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The Coset Incidence Graph of a Finite Group: Structure, Edge Ideals, and Spectrum

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Abstract: For a finite group G with subgroups H and K, the coset incidence graph $\Gamma(G; H, K)$ joins aH to bK when the cosets intersect; it is the Levi graph of a rank-two coset geometry. We show that its combinatorial, homological and spectral invariants are governed by products of H and K. Alternating products $HKH\cdots$ determine distances, diameter and girth. We classify when the graph is Cohen–Macaulay, unmixed or sequentially Cohen–Macaulay, compute the multiplicity of the edge ideal and, when $HK = KH$, all its Betti numbers, and determine the regularity under a packing condition in G. The biadjacency matrix is the Hecke operator of the double coset HK; its spectrum is described by principal angles between fixed subspaces and, for Gelfand pairs, by character averages over HK. The Heawood and Tutte–Coxeter graphs illustrate the results.

Key-Words: Bipartite graph, coset geometry, girth, edge ideal, Castelnuovo–Mumford regularity, Cohen–Macaulay graph, Hecke algebra, Gelfand pair, principal angles, generalized polygon.

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1 Introduction

Let G be a finite group and $H, K \leq G$.

Definition 1.1. The *coset incidence graph* $\Gamma(G; H, K)$ is the bipartite graph with colour classes $X = G/H$ and $Y = G/K$ (left cosets) in which $aH \in X$ and $bK \in Y$ are adjacent if and only if $aH \cap bK = \varnothing$.

Throughout we write

$$D = H\cap K, \qquad d_X = [H : D], \qquad d_Y = [K : D],$$

$$L = \langle H, K\rangle, \qquad U = H\cup K, \qquad S = HK\cup KH.$$

1.1 Context and related work

The graph $\Gamma(G; H, K)$ is classical. It is the incidence (Levi) graph of the rank-two *coset geometry* of Tits [37, 38], whose elements are the cosets of H and K, two of them incident when they meet; see Buekenhout [6] and Buekenhout–Cohen [7]. When (G, H, K) comes from a BN-pair of rank two with H, K the maximal parabolic subgroups, Γ is the incidence graph of a generalized polygon [38, 40]. Replacing left cosets by right cosets, $\Gamma(G; H, K)$ is the bipartite coset graph $\mathrm{Cos}(G, H, K)$ used by Goldschmidt [18] and by Giudici, Li and Praeger [17] to study locally s-arc-transitive graphs; every bipartite graph admitting a group of automorphisms that preserves the two parts and is transitive on edges is of this form ([17, Lemma 3.7]; Proposition 3.3 below). It should be distinguished from the *Sabidussi coset graphs* [29, 33], which have vertex set G/H only and model vertex-transitive graphs. The works on $\mathrm{Cos}(G, H, K)$ and on generalized polygons use the graph to construct or classify symmetric graphs and geometries. They do not study its edge ideal, its girth and diameter as functions of the products $HKH\cdots$ for arbitrary pairs (H, K), or its spectrum through principal angles, which are the subjects of this paper. For rank-two BN-pairs, the behaviour of these products is Tits' construction of generalized polygons, which we recover after Corollary 4.6.

The Hecke algebras $\mathrm{End}_G(\mathbb{C}[G/H])$ and their bases indexed by double cosets are classical [12]. For the parabolic subgroups of a group with a BN-pair, they are the Iwahori–Hecke algebras. Gelfand pairs and their spherical functions are treated in [9, 14]. When (G, H) is a Gelfand pair, the orbitals of G on G/H form a commutative association scheme [1, 4], and the operator BB^{T} studied in Section 6 lies in its Bose–Mesner algebra. The bipartite divisor graph of Iranmanesh and Praeger [25] is a different construction in the same spirit, in which a bipartite graph encodes arithmetic data of a group.

Edge ideals of bipartite graphs have been

studied intensively since Villarreal [43, 45]; Cohen–Macaulay, unmixed and sequentially Cohen–Macaulay bipartite graphs were characterized by Herzog and Hibi [22], Villarreal [44] and Van Tuyl and Villarreal [42], and regularity is bounded by matching-type invariants [20, 27, 48]. To our knowledge these tools have not been applied systematically to the family $\Gamma(G; H, K)$, and the point of this paper is that for this family the relevant combinatorial hypotheses can be decided by elementary computations with H and K.

1.2 Main results

The following theorems summarize the main results; precise statements are given in the body of the paper.

Theorem A (Distances and girth; Theorems 4.2, 4.4, Corollary 4.3)**.** *Let $\Pi_r(H,K) = HKH\cdots$ be the alternating product of $r+1$ factors starting with H. Then $d(eH, aH)$ and $d(eH, bK)$ are the least r of the appropriate parity with $a \in \Pi_r$, respectively $b \in \Pi_r$. If $G = \langle H, K\rangle$ then $\operatorname{diam}\Gamma = 1 + \max\{\ell(H,K), \ell(K,H)\}$, where $\ell(H,K)$ is the least r with $\Pi_r(H,K) = G$, and $\ell(H,K) + 2 \le |H\backslash G/H| + |H\backslash G/K|$, with equality exactly when H is transitive on every sphere about eH. The girth of Γ is determined by the cardinalities $|\Pi_r(H,K)|$, and it equals 4 if and only if $HK \cap KH \neq H \cup K$.*

Theorem B (Cohen–Macaulay type; Theorem 5.3)**.** *The graph $\Gamma(G; H, K)$ is Cohen–Macaulay if and only if $H = K$; it is unmixed if and only if $HK = KH$ and $|H| = |K|$; and it is sequentially Cohen–Macaulay (equivalently shellable, equivalently vertex decomposable) if and only if $H \le K$ or $K \le H$.*

Theorem C (Homological invariants; Proposition 5.2, Theorems 5.4, 5.6)**.** *The edge ideal has height $\min\{[G:H], [G:K]\}$, and $R/I(\Gamma)$ has multiplicity 1 if $|H| \neq |K|$ and $2^{[G:L]}$ if $|H| = |K|$. If $HK = KH$, the graded Betti numbers and the Hilbert series are given by explicit closed formulas, and $\operatorname{reg} R/I(\Gamma) = \operatorname{depth} R/I(\Gamma) = [G : HK]$. In general*

$$\nu_{\mathrm{ind}}(\Gamma) \;\le\; \frac{|G|}{|H \cup K|} \;\le\; \mathrm{m}(\Gamma),$$
$$\nu_{\mathrm{ind}}(\Gamma) \le \operatorname{reg} R/I(\Gamma) \le \mathrm{m}(\Gamma),$$

where ν_{ind} and m are the induced and minimum maximal matching numbers, both described as packing/covering numbers in G. If some $T \subseteq G$ with $|T| = |G|/|H \cup K|$ satisfies $t^{-1}t' \notin HK \cup KH$ for all $t \neq t'$ in T (for instance a subgroup of that order meeting $HK \cup KH$ trivially), then $\operatorname{reg} R/I(\Gamma) = |G|/|H \cup K|$.

Theorem D (Spectrum; Proposition 6.1, Theorems 6.2, 6.4, Corollary 6.3(a))**.** *The biadjacency matrix B of Γ is the Hecke operator of the double coset HK, and $BB^{\mathsf{T}} = \sum_{HxH} \mu(x)\, T_{HxH}$ with $\mu(x) = |HK \cap xHK|/|K|$. For every irreducible representation V of G, let $c_1 \ge \cdots \ge c_{\dim V^H}$ be the squared cosines of the principal angles between V^H and V^K. Then the eigenvalues of BB^{T} are the numbers $d_X d_Y\, c_j$, each with multiplicity $\dim V$. Moreover $\sum_j c_j$ is the average over HK of the character of V. Hence, for a Gelfand pair (G,H), the eigenvalue of BB^{T} on the constituent with character χ is*

$$\lambda_\chi = d_X d_Y \cdot \frac{1}{|HK|} \sum_{x \in HK} \chi(x).$$

Consequences include the spectrum in the 2-transitive case, which is that of a 2-design (Corollary 6.5, with a Fisher-type inequality $|K| \le |H|$ whenever $KH \neq G$), and a spectral-gap bound in terms of the least degree of a nontrivial constituent of $\mathbb{C}[G/H]$ (Proposition 6.6).

1.3 What is new

We state precisely what is known and what is new. The construction itself is classical (Section 1.1). The facts in Section 3, namely the incidence criterion, biregularity, the description of the connected components and of the complete bipartite case, are elementary and largely folklore. Proposition 3.3(b) is [17, Lemma 3.7], and Theorem 5.1 is an instance of the standard fact that a biregular bipartite graph has a matching saturating its smaller side. The dictionary in Corollary 3.7 is elementary, the Perron part of Corollary 6.3(b) is standard, and the bound reg $\le$ m in Lemma 2.8 is due to Woodroofe [48]. We include proofs because the later sections use them. Theorem 3.6 (complete bipartite components) and Theorem 5.4 (Betti numbers in the permutable case) are elementary consequences of the structure of $K_{p,q}$ and are recorded for completeness.

Several further results are close to known criteria, and we point this out explicitly. Theorem 4.2 and Corollary 4.6 are related to the classical group-theoretic criteria for coset geometries to be generalized polygons [26, 40]. Equality in Corollary 4.3 is local distance-transitivity in the sense of [13]. Theorem 5.3(ii) can also be derived from Ravindra's description of well-covered bipartite graphs [32]. Theorem 6.2 rests on the Wedderburn decomposition of $\mathbb{C}[G]$, and Proposition 6.6 uses the multiplicity argument of Sarnak–Xue and Gowers [19, 34]. Theorem 5.3(i) and (iii) follow quickly from the leaf criteria of Herzog–Hibi and Van Tuyl–Villarreal. To our knowledge, however, none of these statements has been formulated for the family $\Gamma(G; H, K)$. The same holds for the girth criterion in Theorem 4.4, Proposition 5.2, the regularity results of Theorem 5.6, the explicit Hecke expansion in

Proposition 6.1, the character-average formula of Theorem 6.4, and Corollary 6.5 as a statement about subgroup pairs. The new content of the paper is the uniform reduction of these criteria to conditions on (H, K), and the consequences drawn from it in Section 7. These include the values of $\operatorname{reg} R/I(\Gamma)$ for the Tutte–Coxeter graph (9), the subdivided cube (6) and the subdivision of K_n $(n-1)$, which are proved in the text, and for the Heawood graph (4), which was computed.

