lies in the rectifying plane of β_1 for every choice of ω_1. The role played by ω in Section 3 is taken here by the difference $\omega_1 - \kappa$.

Lemma 3. *For a timelike B-magnetic curve the following are equivalent: (i) $\omega_1 = \kappa$; (ii) $\phi(T) = \nabla_T T$, that is, β_1 is simultaneously a $T-$magnetic and a B-magnetic trajectory of V_1; (iii) V_1 coincides with the N-magnetic field* (6) *with $\omega \equiv 0$, so that β_1 is at the same time a $T-$, an N- and a B-magnetic curve of the same field.*

Proof. By (59) we have $\phi(T) = \omega_1 N$, while $\nabla_T T = \kappa N$; this gives (i)$\Leftrightarrow$(ii). If $\omega_1 = \kappa$ then $V_1 = -\tau T + \kappa B$, which is exactly (6) with $\omega = 0$, and Lemma 1 applies.

The normalised magnetic field of β_1 is

$$W_1 = \frac{1}{\delta_1}(-\tau T + \omega_1 B), \qquad \delta_1 = \sqrt{\left|-\tau^2 + \omega_1^2\right|}. \tag{60}$$

Throughout this section we assume

$$\tau^2 - \omega_1^2 > 0, \qquad \omega_1 \neq 0, \tag{61}$$

so that W_1 is a unit timelike vector field and $\tau \neq 0$, where

$$\delta_1 = \sqrt{\tau^2 - \omega_1^2}. \tag{62}$$

(If $\omega_1 = 0$ then $W_1 = \mp T$ and the direction curve of W_1 is β_1 itself.) As before, let γ_1 be the direction curve of W_1, i.e. the curve whose unit tangent satisfies $T^* = \pm W_1$, with arclength parameter s^* and Frenet apparatus $\{T^*,\ N^*,\ B^*,\ \kappa^*,\ \tau^*\}$.

Lemma 4. *Along a timelike B-magnetic curve,*

$$W_1' = -\left(\frac{\tau}{\delta_1}\right)' T + \frac{\tau(\omega_1 - \kappa)}{\delta_1} N + \left(\frac{\omega_1}{\delta_1}\right)' B. \tag{63}$$

Consequently: (a) $\langle T^*, N\rangle = 0$ holds identically; (b) the N-component of W_1' vanishes if and only if $\omega_1 = \kappa$, in which case $N^* \in \operatorname{span}\{T, B\}$ and $B^* \parallel N$; (c) the $T-$ and B-components of W_1' vanish simultaneously if and only if τ/δ_1 and ω_1/δ_1 are constant, in which case $N^* \parallel N$ provided $\omega_1 \neq \kappa$.

Proof. Write $W_1 = AT + CB$ with $A = -\tau/\delta_1$ and $C = \omega_1/\delta_1$. By (3),

$$W_1' = A'T + (A\kappa + C\tau)\, N + C'B, \tag{64}$$

and, since $A\kappa + C\tau = \tau(\omega_1 - \kappa)/\delta_1$, (64) coincides with (63). Statement (a) is immediate from (60), and (b), (c) follow by inspecting (63), using $\tau \neq 0$ and the fact that $\operatorname{span}\{T, B\}^{\perp} = RN$.

Lemma 4(a) is the precise sense in which the B-magnetic case is less constrained than the N-magnetic one: the orthogonality $\langle T^*, N\rangle = 0$, which in Section 3 could only be achieved by imposing $\omega \equiv 0$, is automatic here.

4.2 Mannheim pairs, straight lines, Bertrand pairs and involutes

Theorem 6. *Let β_1 be a timelike B-magnetic curve satisfying* (61) *and let γ_1 be the direction curve of W_1. Then (β_1, γ_1) is a Mannheim pair if and only if*

$$\omega_1 = \kappa \quad \text{and} \quad \kappa = \lambda\left(\tau^2 - \kappa^2\right) \quad \text{for a constant } \lambda \neq 0. \tag{65}$$

In particular, if $\omega_1 \neq \kappa$ then β_1 is not a Mannheim curve; and if $\omega_1 = \kappa$ the situation reduces exactly to that of Theorem 1.