1.4 Organization

Section 2 collects background. Section 3 treats local and global structure, Section 4 distances, diameter and girth, Section 5 the edge ideal and Section 6 the spectrum. Section 7 works out examples coming from finite geometries and symmetric groups, and Section 8 concludes.

2 Preliminaries

All graphs are finite, simple and undirected. For subgroups $A, B \le G$ we write $AB = \{\, ab : a \in A,\ b \in B \,\}$, so that $|AB| = |A||B|/|A \cap B|$, and $A\backslash G/B$ for the set of (A, B)-double cosets. Recall that AB is a subgroup if and only if $AB = BA$, in which case $AB = \langle A, B\rangle$ [15] (Section 3.2).

2.1 Bipartite graphs

A bipartite graph Γ with colour classes X, Y is (d_X, d_Y)-*biregular* if every vertex of X has degree d_X and every vertex of Y has degree d_Y; then $|X|d_X = |Y|d_Y = |E(\Gamma)|$. A *matching* is a set of pairwise disjoint edges, and it is *induced* if no edge of Γ joins two of its edges. We write $\nu(\Gamma)$, $\nu_{\mathrm{ind}}(\Gamma)$ and $\mathrm{m}(\Gamma)$ for the largest size of a matching, the largest size of an induced matching, and the smallest size of a maximal (nonextendable) matching. A *vertex cover* is a set of vertices meeting every edge, and $\tau(\Gamma)$ is the least size of one. We use Hall's theorem [21] and König's theorem [28], see [30, 46].

Theorem 2.1 (Hall)**.** *A bipartite graph with classes X, Y has a matching saturating X if and only if $|N(A)| \ge |A|$ for every $A \subseteq X$.*

Theorem 2.2 (König)**.** *For every bipartite graph Γ, $\nu(\Gamma) = \tau(\Gamma)$.*

We also use the following elementary fact.

Lemma 2.3. *Let Δ be a graph and C a shortest cycle of Δ. Then $d_\Delta(x, y) = d_C(x, y)$ for all vertices x, y of C. In particular, if Δ is connected, bipartite and contains a cycle, then* $\operatorname{girth}\Delta \le 2\operatorname{diam}\Delta$.

Proof. Suppose $d_\Delta(x, y) < d_C(x, y)$ and let P be a shortest x–y path. Let A_1, A_2 be the two x–y arcs of C, with $|A_1| = d_C(x, y) \le |A_2|$. Then $P \ne A_1$. The union of two distinct x–y paths contains a cycle of length at most the sum of their lengths, so Δ has a cycle of length at most $|P| + |A_1| < |A_1| + |A_2| = |C|$, a contradiction. For the last statement, take x on a shortest cycle C and y antipodal to it on C. Then $\operatorname{diam}\Delta \ge d_\Delta(x, y) = |C|/2$. □

2.2 Edge ideals

Fix a field k and a graph Γ on $V = \{v_1, \dots, v_N\}$, and let $R = k[v_1, \dots, v_N]$. The *edge ideal* is $I(\Gamma) = (v_i v_j : \{v_i, v_j\} \in E(\Gamma))$. It is the Stanley–Reisner ideal of the *independence complex* $\mathrm{Ind}(\Gamma)$, whose faces are the independent sets of Γ. Standard references are [23, 31, 45].

Proposition 2.4 ([45, Ch. 6], [23, Ch. 9])**.** *The minimal primes of $I(\Gamma)$ are the ideals $P_C = (v : v \in C)$, with C running over the minimal vertex covers of Γ. Hence* $\operatorname{ht} I(\Gamma) = \tau(\Gamma)$ *and* $\dim R/I(\Gamma) = N - \tau(\Gamma)$.

The Hilbert series of a Stanley–Reisner ring (see [36], Chapter II) is

$$\mathrm{HS}\big(R/I(\Gamma); t\big) = \sum_{F \in \mathrm{Ind}(\Gamma)} \Big(\frac{t}{1-t}\Big)^{|F|}. \qquad (1)$$

Its pole at $t = 1$ has order $\dim R/I(\Gamma)$, and the *multiplicity* $e(R/I(\Gamma))$ is the number of independent sets of maximum size, that is, the number of minimum vertex covers. The graded Betti numbers are given by *Hochster's formula* [24], [31, Cor. 5.12]:

$$\beta_{i,j}\big(R/I(\Gamma)\big) = \sum_{W \subseteq V,\ |W| = j} \dim_k \widetilde{H}_{j-i-1}\big(\mathrm{Ind}(\Gamma[W]); k\big), \qquad (2)$$

where $\Gamma[W]$ is the induced subgraph. We set $\operatorname{reg} R/I = \max\{j - i : \beta_{i,j} \ne 0\}$ and $\operatorname{pd} R/I = \max\{i : \beta_{i,j} \ne 0\}$. By the Auslander–Buchsbaum formula, $\operatorname{depth} R/I = N - \operatorname{pd} R/I$.

Disjoint unions. If $\Gamma = \Gamma_1 \sqcup \cdots \sqcup \Gamma_c$, then $R/I(\Gamma) \cong \bigotimes_k R_i/I(\Gamma_i)$ with the R_i in disjoint sets of variables. The tensor product of minimal graded free resolutions is a minimal graded free resolution of the tensor product, so the Betti polynomials $B_{R/I}(s, t) = \sum_{i,j} \beta_{i,j} s^i t^j$ and the Hilbert series are multiplicative, while reg, pd and depth are additive:

$$\begin{aligned} B_{R/I(\Gamma)} &= \prod_i B_{R_i/I(\Gamma_i)}, \\ \operatorname{reg} \frac{R}{I(\Gamma)} &= \sum_i \operatorname{reg} \frac{R_i}{I(\Gamma_i)}, \\ \operatorname{pd} \frac{R}{I(\Gamma)} &= \sum_i \operatorname{pd} \frac{R_i}{I(\Gamma_i)}. \end{aligned} \qquad (3)$$

Cohen–Macaulay type. A graph Γ is *Cohen–Macaulay* (CM), *sequentially Cohen–Macaulay* (SCM), *shellable* or *vertex decomposable* if $R/I(\Gamma)$, respectively $\mathrm{Ind}(\Gamma)$, has that property. It is *unmixed* (well-covered) if all minimal vertex covers have the same size. We use three criteria.

Theorem 2.5 (Herzog–Hibi [22])**.** *A bipartite graph without isolated vertices is CM if and only if its colour classes can be labelled $\{x_1,\dots,x_n\}$ and $\{y_1,\dots,y_n\}$ so that* (i) $x_iy_i \in E$ *for all* i*;* (ii) $x_iy_j \in E$ *implies* $i \le j$*; and* (iii) $x_iy_j, x_jy_k \in E$ *implies* $x_iy_k \in E$.

Theorem 2.6 (Villarreal [44, Thm. 1.1])**.** *A bipartite graph without isolated vertices is unmixed if and only if its colour classes can be labelled $\{x_1,\dots,x_g\}$ and $\{y_1,\dots,y_g\}$ so that* (a) $x_iy_i \in E$ *for all* i*, and* (b) *if* $x_iy_j, x_jy_k \in E$ *with* i,j,k *distinct, then* $x_iy_k \in E$.

Theorem 2.7 (Van Tuyl–Villarreal [42, Thm. 2.13, Lemma 3.9, Thm. 3.10])**.** *Chordal graphs are shellable, and a bipartite graph is shellable if and only if it is SCM. An SCM bipartite graph without isolated vertices has a vertex of degree one.*

Moreover, chordal graphs are vertex decomposable [47], and for bipartite graphs vertex decomposability is equivalent to being SCM [41].

Regularity bounds. For every graph Γ [20,27,48],

$$\nu_{\mathrm{ind}}(\Gamma) \le \operatorname{reg} R/I(\Gamma) \le \operatorname{cochord}(\Gamma) \le \nu(\Gamma), \quad (4)$$

where $\operatorname{cochord}(\Gamma)$ is the least number of subgraphs with chordal complements whose edge sets cover $E(\Gamma)$ [48, Thm. 1].

Lemma 2.8 (Woodroofe [48])**.** *For every graph* Γ*,* $\operatorname{cochord}(\Gamma) \le \mathrm{m}(\Gamma)$. *Hence* $\operatorname{reg} R/I(\Gamma) \le \mathrm{m}(\Gamma)$.

Woodroofe notes this bound as a corollary of his results; we include the short proof for completeness.

Proof. Let F be a maximal matching. For $f = \{u,v\} \in F$ let Δ_f be the subgraph formed by all edges meeting u or v. By maximality the Δ_f cover $E(\Gamma)$. In Δ_f every edge contains u or v, so $W = V(\Delta_f)\setminus\{u,v\}$ is independent in Δ_f and is therefore a clique in the complement $\overline{\Delta_f}$. Since uv is an edge of Δ_f, all neighbours of u in $\overline{\Delta_f}$ lie in W, so u is simplicial; after deleting u the same holds for v; what remains is the clique W. Thus $\overline{\Delta_f}$ has a perfect elimination ordering and is chordal. □

2.3 Permutation modules, Hecke operators and Gelfand pairs

For $A \le G$ the permutation module $\mathbb{C}[G/A]$ has basis G/A with G acting by $g\cdot aA = (ga)A$, and we give it the inner product for which G/A is orthonormal. For $A,B \le G$ let $\mathbb{C}(A\backslash G/B)$ be the space of functions $f\colon G\to\mathbb{C}$ that are constant on (A,B)-double cosets. Each such f defines

$$T_f\colon \mathbb{C}[G/B]\to\mathbb{C}[G/A],\qquad T_f(bB) = \sum_{aA\in G/A} f(a^{-1}b)\,aA, \quad (5)$$

which is well defined by the bi-invariance of f. A direct computation shows that T_f is a G-morphism and that, for $f\in\mathbb{C}(A\backslash G/B)$ and $f'\in\mathbb{C}(B\backslash G/C)$,

$$T_fT_{f'} = T_{f*_Bf'},\qquad (f*_B f')(x) = \frac{1}{|B|}\sum_{y\in G} f(y)f'(y^{-1}x). \quad (6)$$

The map $f\mapsto T_f$ is a linear isomorphism $\mathbb{C}(A\backslash G/B)\to \mathrm{Hom}_G(\mathbb{C}[G/B],\mathbb{C}[G/A])$ [12, §11D]. For $A=B=H$ this identifies the *Hecke algebra* $\mathcal{H}(G,H)=\mathrm{End}_G(\mathbb{C}[G/H])$ with the convolution algebra $\mathbb{C}(H\backslash G/H)$, which has basis the operators T_{HxH} attached to the indicator functions of the double cosets. The pair (G,H) is a *Gelfand pair* if $\mathcal{H}(G,H)$ is commutative, equivalently if $\mathbb{C}[G/H]$ is multiplicity-free [9, 14]. Every 2-transitive action gives a Gelfand pair, with $\mathbb{C}[G/H]=\mathbf{1}\oplus V$ and V irreducible.

Let $\mathrm{Irr}(G)$ be the set of irreducible characters. For $\chi\in\mathrm{Irr}(G)$ let V_χ be a unitary representation affording χ. For $A\le G$ put $e_A = |A|^{-1}\sum_{a\in A} a \in \mathbb{C}[G]$; then $\rho_\chi(e_A)=P_A$ is the orthogonal projection of V_χ onto the fixed space V_χ^A. By Frobenius reciprocity, $m_\chi(A) := \dim V_\chi^A$ is the multiplicity of χ in $\mathbb{C}[G/A]$ [12, 35]. If (G,H) is a Gelfand pair and $m_\chi(H)=1$, with $V_\chi^H=\mathbb{C}u$ and $\|u\|=1$, the *spherical function* of χ is $\omega_\chi(g)=\langle gu,u\rangle = |H|^{-1}\sum_{h\in H}\chi(gh)$ [9, Ch. 4]. Finally, for subspaces P,Q of a Hilbert space, the *principal angles* $0\le\theta_1\le\cdots\le\theta_{\dim P}\le\pi/2$ are defined by the condition that $\cos\theta_j$ are the singular values of the restriction to P of the orthogonal projection onto Q, padded with zeros [3].

3 Local and global structure

Lemma 3.1. *For $a,b\in G$ we have $aH\cap bK\neq\varnothing \iff a^{-1}b\in HK \iff b^{-1}a\in KH$.*

Proof. A common element is an equation $ah=bk$ with $h\in H$, $k\in K$, and such h,k exist if and only if $a^{-1}b = hk^{-1}\in HK$. Inverting gives $b^{-1}a\in KH$. □

Lemma 3.2 (Biregularity)**.** *The graph $\Gamma(G;H,K)$ is (d_X,d_Y)-biregular, $|X|=[G:H]$, $|Y|=[G:K]$, and $|E(\Gamma)|=[G:D]$.*

Proof. The neighbours of aH are the cosets ahK, $h \in H$, and $ah_1K = ah_2K$ if and only if $h_1^{-1}h_2 \in H \cap K = D$. Hence aH has $[H : D] = d_X$ neighbours. The claim for Y is symmetric. Finally $|E| = [G : H]\,[H : D] = [G : D]$. □

Proposition 3.3. (a) *G acts on $\Gamma = \Gamma(G; H, K)$ by $g \cdot aH = gaH$, $g \cdot bK = gbK$. The action preserves X and Y, is transitive on X, on Y and on $E(\Gamma)$, and $gD \mapsto \{gH, gK\}$ is a G-equivariant bijection $G/D \to E(\Gamma)$. The kernel of the action is* $\mathrm{core}_G(H) \cap \mathrm{core}_G(K)$.

(b) ([17, Lemma 3.7]) *Conversely, let Δ be a bipartite graph with parts X', Y' and no isolated vertices, and let $G \le \mathrm{Aut}(\Delta)$ preserve X' and Y' and act transitively on edges. Then for every edge $\{v, w\}$ with $v \in X'$ we have $\Delta \cong \Gamma(G; G_v, G_w)$ via $gG_v \mapsto gv$, $gG_w \mapsto gw$.*

(c) *If $N \trianglelefteq G$ and $N \le D$, then $\Gamma(G; H, K) \cong \Gamma(G/N; H/N, K/N)$.*

Proof. (a) Adjacency is preserved because $g(aH \cap bK) = gaH \cap gbK$. If $aH \cap bK \ni g$, then $aH = gH$ and $bK = gK$, so every edge has the form $\{gH, gK\}$, and $\{gH, gK\} = \{g'H, g'K\}$ if and only if $g^{-1}g' \in D$. The element g fixes every vertex if and only if $g \in \bigcap_a aHa^{-1} \cap \bigcap_b bKb^{-1}$.

(b) Since there are no isolated vertices, edge-transitivity implies that G is transitive on X' and on Y', so both maps are bijections by the orbit–stabilizer theorem. If $x \in gG_v \cap g'G_w$, then $gv = xv$ and $g'w = xw$, so $\{gv, g'w\} = x\{v, w\}$ is an edge. Conversely, if $\{gv, g'w\}$ is an edge, then it equals $x\{v, w\}$ for some $x \in G$, and $x \in gG_v \cap g'G_w$.

(c) The cosets aH and $(aN)(H/N)$ correspond bijectively, and the correspondence preserves nonempty intersections. □

Theorem 3.4 (Connected components)**.** *The component of $\Gamma(G; H, K)$ containing aH consists of the cosets $a'H$ and bK with $a', b \in aL$. Hence Γ has $[G : L]$ components, each isomorphic to $\Gamma(L; H, K)$, and Γ is connected if and only if $G = \langle H, K\rangle$.*

Proof. If $aH \sim bK$, then $a^{-1}b \in HK \subseteq L$ by Lemma 3.1, so every vertex reachable from aH is a coset contained in aL. Conversely, by Proposition 3.3(a) we may take $a = e$. Let $w = s_1 \cdots s_r \in L$ with $s_i \in H \cup K$ and put $p_j = s_1 \cdots s_j$, $p_0 = e$. Then $p_jH \sim p_jK$ for all j. Moreover $p_{j-1}H = p_jH$ if $s_j \in H$, and $p_{j-1}K = p_jK$ if $s_j \in K$. Following these equalities and adjacencies gives a walk from eH to wH and to wK. The cosets inside L span a copy of $\Gamma(L; H, K)$, because the incidence criterion of Lemma 3.1 only involves elements of L. □

Theorem 3.5 (Complete bipartite case)**.** *$\Gamma(G; H, K) \cong K_{[G:H],[G:K]}$ if and only if $G = HK$, equivalently $|H||K| = |G||D|$.*

Proof. By Lemma 3.1, Γ is complete bipartite if and only if $a^{-1}b \in HK$ for all a, b, that is, if and only if $HK = G$. □

Theorem 3.6 (Permutable case)**.** *The following are equivalent:* (i) *$HK = KH$;* (ii) *HK is a subgroup;* (iii) *$HK = L$;* (iv) *every component of $\Gamma(G; H, K)$ is complete bipartite. When they hold, Γ is the disjoint union of $[G : HK]$ copies of K_{d_X,d_Y}.*

Proof. (i)⇔(ii)⇔(iii) is classical [15] (Section 3.2). By Theorem 3.4, each component is $\Gamma(L; H, K)$, and by Theorem 3.5 applied in L this is complete bipartite if and only if $L = HK$. In that case the component has $[L : H] = |K|/|D| = d_Y$ vertices in X and $[L : K] = d_X$ vertices in Y, and there are $[G : HK]$ components. □

The following dictionary collects the cases in which the shape of Γ is decided by an elementary condition on (H, K); part (viii) is proved in Theorem 4.4. Figure 1 (Appendix) shows two small cases.

Corollary 3.7. (i) *Γ is connected $\iff$ $G = \langle H, K\rangle$.*

(ii) *Γ is complete bipartite $\iff$ $G = HK$.*

(iii) *Γ is a disjoint union of complete bipartite graphs $\iff$ $HK = KH$.*

(iv) *Γ is regular $\iff$ $|H| = |K|$.*

(v) *Γ is a perfect matching $\iff$ $H = K$.*

(vi) *Γ is a forest $\iff$ $H \le K$ or $K \le H$. If $H \le K$, then Γ is the disjoint union of $[G : K]$ stars $K_{1,[K:H]}$.*

(vii) *Γ is a cycle $\iff$ $G = \langle H, K\rangle$ and $d_X = d_Y = 2$, and then $\Gamma \cong C_{2[G:H]}$.*

(viii) *Γ has girth at least 6, or is a forest (the coset geometry is a partial linear space) $\iff$ $HK \cap KH = H \cup K$.*

Proof. Parts (i)–(iii) were proved above. For (iv) and (v), note that $d_X = d_Y$ if and only if $|H| = |K|$, and $d_X = d_Y = 1$ if and only if $H = K$. For (vi), if $d_X, d_Y \ge 2$ then every vertex has degree at least two, so Γ contains a cycle. If $d_X = 1$, that is $H \le K$, every vertex of X is a leaf, and each bK is the centre of a star whose leaves are the $[K : H]$ cosets $aH \subseteq bK$. Part (vii) holds because a connected 2-regular graph is a cycle. □

4 Distances, diameter and girth

Definition 4.1. For $r \ge 0$ let $F_r = H$ if r is even and $F_r = K$ if r is odd, and put

$$\Pi_r = \Pi_r(H, K) = F_0F_1 \cdots F_r = \underbrace{HKHK\cdots}_{r+1 \text{ factors}},$$

and $\Pi_{-1} = \Pi_{-2} = \varnothing$. Let $\ell(H, K)$ be the least $r \ge 0$ with $\Pi_r(H, K) = L$. Define $\Pi_r(K, H)$ and $\ell(K, H)$ by exchanging H and K.

Since $e \in H \cap K$, we have $\Pi_{r-1} \subseteq \Pi_r$ and $\Pi_{r-2} \subseteq \Pi_r$. Moreover Π_r is a union of left cosets of F_r, and every element of L lies in some Π_r, so $\ell(H,K)$ is well defined. The sets Π_r measure distances in Γ.

Theorem 4.2. *Let $\Gamma = \Gamma(G; H, K)$ and $r \geq 0$. Write $\mathcal{S}_r(v)$ for the set of vertices at distance exactly r from v.*

(a) *For $x \in G$, the coset xF_r is at distance at most r from eH if and only if $x \in \Pi_r$. Hence*

$$d(eH, aH) = \min\{r \text{ even} : a \in \Pi_r\},$$
$$d(eH, bK) = \min\{r \text{ odd} : b \in \Pi_r\},$$

and $d(aH, a'H) = d(eH, a^{-1}a'H)$, $d(aH, bK) = d(eH, a^{-1}bK)$.