Proof. Write $\gamma_1 = \beta_1 + \lambda N$ with λ a non-zero differentiable function. Differentiating,

$$T^* \frac{ds^*}{ds} = (1 + \lambda\kappa)T + \lambda' N - \lambda\tau B. \tag{66}$$

By Lemma 4(a) the inner product of (66) with N vanishes identically, so $\lambda' = 0$ and the distance is automatically constant. The Mannheim condition $B^* \parallel N$ is equivalent to $N^* \perp N$, i.e. to the vanishing of the N-component of W_1', which by Lemma 4(b) holds if and only if $\omega_1 = \kappa$. Substituting $T^* = \frac{1}{\delta_1}(-\tau T + \omega_1 B)$ into (66) and comparing the $T-$ and B-components gives

$$\omega_1(1 + \lambda\kappa) = \lambda\tau^2. \tag{67}$$

Together with $\omega_1 = \kappa$, (67) is precisely $\kappa = \lambda(\tau^2 - \kappa^2)$. Conversely, if (65) holds then $V_1 = -\tau T + \kappa B$ and $\delta_1 = \delta$, so $W_1 = W$ and Theorem 1 gives the conclusion.

Theorem 7. *γ_1 is a straight line if and only if*

$$\omega_1 = \kappa \qquad \text{and} \qquad \frac{\tau}{\kappa} \text{ is constant,} \tag{68}$$

that is, if and only if β_1 is a general helix which is simultaneously a $T-$, N- and B-magnetic curve.

Proof. The curve γ_1 is a straight line precisely when $W_1' = 0$. By (63) and $\tau \neq 0$ this forces $\omega_1 = \kappa$, together with τ/δ_1 and ω_1/δ_1 constant. Since $\omega_1 = \kappa$, (62) gives $\delta_1 = \sqrt{\tau^2 - \kappa^2} = \delta$, so both τ/δ and κ/δ are constant; by (12) this means ψ is constant, i.e. τ/κ is constant. The converse is immediate. The last assertion follows from Lemma 3.

Remark 6. Theorem 7 is consistent with Theorem 3 and Remark 2: for $\omega \equiv 0$ the straight-line condition of Section 3 reduces to $\psi' = 0$, which is exactly (68).

The next theorem is the main point of this section. In contrast with Theorem 4, Bertrand pairs *do* exist in the B-magnetic setting.

Theorem 8. *Let β_1 be a timelike B-magnetic curve satisfying* (61)*. Then (β_1, γ_1) is a Bertrand pair if and only if $\omega_1 \neq \kappa$, the quotients τ/δ_1 and ω_1/δ_1 are constant, and*

$$\omega_1 = \xi\left(\tau^2 - \omega_1\kappa\right) \quad \text{for a constant } \xi \neq 0. \tag{69}$$

Equivalently, writing $-\tau/\delta_1 = \cosh\chi$ and $\omega_1/\delta_1 = \sinh\chi$ with $\chi \neq 0$ constant, the Bertrand B-magnetic curves are exactly those with

$$\tau = -\delta_1\cosh\chi, \qquad \omega_1 = \delta_1\sinh\chi, \qquad \kappa = \frac{\delta_1\cosh^2\chi}{\sinh\chi} - \frac{1}{\xi}, \tag{70}$$

where $\delta_1 > 0$ is an arbitrary function with $\kappa > 0$ and $\delta_1 \neq \sinh\chi/\xi$, this last condition being the parametric form of $\omega_1 \neq \kappa$, since

$$\omega_1 - \kappa = \frac{1}{\xi} - \frac{\delta_1}{\sinh\chi}; \tag{71}$$

the constant ξ is the distance between corresponding points.