(b) *$\mathcal{S}_r(eH) = \{\, xF_r : x \in \Pi_r \setminus \Pi_{r-2} \,\}$ and $|\mathcal{S}_r(eH)| = |\Pi_r \setminus \Pi_{r-2}|/|F_r|$.*

(c) *If $G = \langle H, K\rangle$, then $\operatorname{ecc}(eH) = 1 + \ell(H,K)$, $\operatorname{ecc}(eK) = 1 + \ell(K,H)$, and*

$$\operatorname{diam}\Gamma = 1 + \max\{\ell(H,K), \ell(K,H)\}.$$

Proof. (a) We argue by induction on r. For $r = 0$, $xH = eH$ if and only if $x \in H = \Pi_0$. Let $r \geq 0$. A vertex xF_{r+1} lies at distance at most $r+1$ from eH if and only if it is adjacent to a vertex yF_r at distance at most r. (When $r+1$ is even and $xF_{r+1} = eH$, the vertex eK is such a neighbour, since $r \geq 1$.) By induction this means $y \in \Pi_r$. By Lemma 3.1, $yF_r \sim xF_{r+1}$ if and only if $y^{-1}x \in F_rF_{r+1}$. (For $F_r = K$ we use $yK \sim xH \iff x^{-1}y \in HK \iff y^{-1}x \in KH$.) Hence the condition is $x \in \Pi_r F_r F_{r+1} = \Pi_{r+1}$, because $\Pi_r F_r = \Pi_r$. Since Γ is bipartite, the distance from eH to an F_r-coset has the parity of r, which gives the displayed formulas. The last assertion follows from Proposition 3.3(a).

(b) This follows from (a), because Π_r and Π_{r-2} are unions of left cosets of $F_r = F_{r-2}$.

(c) Let $r \geq 1$. All vertices lie within distance r of eH if and only if all F_r-cosets lie within distance r and all F_{r-1}-cosets lie within distance r, hence by parity within distance $r-1$. By (a) this happens if and only if $\Pi_r = \Pi_{r-1} = G$, that is, if and only if $\Pi_{r-1} = G$. Thus $\operatorname{ecc}(eH) = 1 + \ell(H,K)$. By symmetry $\operatorname{ecc}(eK) = 1 + \ell(K,H)$, and G is transitive on X and on Y. □

Corollary 4.3 (Orbit bound)**.** *Suppose $G = \langle H, K\rangle$. Then*

$$\ell(H,K) + 2 \leq |H\backslash G/H| + |H\backslash G/K|.$$

Equality holds if and only if H acts transitively on every sphere $\mathcal{S}_r(eH)$.

Proof. H is the stabilizer of the vertex eH, so it preserves each sphere $\mathcal{S}_r(eH)$. Its orbits on X and Y correspond to $H\backslash G/H$ and $H\backslash G/K$ via $aH \mapsto HaH$ and $bK \mapsto HbK$. The $\operatorname{ecc}(eH) + 1$ nonempty spheres partition the vertex set into H-invariant sets, so their number is at most the number of H-orbits, with equality exactly when each sphere is a single orbit. Now apply Theorem 4.2(c). □

For the girth we compare $|\mathcal{S}_r(eH)|$ with the sphere sizes of the biregular tree. For $r \geq 1$ put

$$t_r = d_X\,(d_Y - 1)^{\lceil (r-1)/2 \rceil}\,(d_X - 1)^{\lfloor (r-1)/2 \rfloor},$$

the number of vertices at distance r from a vertex of degree d_X in the (d_X, d_Y)-biregular tree.

Theorem 4.4 (Girth)**.** *Assume $H \not\leq K$ and $K \not\leq H$, so that $d_X, d_Y \geq 2$ and Γ contains cycles.*

(a) *$|\Pi_r \setminus \Pi_{r-2}| \leq t_r\,|F_r|$ for every $r \geq 1$, with equality for $r = 1$.*

(b) *$\operatorname{girth}\Gamma = 2 \cdot \min\{\, r \geq 2 : |\Pi_r \setminus \Pi_{r-2}| < t_r\,|F_r| \,\}$.*

(c) *$\operatorname{girth}\Gamma = 4$ if and only if $HK \cap KH \neq H \cup K$.*

Proof. Write $\mathcal{S}_r = \mathcal{S}_r(eH)$. By Theorem 4.2(b), (a) and (b) are statements about $|\mathcal{S}_r|$. Call $w \in \mathcal{S}_{r-1}$ a *parent* of $v \in \mathcal{S}_r$ if $v \sim w$. Since Γ is bipartite, every neighbour of $v \in \mathcal{S}_r$ lies in $\mathcal{S}_{r-1} \cup \mathcal{S}_{r+1}$.

(a) Every vertex of $\mathcal{S}_r$ has at least one parent, and for $r \geq 2$ every vertex of $\mathcal{S}_{r-1}$ has a parent and hence at most $\deg - 1$ neighbours in $\mathcal{S}_r$. So $|\mathcal{S}_1| = d_X$ and $|\mathcal{S}_r| \leq (\deg_{r-1} - 1)|\mathcal{S}_{r-1}|$, where $\deg_{r-1}$ is d_Y or d_X according as $r-1$ is odd or even. By induction $|\mathcal{S}_r| \leq t_r$.

(b) We first show that, for $j \geq 1$, every vertex of $\mathcal{S}_1 \cup \cdots \cup \mathcal{S}_j$ has a unique parent if and only if $|\mathcal{S}_i| = t_i$ for $1 \leq i \leq j$. Suppose every vertex of $\mathcal{S}_1, \ldots, \mathcal{S}_{i-1}$ has a unique parent. Then every vertex of $\mathcal{S}_{i-1}$ has exactly $\deg_{i-1} - 1$ neighbours in $\mathcal{S}_i$. So the number of edges between $\mathcal{S}_{i-1}$ and $\mathcal{S}_i$ is $(\deg_{i-1} - 1)|\mathcal{S}_{i-1}|$, which equals t_i by induction, and this equals $|\mathcal{S}_i|$ if and only if every vertex of $\mathcal{S}_i$ has a unique parent.

Next we show that $\operatorname{girth}\Gamma > 2j$ if and only if every vertex of $\mathcal{S}_1 \cup \cdots \cup \mathcal{S}_j$ has a unique parent. If some $v \in \mathcal{S}_i$ with $i \leq j$ has two parents, then there are two distinct geodesics from eH to v, and their union contains a cycle of length at most $2i \leq 2j$. Conversely, suppose $\operatorname{girth}\Gamma = 2m \leq 2j$ and let C be a shortest cycle. Since G is transitive on X and C meets X, we may assume $eH \in C$. Let v be the vertex of C opposite to eH. By Lemma 2.3, $d(eH, v) = m$, and the two neighbours of v on C are at distance $m-1$ from eH. They are distinct, so $v \in \mathcal{S}_m$ has two parents, and $m \leq j$. Combining the two equivalences gives (b).

(c) Since G is transitive on edges, a 4-cycle may be assumed to contain the edge $eH \sim eK$; let it be $eH \sim eK \sim aH \sim bK \sim eH$ with $aH \neq eH$ and $bK \neq eK$. By Lemma 3.1, $aH \sim eK$ means $a \in KH$, so we may take $a = k \in K \setminus H$. Likewise

$bK \sim eH$ allows $b = h \in H \setminus K$. The remaining adjacency $kH \sim hK$ means $k^{-1}h \in HK$. Since k^{-1} also ranges over $K \setminus H$, a 4-cycle exists if and only if there are $k \in K \setminus H$ and $h \in H \setminus K$ with $kh \in HK$. Every element $x \in KH$ has the form $x = kh$, and all such expressions are $(kd)(d^{-1}h)$ with $d \in D$. Since $kd \in H \iff k \in H$ and $d^{-1}h \in K \iff h \in K$, we have $x = kh$ with $k \notin H$, $h \notin K$ if and only if $x \in KH \setminus (H \cup K)$. Therefore a 4-cycle exists if and only if $HK \cap KH \not\subseteq H \cup K$. As $H \cup K \subseteq HK \cap KH$ always, this is (c). □

Remark 4.5. The condition $HK \cap KH = H \cup K$ says that two distinct H-cosets meet at most one common K-coset, that is, the coset geometry is a partial linear space. Theorem 4.4(b) shows that the girth, like the diameter, depends only on the cardinalities of the alternating products.

Corollary 4.6 (Generalized polygons)**.** *Assume $G = \langle H, K\rangle$, $H \not\leq K$, $K \not\leq H$, and let $n \geq 2$. Then $\Gamma(G; H, K)$ is the incidence graph of a (possibly thin) generalized n-gon, that is, it has diameter n and girth $2n$, if and only if* $\max\{\ell(H,K), \ell(K,H)\} = n - 1$ *and* $|\Pi_r(H,K) \setminus \Pi_{r-2}(H,K)| = t_r|F_r|$ *for* $1 \leq r \leq n-1$.

Proof. The first condition is $\operatorname{diam}\Gamma = n$ (Theorem 4.2). The second is $\operatorname{girth}\Gamma \geq 2n$ (Theorem 4.4(b)), and $\operatorname{girth}\Gamma \leq 2\operatorname{diam}\Gamma$ by Lemma 2.3. □

For the parabolic subgroups of a rank-two BN-pair, the products Π_r are unions of Bruhat cells and the conditions of Corollary 4.6 hold; this is Tits' construction of generalized polygons [38]. The Heawood and Tutte–Coxeter graphs in Section 7 illustrate Theorems 4.2 and 4.4, and attain equality in Corollary 4.3; Figure 2 (Appendix) shows the distance layers of the Heawood graph.

5 Matchings, covers and the edge ideal

Let $R = k[X \sqcup Y]$ be the polynomial ring in the $N = [G : H] + [G : K]$ vertices of $\Gamma = \Gamma(G; H, K)$.

Theorem 5.1. $\nu(\Gamma) = \tau(\Gamma) = \min\{[G : H], [G : K]\}$.

Proof. The statement is symmetric in H and K, so assume $|H| \geq |K|$. Then $|X| \leq |Y|$ and $d_X \geq d_Y$. For $A \subseteq X$, counting the edges leaving A gives $d_X|A| \leq d_Y|N(A)|$, so $|N(A)| \geq |A|$. By Hall's theorem there is a matching saturating X, so $\nu = |X|$, and $\tau = \nu$ by König's theorem. □

Proposition 5.2 (Minimum covers and multiplicity)**.** *The following hold.*

(a) *The height and dimension are*

$$\operatorname{ht} I(\Gamma) = \min\{[G : H], [G : K]\},$$
$$\dim R/I(\Gamma) = \max\{[G : H], [G : K]\}.$$

(b) *If $|H| > |K|$, then X is the unique minimum vertex cover; if $|H| < |K|$, it is Y. If $|H| = |K|$, the minimum vertex covers are exactly the sets obtained by choosing one colour class in each connected component.*

(c) *Consequently the multiplicity of $R/I(\Gamma)$ is*

$$e\big(R/I(\Gamma)\big) = \begin{cases} 1, & |H| \neq |K|, \\ 2^{[G:L]}, & |H| = |K|. \end{cases}$$

Proof. (a) follows from Proposition 2.4 and Theorem 5.1. For (b), assume $|H| \geq |K|$. A vertex cover has the form $C = (X \setminus A) \cup B$ with $A \subseteq X$, $B \subseteq Y$ and $N(A) \subseteq B$, so $|C| \geq |X| - |A| + |N(A)|$. If $|H| > |K|$ then $d_X > d_Y$, and $d_X|A| \leq d_Y|N(A)|$ gives $|N(A)| > |A|$ whenever $A \neq \varnothing$; hence X is the only cover of size $|X| = \tau$. If $|H| = |K|$, a cover of size τ requires $B = N(A)$ and $|N(A)| = |A|$. Equality in $d|A| \leq d|N(A)|$ means that every edge at $N(A)$ ends in A, so $A \cup N(A)$ is a union of components; conversely, every such choice gives a minimum cover. By Theorem 3.4 there are $[G : L]$ components. Part (c) follows from (1): the multiplicity is the number of maximum independent sets, which are the complements of the minimum vertex covers. □

Theorem 5.3 (Cohen–Macaulay type)**.** *Let $\Gamma = \Gamma(G; H, K)$.*

(i) *Γ is Cohen–Macaulay if and only if $H = K$. In that case $I(\Gamma)$ is a complete intersection.*

(ii) *Γ is unmixed if and only if $HK = KH$ and $|H| = |K|$, that is, if and only if Γ is a disjoint union of copies of K_{d_X,d_X}.*

(iii) *The following are equivalent: Γ is sequentially Cohen–Macaulay; Γ is shellable; Γ is vertex decomposable; Γ is a forest; $H \leq K$ or $K \leq H$.*

Proof. Γ has no isolated vertices, since $d_X, d_Y \geq 1$.

(i) If $H = K$, then Γ is the perfect matching joining each $aH \in X$ to the same coset $aH \in Y$. Its edge ideal is generated by the regular sequence of the $[G : H]$ monomials $v_{aH}w_{aH}$ in disjoint variables, so it is a complete intersection and Γ is CM. Conversely, let Γ be CM and take a labelling as in Theorem 2.5. By (ii) of that theorem, x_n is adjacent only to y_n and y_1 is adjacent only to x_1. So $d_X = d_Y = 1$ by biregularity, that is, $H \leq K$ and $K \leq H$.

(ii) If $HK = KH$ and $|H| = |K|$, then Γ is a disjoint union of copies of $K_{d,d}$ (Theorem 3.6). The minimal vertex covers of $K_{d,d}$ are its two sides: a cover omitting a vertex must contain all of its neighbours, that is, the whole opposite side. Both sides have size d, so Γ is unmixed. Conversely, suppose Γ is unmixed. The sets X and Y are minimal vertex covers, since every vertex has an edge. Hence $|X| = |Y|$, so $|H| = |K|$ and $d_X = d_Y =:$

d. Take a labelling as in Theorem 2.6 and write $i \preceq j$ when $x_i y_j \in E$. Condition (a) makes $\preceq$ reflexive, and (b) makes it transitive (the cases where two of i, j, k coincide are trivial). So $\preceq$ is a preorder on $\{1, \dots, g\}$ in which every element has exactly d successors ($\deg x_i = d$) and exactly d predecessors ($\deg y_i = d$). Let C be a minimal equivalence class. The predecessors of $i \in C$ are exactly C, so $|C| = d$. The successors of i are C together with all elements of strictly larger classes, and there are $d = |C|$ of them, so no class lies strictly above C. Every class lies above a minimal class, so every class is both minimal and maximal. Hence $\preceq$ is an equivalence relation with classes of size d, and $x_i \sim y_j$ exactly when i, j lie in the same class. Thus Γ is a disjoint union of copies of $K_{d,d}$, and $HK = KH$ by Theorem 3.6.

(iii) If $H \le K$ or $K \le H$, then Γ is a forest (Corollary 3.7(vi)), hence chordal, hence vertex decomposable [47] and shellable (Theorem 2.7). Vertex decomposable implies shellable, which implies SCM. Conversely, if Γ is SCM, then by Theorem 2.7 it has a vertex of degree one, so $d_X = 1$ or $d_Y = 1$, that is, $H \le K$ or $K \le H$. □

Theorem 5.4 (Betti numbers in the permutable case)**.** *Suppose $HK = KH$, and put $c = [G : HK]$, $p = d_X$, $q = d_Y$. For $i \ge 1$ let*

$$b_i = \binom{p+q}{i+1} - \binom{p}{i+1} - \binom{q}{i+1}.$$

(a) *The Betti polynomial of $R/I(\Gamma)$ is*

$$\sum_{i,j} \beta_{i,j}\big(R/I(\Gamma)\big) s^i t^j = \Big(1 + \sum_{i=1}^{p+q-1} b_i\, s^i\, t^{i+1}\Big)^c.$$

(b) $\mathrm{HS}\big(R/I(\Gamma); t\big) = \Big((1-t)^{-p} + (1-t)^{-q} - 1\Big)^c$.

(c) $\operatorname{reg} R/I(\Gamma) = c$, $\operatorname{pd} R/I(\Gamma) = c(p+q-1)$ *and* $\operatorname{depth} R/I(\Gamma) = c$.

(d) *$I(\Gamma)$ has a linear resolution if and only if $c = 1$, that is, $G = HK$. Since $I(\Gamma)$ is generated in a single degree, the same holds for the property of being componentwise linear.*

Proof. By Theorem 3.6, Γ is the disjoint union of c copies of $K_{p,q}$, so by (3) it suffices to treat $\Gamma = K_{p,q}$ with parts P, Q of sizes p, q and $c = 1$. (By Theorem 3.6, P lies in Y and Q in X; the formulas are symmetric in p, q.) Let W be a set of vertices. If W lies in one part, then $\Gamma[W]$ has no edges and $\mathrm{Ind}(\Gamma[W])$ is a simplex, which is acyclic. If W meets both parts, then $\Gamma[W] = K_{a,b}$ with $a, b \ge 1$, and $\mathrm{Ind}(\Gamma[W])$ is the disjoint union of two simplices. Its only nonzero reduced homology is $\widetilde{H}_0 \cong k$. By Hochster's formula (2), $\beta_{i,j} = 0$ unless $j = i+1$, and $\beta_{i,i+1}$ is the number of $(i+1)$-subsets of $P \sqcup Q$ meeting both parts, which is b_i. This proves (a). Since $b_{p+q-1} = 1$ and $b_i = 0$ for $i \ge p+q$, we get $\operatorname{reg} = 1$ and $\operatorname{pd} = p+q-1$, and hence $\operatorname{depth} = (p+q) - (p+q-1) = 1$. The independent sets of $K_{p,q}$ are the subsets of P and the subsets of Q, and the empty set is counted twice, so (1) gives $\mathrm{HS} = \big(1 + \frac{t}{1-t}\big)^p + \big(1 + \frac{t}{1-t}\big)^q - 1$, which is (b). For general c, (3) gives

$$\operatorname{reg} \frac{R}{I(\Gamma)} = \sum_{i=1}^{c} 1 = c, \qquad \operatorname{depth} \frac{R}{I(\Gamma)} = \sum_{i=1}^{c} 1 = c,$$
$$\operatorname{pd} \frac{R}{I(\Gamma)} = \sum_{i=1}^{c} (p+q-1) = c(p+q-1),$$

which is (c). Finally, $I(\Gamma)$ is generated in degree 2, so it has a linear resolution if and only if $\operatorname{reg} R/I(\Gamma) = 1$, that is, $c = 1$. □

Remark 5.5. The linearity of the resolution of $I(K_{p,q}) = (x_1, \dots, x_p)(y_1, \dots, y_q)$ can also be deduced from the fact that a product of ideals generated by linear forms has a linear resolution [10]. A direct argument is also available: ordered lexicographically, the generators $x_i y_j$ have linear quotients, because the colon ideal of the earlier generators by $x_i y_j$ is $(x_1, \dots, x_{i-1}, y_1, \dots, y_{j-1})$ [23, Ch. 8]. By Fröberg's theorem [16], the linearity also follows from the fact that the complement $K_p \sqcup K_q$ is chordal. When $c \ge 2$, the complement contains an induced 4-cycle, which accounts for (d).

Outside the permutable case, the components of Γ are not complete bipartite and the homological invariants depend on the finer structure of (H, K). The following theorem translates the matching invariants that control the regularity into packing and covering problems in G. Recall that $U = H \cup K$ and $S = HK \cup KH$, and note that $|U| = |H| + |K| - |D|$ and $UU^{-1} = S$.