Proof. Let $\gamma_1 = \beta_1 + \xi N$ with ξ a non-zero differentiable function. Differentiating,

$$T^*\frac{ds^*}{ds} = (1 + \xi\kappa)T + \xi' N - \xi\tau B, \tag{72}$$

and by Lemma 4(a) the N-component of the left-hand side vanishes, so $\xi' = 0$. The Bertrand condition $N^* \parallel N$ means that W_1' is proportional to N, which by Lemma 4(c) holds if and only if τ/δ_1 and ω_1/δ_1 are constant and $\omega_1 \neq \kappa$ (the latter guaranteeing $W_1' \neq 0$, i.e. that γ_1 is not a straight line). Substituting $T^* = \pm W_1$ into (72) and comparing the $T-$ and B-components gives

$$\frac{1 + \xi\kappa}{-\tau} = \frac{-\xi\tau}{\omega_1}. \tag{73}$$

Cross-multiplying (73) gives $\omega_1(1 + \xi\kappa) = \xi\tau^2$, which is (69). Conversely, if these conditions hold, then (β_1, γ_1) is a Bertrand pair with constant distance ξ. Finally, $\tau^2 - \omega_1^2 = \delta_1^2$ shows that $-\tau/\delta_1$ and ω_1/δ_1 may be written as $\cosh\chi$ and $\sinh\chi$; solving (69) for κ then gives (70).

Example 3. Take $\sinh\chi = 1$ (so $\cosh\chi = \sqrt{2}$), $\xi = 1$ and $\delta_1(s) = \dfrac{6}{5} + \dfrac{s}{10}$. Then (70) gives

$$\kappa = \frac{7}{5} + \frac{s}{5}, \qquad \tau = -\sqrt{2}\left(\frac{6}{5} + \frac{s}{10}\right), \qquad \omega_1 = \frac{6}{5} + \frac{s}{10}, \tag{74}$$

so that $\tau^2 - \omega_1^2 = \delta_1^2 > 0$ and, by (71), $\omega_1 - \kappa = -\frac{1}{5} - \frac{s}{10} \neq 0$ for $s \geq 0$. Condition (69) is satisfied identically, since $\tau^2 - \omega_1\kappa = 2\delta_1^2 - \delta_1(2\delta_1 - 1) = \delta_1 = \omega_1$. Integrating the Frenet system with the curvature and torsion in (74) and $\beta_1(0) = 0$, $T(0) = (0,0,1)$, $N(0) = (1,0,0)$, $B(0) = (0,1,0)$ produces the timelike B-magnetic curve of Fig. 2 together with its Bertrand mate $\gamma_1 = \beta_1 + N$, at constant distance $\xi = 1$. Note that $\kappa - \omega_1 = \frac{1}{5} + \frac{s}{10} \geq \frac{1}{5}$ throughout, so by Lemma 4(c) the pair is non-degenerate and γ_1 is not a straight line.

Theorem 9. *There is no timelike B-magnetic curve β_1 such that γ_1 is an involute of β_1.*

Proof. Assume that $\gamma_1 = \beta_1 + \vartheta T$ is an involute of β_1, so that $\langle T^*, T\rangle = 0$. Differentiating,

$$T^*\frac{ds^*}{ds} = (1 + \vartheta')T + \vartheta\kappa N, \tag{75}$$

and taking the inner product of (75) with T gives $1 + \vartheta' = 0$, so that $\gamma_1' = \vartheta\kappa N$ and $T^* \parallel N$. But $T^* =$

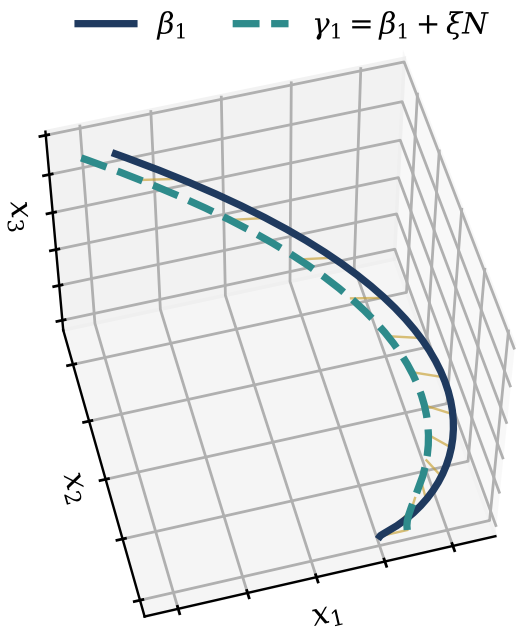


Figure 2: The timelike B-magnetic curve β_1 of Example 3 (solid) and its Bertrand mate $\gamma_1 = \beta_1 + \xi N$ (dashed), for $s \in [0, 6]$. The connecting segments join corresponding points and show the constant distance $\xi = 1$; the principal normal lines of the two curves coincide along these segments.