Theorem 5.6 (Regularity)**.** *For $T \subseteq G$ let $E_T = \{\, \{tH, tK\} : t \in T \,\}$, and write $T^{-1}T \cap Z = \{e\}$ to mean that $t^{-1}t' \notin Z$ for all distinct $t, t' \in T$.*

(a) *E_T is a matching of size $|T|$ if and only if $T^{-1}T \cap U = \{e\}$; an induced matching if and only if $T^{-1}T \cap S = \{e\}$; and a maximal matching if and only if moreover $TU = G$. Every matching arises in this way, so that*

$$\nu(\Gamma) = \max\{\, |T| : T^{-1}T \cap U = \{e\} \,\},$$
$$\nu_{\mathrm{ind}}(\Gamma) = \max\{\, |T| : T^{-1}T \cap S = \{e\} \,\},$$
$$\mathrm{m}(\Gamma) = \min\{\, |T| : T^{-1}T \cap U = \{e\},\ TU = G \,\}.$$

(b) *The following inequalities hold:*

$$\frac{|G|}{|S|} \;\le\; \nu_{\text{ind}}(\Gamma) \;\le\; \frac{|G|}{|U|} \;\le\; \mathrm{m}(\Gamma),$$
$$\frac{|G|}{|U|} = \frac{[G:D]}{d_X + d_Y - 1},$$

and $\nu_{\text{ind}}(\Gamma) \le [G:J]$ *for every subgroup* $J \le G$ *with* $J \subseteq S$.

(c) $[G:L] \le \nu_{\text{ind}}(\Gamma) \le \operatorname{reg} R/I(\Gamma) \le \mathrm{m}(\Gamma) \le \nu(\Gamma) = \min\{[G:H],[G:K]\}$.

(d) *If there is* $T \subseteq G$ *with* $|T| = |G|/|U|$ *and* $T^{-1}T \cap S = \{e\}$*, for instance a subgroup* $Q \le G$ *of order* $|G|/|U|$ *with* $Q \cap S = \{e\}$*, then*

$$\operatorname{reg} R/I(\Gamma) = \nu_{\text{ind}}(\Gamma) = \mathrm{m}(\Gamma) = \frac{|G|}{|H \cup K|}.$$

Proof. (a) By Proposition 3.3(a), every edge is $\{tH, tK\}$ for some t. Let $t \neq t'$ and $x = t^{-1}t'$. The edges $\{tH, tK\}$ and $\{t'H, t'K\}$ are disjoint if and only if $tH \neq t'H$ and $tK \neq t'K$, that is, $x \notin U$. In that case they are distinct, so $|E_T| = |T|$. They form an induced matching if, in addition, $tH \not\sim t'K$ and $t'H \not\sim tK$. By Lemma 3.1 this means $x \notin HK$ and $x^{-1} \notin HK$, that is, $x \notin S$. Note $U \subseteq S$. An edge $\{gH, gK\}$ meets $\{tH, tK\}$ if and only if $g \in tH \cup tK = tU$, so a matching E_T is maximal if and only if $TU = G$.

(b) Let $T^{-1}T \cap S = \{e\}$. If $tu = t'u'$ with $u, u' \in U$, then $t^{-1}t' = uu'^{-1} \in UU^{-1} = S$, so $t = t'$. Hence the translates tU are disjoint and $|T||U| \le |G|$. If $TU = G$, then $|T||U| \ge |G|$. If T is maximal with $T^{-1}T \cap S = \{e\}$, then $TS = G$: otherwise any $g \notin TS$ could be added to T, since $S = S^{-1}$. Hence $|T||S| \ge |G|$. If $J \le G$ and $J \subseteq S$, then two elements of T in one left coset gJ would satisfy $t^{-1}t' \in J$, so $|T| \le [G:J]$. Finally $|G|/|U| = [G:D]/(|U|/|D|) = [G:D]/(d_X + d_Y - 1)$.

(c) Choosing one edge in each of the $[G:L]$ components gives an induced matching, so $[G:L] \le \nu_{\text{ind}}$. The other inequalities are (4), Lemma 2.8 and Theorem 5.1.

(d) By (a), E_T is an induced matching. By (b), the translates tU are disjoint, and $|T||U| = |G|$ forces $TU = G$, so E_T is also a maximal matching. Hence $|T| \le \nu_{\text{ind}} \le \text{reg} \le \mathrm{m} \le |T|$. □

Remark 5.7. In graph-theoretic terms, the hypothesis of Theorem 5.6(d) says that Γ has an *efficient edge dominating set* [8], that is, an induced matching that is also a maximal matching. The theorem shows that the existence of such a set reduces to a packing condition on G with respect to $HK \cup KH$, and that the regularity is then $[G:D]/(d_X + d_Y - 1)$. The bounds in (b) and (c) can be strict. For the Heawood graph (Example 7.2) we have $\nu_{\text{ind}} = 3 < \text{reg} = 4 < \mathrm{m} = 5 < \nu = 7$, and $|G|/|U| = 4.2$ is not an integer.

6 The spectrum

Let B be the biadjacency matrix of $\Gamma = \Gamma(G; H, K)$, with rows indexed by G/H and columns indexed by G/K, and let $M = BB^{\mathsf{T}}$. The adjacency matrix is $A = \left(\begin{smallmatrix} 0 & B \\ B^{\mathsf{T}} & 0 \end{smallmatrix}\right)$, so the spectrum of Γ consists of the numbers $\pm\sqrt{\lambda}$, for λ a nonzero eigenvalue of M (with the multiplicity of λ), together with 0 of multiplicity $|X| + |Y| - 2\operatorname{rank} B$ [2, 5]. We write spectra as multisets, $\{\lambda_1^{(m_1)}, \lambda_2^{(m_2)}, \dots\}$, with multiplicities in parentheses.

6.1 The biadjacency matrix as a Hecke operator

Proposition 6.1. *In the notation of* (5)*,* $B = T_{\mathbf{1}_{HK}}$ *and* $B^{\mathsf{T}} = T_{\mathbf{1}_{KH}}$*, where* $\mathbf{1}_Z$ *is the indicator function of* Z*. Consequently* $M = T_\mu \in \mathcal{H}(G, H)$*, where*

$$\mu = \mathbf{1}_{HK} *_K \mathbf{1}_{KH},$$
$$\mu(x) = \frac{|HK \cap xHK|}{|K|} = M_{eH,xH} \qquad (x \in G).$$

Thus $\mu(x)$ *is the number of common neighbours of* eH *and* xH*. The function* μ *is supported on* HKH*,* $\mu(e) = d_X$*, and in the double-coset basis*

$$M = \sum_{HxH \subseteq HKH} \frac{|HK \cap xHK|}{|K|}\, T_{HxH}.$$

Proof. By Lemma 3.1, $B(bK) = \sum_{aH} \mathbf{1}_{HK}(a^{-1}b)\, aH$ and $B^{\mathsf{T}}(aH) = \sum_{bK} \mathbf{1}_{KH}(b^{-1}a)\, bK$. Since HK is left H-invariant and right K-invariant, $\mathbf{1}_{HK} \in \mathbb{C}(H\backslash G/K)$, and similarly $\mathbf{1}_{KH} \in \mathbb{C}(K\backslash G/H)$. By (6), $\mu(x) = |K|^{-1}\,|\{\, y \in HK : y^{-1}x \in KH \,\}|$. Moreover $y^{-1}x \in KH \iff x^{-1}y \in HK \iff y \in xHK$, so $\mu(x) = |HK \cap xHK|/|K|$. The set $HK \cap xHK$ is nonempty if and only if $x \in HK(HK)^{-1} = HKH$, and $\mu(e) = |HK|/|K| = d_X$. □

6.2 Principal angles

For $\chi \in \mathrm{Irr}(G)$ write $m_\chi = \dim V_\chi^H$ and $n_\chi = \dim V_\chi^K$, and let

$$1 \ge c_1(\chi) \ge c_2(\chi) \ge \cdots \ge c_{m_\chi}(\chi) \ge 0$$

be the eigenvalues of the positive semidefinite operator $P_H P_K P_H$ restricted to V_χ^H. Equivalently, $c_j(\chi) = \cos^2\theta_j(\chi)$, where the $\theta_j(\chi)$ are the principal angles between V_χ^H and V_χ^K.

Theorem 6.2. *The spectrum of* $M = BB^{\mathsf{T}}$ *is the multiset union*

$$\mathrm{Spec}(M) = \biguplus_{\chi \in \mathrm{Irr}(G)} \left\{ \left(d_X d_Y\, c_j(\chi)\right)^{(\chi(1))} : 1 \le j \le m_\chi \right\}.$$

Hence the nonzero eigenvalues of Γ *are the numbers* $\pm\sqrt{d_X d_Y}\cos\theta_j(\chi)$ *with* $\theta_j(\chi) < \pi/2$*, each contributing multiplicity* $\chi(1)$*, and*

$$\operatorname{rank} B = \sum_{\chi} \chi(1)\cdot \#\{\, j : c_j(\chi) > 0 \,\}.$$

Proof. Step 1. For $A \le G$, the map $\varphi_A\colon \mathbb{C}[G/A] \to \mathbb{C}[G]e_A$, $aA \mapsto ae_A$, is an isomorphism of left $\mathbb{C}[G]$-modules: it is well defined because $a'e_A = e_A$ for $a' \in A$, and both sides have dimension $[G:A]$.

Step 2. We claim that $\varphi_H B \varphi_K^{-1}(y) = d_Y\, y e_H$ for $y \in \mathbb{C}[G]e_K$. Indeed,

$$be_K e_H = \frac{1}{|K|}\sum_{k\in K}(bk)e_H = \frac{|D|}{|K|}\sum_{aH\subseteq bKH} ae_H,$$

because $bk_1H = bk_2H$ if and only if $k_1^{-1}k_2 \in D$. Moreover $aH \subseteq bKH$ if and only if $b^{-1}a \in KH$, if and only if $aH \sim bK$, by Lemma 3.1. Similarly $\varphi_K B^{\mathsf{T}} \varphi_H^{-1}(x) = d_X\, x e_K$. Hence, for $x \in \mathbb{C}[G]e_H$,

$$\begin{aligned}\varphi_H M \varphi_H^{-1}(x) &= d_X d_Y\, x\, e_K e_H \\ &= d_X d_Y\, x\,(e_H e_K e_H).\end{aligned}$$

Step 3. The Fourier transform $x \mapsto (\rho_\chi(x))_\chi$ is an algebra isomorphism $\mathbb{C}[G] \to \bigoplus_\chi \operatorname{End}(V_\chi)$ [35]. It maps $\mathbb{C}[G]e_H$ onto $\bigoplus_\chi \operatorname{End}(V_\chi)P_H$ and right multiplication by $e_He_Ke_H$ to right multiplication by $Q_\chi = P_HP_KP_H$ in each summand.

Step 4. Restriction to V_χ^H identifies $\operatorname{End}(V_\chi)P_H$ with $\operatorname{Hom}(V_\chi^H, V_\chi)$, and right multiplication by Q_χ becomes $T \mapsto T\circ Q_\chi|_{V_\chi^H}$. Let $u_1,\dots,u_{m_\chi}$ be an orthonormal eigenbasis of $Q_\chi|_{V_\chi^H}$ with eigenvalues $c_j(\chi)$. The rank-one maps $v \mapsto \langle v, u_j\rangle w$, for $w \in V_\chi$, span a $\chi(1)$-dimensional eigenspace with eigenvalue $c_j(\chi)$, and together these spaces exhaust $\operatorname{Hom}(V_\chi^H, V_\chi)$. Finally $Q_\chi|_{V^H} = (P_K|_{V^H})^*(P_K|_{V^H})$, so the $c_j(\chi)$ are the squared singular values of the projection $V^H \to V^K$, that is, the squared cosines of the principal angles. The statements about Γ and $\operatorname{rank} B = \operatorname{rank} M$ follow. □

Corollary 6.3. (a) (Trace formula.) *For every* $\chi \in \operatorname{Irr}(G)$*,*

$$\sum_{j=1}^{m_\chi} c_j(\chi) = \frac{1}{|HK|}\sum_{x\in HK}\chi(x).$$

(b) (Perron eigenvalue.) $c_j(\chi) = 1$ *if and only if the corresponding eigenvector lies in* $V_\chi^H \cap V_\chi^K = V_\chi^L$*. Hence* d_Xd_Y *is the largest eigenvalue of* M*, with multiplicity* $\sum_\chi \chi(1)\dim V_\chi^L = [G:L]$*, and the spectral radius of* Γ *is* $\sqrt{d_Xd_Y}$*, with multiplicity equal to the number of components.*

(c) (Zero eigenvalues.) $\#\{\, j : c_j(\chi) > 0\,\} \le \min\{m_\chi, n_\chi\}$*. In particular a constituent* χ *of* $\mathbb{C}[G/H]$ *with* $V_\chi^K = 0$ *contributes only the eigenvalue* 0*. Moreover*

$$\operatorname{rank} B \le \sum_{\chi\in\operatorname{Irr}(G)} \chi(1)\,\min\{m_\chi, n_\chi\}.$$

(d) (Multiplicities.) *Every eigenvalue of* M *has multiplicity equal to a sum of degrees of irreducible constituents of* $\mathbb{C}[G/H]$*.*

Proof. (a) Q_χ vanishes on $(V^H)^\perp$ and maps into V^H, so $\sum_j c_j = \operatorname{tr}_V(P_HP_KP_H) = \operatorname{tr}_V(P_KP_H) = \frac{1}{|H||K|}\sum_{h,k}\chi(kh)$. The multiplication map $K\times H \to KH$ is $|D|$-to-one, $|H||K| = |D||KH|$, and $\chi(kh) = \chi(hk)$; this gives the average over KH, which equals the average over HK.

(b) $\|P_Ku\| = \|u\|$ if and only if $u \in V^K$. The multiplicity formula is $\dim \mathbb{C}[G/L] = [G:L]$, by Frobenius reciprocity.

(c) and (d) are immediate from Theorem 6.2. □

Corollary 6.3(b) gives a second proof, via representation theory, of the count of components in Theorem 3.4.

6.3 Gelfand pairs

Theorem 6.4. *Let* (G,H) *be a Gelfand pair and* $\mathbb{C}[G/H] = \bigoplus_\chi V_\chi$ *its decomposition into pairwise nonisomorphic irreducibles. Then* M *acts on* V_χ *as the scalar*

$$\begin{aligned}\lambda_\chi &= d_Xd_Y\cdot\frac{1}{|HK|}\sum_{x\in HK}\chi(x)\\ &= d_Xd_Y\cdot\frac{1}{|K|}\sum_{k\in K}\omega_\chi(k)\\ &= \sum_{HxH\subseteq HKH}\mu(x)\,\frac{|HxH|}{|H|}\,\omega_\chi(x),\end{aligned}$$

where ω_χ *is the spherical function and* μ *is as in Proposition 6.1. The spectrum of* Γ *is* $\{\pm\sqrt{\lambda_\chi}^{(\chi(1))} : \lambda_\chi \ne 0\} \cup \{0^{(z)}\}$*, with* $z = [G:H] + [G:K] - 2\sum_{\lambda_\chi\ne 0}\chi(1)$*.*

Proof. Here $m_\chi = 1$, so the first expression is Corollary 6.3(a). For the second, $c_1(\chi) = \|P_Ku\|^2 = \langle P_Ku, u\rangle = |K|^{-1}\sum_k\langle ku, u\rangle$. For the third, the spherical function satisfies the functional equation $|H|^{-1}\sum_{h\in H}\omega(xhy) = \omega(x)\omega(y)$, because $|H|^{-1}\sum_h\langle xhyu, u\rangle = \langle xP_Hyu, u\rangle$ and $P_H = \langle\cdot, u\rangle u$. Viewing $\omega = \omega_\chi$ as the vector

$\sum_{aH}\omega(a)\,aH$, we obtain

$$\begin{aligned}(T_\mu\omega)(aH) &= \frac{1}{|H|}\sum_{y\in G}\mu(y)\,\omega(ay)\\ &= \frac{1}{|H|^2}\sum_{y}\sum_{h}\mu(y)\,\omega(ahy)\\ &= \Big(\frac{1}{|H|}\sum_{y}\mu(y)\omega(y)\Big)\omega(a),\end{aligned}$$

using the left H-invariance of μ. Thus ω_χ is an eigenvector of $M = T_\mu$ with the stated eigenvalue. It is nonzero and lies in the isotypic component of $\bar\chi$. Since the permutation character of $\mathbb{C}[G/H]$ is real-valued, $\bar\chi$ is also a constituent of $\mathbb{C}[G/H]$. Since λ_χ is real, the first formula gives $\lambda_{\bar\chi} = \overline{\lambda_\chi} = \lambda_\chi$. □

When (G, H) is a Gelfand pair, the operators T_{HxH} span the Bose–Mesner algebra of the association scheme of G-orbitals on G/H [1, 4]. Theorem 6.4 evaluates the element M of this algebra explicitly: on each eigenspace its eigenvalue is $d_X d_Y$ times the average over the single double coset HK of the corresponding character. The most important special case is that of 2-transitive actions. When G is 2-transitive on G/H, the neighbourhoods of the K-cosets form the blocks of a 2-design, and the classical eigenvalue $r_\mathcal{D} - \lambda_\mathcal{D}$ of 2-designs appears.

Corollary 6.5 (2-transitive actions)**.** *Suppose G acts 2-transitively on G/H, let $n = [G : H] \geq 2$, and assume $KH \neq G$ (that is, Γ is not complete bipartite). Then:*

(a) *The neighbourhoods of the vertices of Y are the blocks of a 2-$(n, d_Y, \lambda_\mathcal{D})$ design (possibly with repeated blocks) with replication number $r_\mathcal{D} = d_X$ and $\lambda_\mathcal{D} = d_X(d_Y-1)/(n-1)$, and $BB^{\mathsf T} = (r_\mathcal{D} - \lambda_\mathcal{D})I + \lambda_\mathcal{D}J$.*

(b) *The spectrum of Γ is*

$$\Big\{\pm\sqrt{d_X d_Y},\ \Big(\pm\sqrt{\tfrac{d_X(n-d_Y)}{n-1}}\Big)^{(n-1)},\ 0^{([G:K]-n)}\Big\}.$$

(c) (Fisher-type inequality.) *$[G : K] \geq [G : H]$, that is, $|K| \leq |H|$.*

Proof. By 2-transitivity, the number of K-cosets adjacent to two distinct H-cosets is a constant $\lambda_\mathcal{D}$, and counting gives $\lambda_\mathcal{D}(n-1) = d_X(d_Y-1)$. So M has diagonal entries d_X and off-diagonal entries $\lambda_\mathcal{D}$, and its eigenvalues are $d_X + (n-1)\lambda_\mathcal{D} = d_X d_Y$ (on $\mathbf{1}$) and $d_X - \lambda_\mathcal{D} = d_X(n-d_Y)/(n-1)$ (on $\mathbf{1}^\perp$, with multiplicity $n-1$). The latter is positive because $d_Y = |KH|/|H| < n$. Hence $\operatorname{rank} B = n \leq |Y|$. The same values follow from Theorem 6.4, or from the trace identity $\operatorname{tr} M = nd_X = d_X d_Y + (n-1)(d_X - \lambda_\mathcal{D})$. □

For example, for $G = S_n$ and $H = S_{n-1}$, part (c) says that an intransitive subgroup of S_n has order at most $(n-1)!$. For the Fano plane it gives $\{\pm 3, (\pm\sqrt{2})^{(6)}\}$ (Example 7.2).

6.4 A spectral gap

Proposition 6.6. *Assume $G = \langle H, K\rangle$ and $H \neq G$, and let δ_H be the least degree of a nontrivial irreducible constituent of $\mathbb{C}[G/H]$. Then every eigenvalue $\lambda \neq d_X d_Y$ of M has multiplicity at least δ_H, and the second largest eigenvalue satisfies*

$$\lambda_2(M) \leq \frac{d_X\big([G:H] - d_Y\big)}{\delta_H}.$$

In particular $\lambda_2(M) \leq d_X([G : H] - d_Y)/\delta(G)$, where $\delta(G)$ is the least degree of a nontrivial irreducible representation of G.

Proof. Since Γ is connected, the trivial character occurs once in $\mathbb{C}[G/H]$ and carries the eigenvalue $d_X d_Y$ (Corollary 6.3(b)). The eigenspace of any other eigenvalue λ is a G-submodule of $\mathbf{1}^\perp$, which has no trivial constituent, so its dimension is at least δ_H. All eigenvalues are nonnegative, so $\delta_H\lambda_2 \leq \operatorname{tr} M - d_X d_Y = [G : H]d_X - d_X d_Y$. □

For groups with no small nontrivial representations (the quasirandom groups of Gowers [19], such as $\mathrm{PSL}_2(q)$ with $\delta(\mathrm{PSL}_2(q)) \geq (q-1)/2$), Proposition 6.6 bounds the second singular value of B, and hence the expansion of the bipartite graph Γ. For the Fano plane, $\delta_H = 6$ and the bound $\lambda_2 \leq 3 \cdot 4/6 = 2$ is attained. It is also attained for the subdivided complete graphs Γ_n with $n \geq 5$ (Proposition 7.4): there $\delta_H = n-1$, and the bound equals $n-2 = \lambda_{(n-1,1)}$.