Source: created by the authors.

$\pm W_1 \in \text{span}\{T, B\}$ by (60), while $N \perp \text{span}\{T, B\}$. This is impossible because $\kappa > 0$ and $\vartheta \neq 0$.

Remark 7. The proof of Theorem 9 is shorter than that of Theorem 5 because W_1 never has an N-component, so the contradiction is reached at first order.

Remark 8. Comparing the two sections, the picture is the following. In both settings the requirement $\langle T^*, N \rangle = 0$ is what drives every statement. For an N-magnetic curve it is a genuine restriction, equivalent to $\omega \equiv 0$, and once imposed it forces the whole Frenet frame of γ into the rectifying plane of the trajectory; this makes Mannheim pairs possible and Bertrand pairs impossible (Theorems 1 and 4). For a B-magnetic curve the same orthogonality is automatic, and the relevant dichotomy is instead $\omega_1 = \kappa$ versus $\omega_1 \neq \kappa$: the first alternative reproduces the N-magnetic Mannheim theory verbatim (Theorem 6 and Lemma 3), while the second, which is the generic one, is exactly what allows Bertrand pairs to exist (Theorem 8). Involutes are excluded in both settings. In this sense Mannheim pairs occur precisely on the overlap where the trajectory is simultaneously a $T-$, N- and B-magnetic curve, and Bertrand pairs occur precisely off that overlap and only in the B-magnetic family.

5 Conclusion

In this study, we developed an associated-curve theory for timelike N-magnetic and B-magnetic curves in Minkowski 3-space. The results show that the two magnetic structures are governed by essentially different geometric compatibility conditions. For timelike N-magnetic curves, the Mannheim correspondence occurs precisely when the magnetic field lies in the rectifying plane of the trajectory and

$$\kappa = \lambda(\tau^2 - \kappa^2), \qquad \lambda \neq 0. \tag{76}$$

Under condition (76), the Mannheim structure is of codimension one with respect to the magnetic field and is therefore destroyed by a generic transverse perturbation. Bertrand pairs and involute-evolute pairs do not occur in the timelike N-magnetic setting.

For timelike B-magnetic curves, the orthogonality condition that drives the associated-curve construction is automatically satisfied. The relation $\omega_1 = \kappa$ reproduces the Mannheim structure, whereas the generic case $\omega_1 \neq \kappa$ permits Bertrand pairs under the necessary and sufficient conditions established in Theorem 8. Involute-evolute pairs are excluded in this setting as well. Consequently, the existence of Bertrand pairs provides a fundamental distinction between the timelike N-magnetic and B-magnetic families.

A further contribution is the reciprocal relation between the slant-helix invariant of a timelike N-magnetic trajectory and the general-helix invariant of its Mannheim partner. The explicit Mannheim and Bertrand examples demonstrate that the derived conditions are constructive and not merely necessary restrictions. These results show that the associated-curve structure reflects a specific compatibility between the geometry of the trajectory and the orientation of the magnetic field, rather than a direct extension of the corresponding spacelike theory.

The analysis is restricted to the non-degenerate regimes in which the normalised magnetic fields are timelike and the required Frenet apparatus is well defined. Moreover, the structural instability discussed here concerns the persistence of the associated-curve constraint and does not establish Lyapunov stability or instability of the magnetic trajectory. Future studies may investigate the spacelike and lightlike normalisation regimes, the behaviour of these curve pairs under prescribed electromagnetic fields, and the stability of nearby solutions of the Lorentz force equation. Extensions to higher dimensional Lorentzian manifolds and

applications to relativistic particle dynamics and plasma-related geometric models also constitute natural directions for further research.

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