7 Examples

We now apply the results above to several examples. Table 2 collects the outcome. Three kinds of value in this section were obtained by computer: the Betti numbers and the numbers ν_{ind} and m in Example 7.2, and the projective dimensions quoted after Proposition 7.4. The Betti numbers come from Hochster's formula (2), computed over $\mathbb{F}_2$ and over $\mathbb{F}_{32003}$ with the same result. All other values are derived in the text; they were also checked by computer.

Example 7.1 (Two small cases; Figure 1 (Appendix))**.** (a) $G = S_3$, $H = \langle(1\,2)\rangle$, $K = \langle(1\,3)\rangle$. Here $D = 1$, $d_X = d_Y = 2$ and $G = \langle H, K\rangle$, so $\Gamma \cong C_6$ (Corollary 3.7(vii)). The action on G/H is 2-transitive with $n = 3$, so Corollary 6.5 gives the eigenvalue $2 \cdot 1/2 = 1$ on $\mathbf{1}^\perp$ and $\operatorname{Spec}\Gamma = \{\pm 2, (\pm 1)^{(2)}\}$. Moreover $HK \cap KH = H \cup K$, because $(1\,2)(1\,3) \neq (1\,3)(1\,2)$, so the girth is 6

(Theorem 4.4(c)). Since $|H| = |K|$ but $HK \neq KH$, the graph is not unmixed, and $e(R/I(\Gamma)) = 2$. Finally $\nu = 3$ and $\nu_{\text{ind}} = \operatorname{reg} R/I(\Gamma) = 2$: two opposite edges of C_6 form an induced matching that is also maximal, so (4) and Lemma 2.8 give $2 \leq \operatorname{reg} \leq 2$.

(b) $G = \mathbb{Z}_6$, $H = \langle 2 \rangle$, $K = \langle 3 \rangle$. Here $HK = G$, so $\Gamma = K_{2,3}$, and Theorem 5.4 with $c = 1$, $p = 3$, $q = 2$ gives the linear Betti numbers $(b_1, b_2, b_3, b_4) = (6, 9, 5, 1)$, $\mathrm{HS} = (1-t)^{-3} + (1-t)^{-2} - 1$, reg $=$ depth $= 1$ and pd $= 4$.

Example 7.2 (The Fano plane and the Heawood graph)**.** Let $G = \mathrm{GL}_3(2) \cong \mathrm{PSL}_2(7)$, of order 168, act on the seven points and seven lines of $\mathrm{PG}(2,2)$. Let H be the stabilizer of a point p_0 and K the stabilizer of a line $L_0 \ni p_0$. Then $|H| = |K| = 24$, and D is the stabilizer of the flag (p_0, L_0), of order 8, so $d_X = d_Y = 3$. By Proposition 3.3(b), $\Gamma(G; H, K)$ is the point–line incidence graph of the Fano plane, the *Heawood graph*.

Permutability. $|HK| = |H||K|/|D| = 72$. Since 72 does not divide 168, Lagrange's theorem shows that no subgroup of G has order 72. So HK is not a subgroup and H, K do not permute (Theorem 3.6).

Distances and girth. By Lemma 3.1, $x \in KH$ if and only if $xp_0 \in L_0$, and $x \in HK$ if and only if $p_0 \in xL_0$. If $x \in HK \cap KH$ lies in neither H nor K, then the distinct points p_0 and xp_0 both lie on the distinct lines L_0 and xL_0, which is impossible. Hence $HK \cap KH = H \cup K$, and the girth is at least 6. The alternating products have sizes $|\Pi_0| = 24$, $|\Pi_1| = 72$, $|\Pi_2| = 168$. So $\ell(H,K) = \ell(K,H) = 2$ and $\operatorname{diam}\Gamma = 3$. Moreover $|\Pi_2 \setminus \Pi_0|/|H| = 6 = t_2$, while $|\Pi_3 \setminus \Pi_1|/|K| = 4 < 12 = t_3$, so $\operatorname{girth}\Gamma = 6$ by Theorem 4.4(b). Since $|H\backslash G/H| = |H\backslash G/K| = 2$, equality holds in Corollary 4.3; see Figure 2 (Appendix).

Spectrum. Here $\mu(e) = 3$ and $\mu(x) = 1$ for $x \notin H$ (Proposition 6.1), so $M = 3T_H + T_{G\backslash H} = 2I + J$. The action is 2-transitive with $n = 7$, and Corollary 6.5 gives the eigenvalue $3 \cdot 4/6 = 2$ on the 6-dimensional constituent. Equivalently, the average of its character $\chi = \mathrm{fix} - 1$ over HK is $2/9$ (Theorem 6.4). Hence

$$\mathrm{Spec}(\Gamma) = \{\pm 3,\ (\pm\sqrt{2})^{(6)}\}.$$

Edge ideal. $\nu = \tau = \operatorname{ht} I = \dim R/I = 7$, and $e(R/I) = 2$ (Proposition 5.2). The graph is neither CM, nor unmixed, nor SCM (Theorem 5.3). For the matching invariants, $|U| = 40$ and $|S| = 104$, so Theorem 5.6 gives $2 \leq \nu_{\text{ind}} \leq 4.2 \leq \mathrm{m}$; an exhaustive computer search gives $\nu_{\text{ind}} = 3$ and $\mathrm{m} = 5$. Since $|G|/|U|$ is not an integer, (d) does not apply. The graded Betti numbers $\beta_{i,i+r}$ of $R/I(\Gamma)$ are listed in Table 1. Thus $\operatorname{reg} R/I(\Gamma) = 4$, pd $= 10$, depth $= 4$, and

$$\mathrm{HS}\big(R/I(\Gamma); t\big) = \frac{1 + 7t + 7t^2 - 21t^3 - 14t^4 + 28t^5 - 6t^7}{(1-t)^7}.$$

The numerator at $t = 1$ equals $2 = e(R/I)$, as predicted by Proposition 5.2.

Example 7.3 (The generalized quadrangle $W(2)$ and the Tutte–Coxeter graph)**.** Let $G = S_6 \cong \mathrm{Sp}_4(2)$ act on the 15 duads (2-subsets of $\{1, \ldots, 6\}$) and the 15 synthemes (partitions into three duads), with incidence given by containment. This is the generalized quadrangle $W(2)$. Let $H = S_{\{1,2\}} \times S_{\{3,4,5,6\}}$ be the stabilizer of $d_0 = \{1, 2\}$ and K the stabilizer of $s_0 = \{12|34|56\}$; then $|H| = |K| = 48$ and $|D| = 16$, so $d_X = d_Y = 3$. The group G is flag-transitive, so $\Gamma(G; H, K)$ is the incidence graph of $W(2)$, the *Tutte–Coxeter graph* (Tutte's 8-cage) [11, 39].

Distances and girth. The products have sizes $|\Pi_r(H,K)| = 48, 144, 336, 720$ for $r = 0, 1, 2, 3$, and the same holds for $\Pi_r(K,H)$. These are $|F_r|$ times the numbers $1, 3, 7, 15$ of duads, respectively synthemes, within distance r of d_0. They follow from the facts that each duad lies in 3 synthemes, each syntheme contains 3 duads, and two duads lie in at most one common syntheme. So $\operatorname{diam}\Gamma = 4$. The sphere sizes $|\mathcal{S}_r(eH)| = 3, 6, 12, 8$ for $r = 1, \ldots, 4$ agree with $t_r = 3, 6, 12$ for $r \leq 3$ and fall below $t_4 = 24$, so $\operatorname{girth}\Gamma = 8$. Since $|H\backslash G/H| = 3$ and $|H\backslash G/K| = 2$, equality holds in Corollary 4.3. By Corollary 4.6, Γ is the incidence graph of a generalized quadrangle.

Spectrum. (G, H) is a Gelfand pair with $\mathbb{C}[G/H] = \chi^{(6)} \oplus \chi^{(5,1)} \oplus \chi^{(4,2)}$, of dimensions $1, 5, 9$. The 5-dimensional constituent lies in the permutation module $\mathbb{C}^6$. Since K is transitive on $\{1, \ldots, 6\}$, $(\mathbb{C}^6)^K$ is one-dimensional, so $V^K_{(5,1)} = 0$ and $\lambda_{(5,1)} = 0$ (Corollary 6.3(c)). The trace identity $\operatorname{tr} M = 15 \cdot 3 = 9 + 5 \cdot 0 + 9\lambda_{(4,2)}$ gives $\lambda_{(4,2)} = 4$, in agreement with the averages over HK in Theorem 6.4. Hence

$$\mathrm{Spec}(\Gamma) = \{\pm 3,\ (\pm 2)^{(9)},\ 0^{(10)}\}.$$

Edge ideal. $\operatorname{ht} I = \dim R/I = 15$ and $e(R/I) = 2$. Here $|U| = 48 + 48 - 16 = 80$ and $|G|/|U| = 9$. Let $Q = \langle (2\,3\,4), (1\,5\,6) \rangle \cong C_3 \times C_3$. For $1 \neq q \in Q$, the duad qd_0 meets both $\{1, 5, 6\}$ and $\{2, 3, 4\}$ and differs from d_0, so $qd_0 \notin s_0$, that is, $q \notin KH$. Likewise qs_0 consists of $qd_0 \neq d_0$ and two duads inside $\{2, 3, 4\}$ and $\{1, 5, 6\}$, so $d_0 \notin qs_0$, that is, $q \notin HK$. Thus $Q \cap S = \{e\}$, and Theorem 5.6(d) gives

$$\operatorname{reg} R/I(\Gamma) = \nu_{\text{ind}}(\Gamma) = \mathrm{m}(\Gamma) = 9.$$

Proposition 7.4 (Subdivided complete graphs)**.** *Let $n \geq 4$, $G = S_n$, $H = S_{\{1,2\}} \times S_{\{3,\dots,n\}}$ and $K = S_{\{2,\dots,n\}}$ (the stabilizer of 1). Then $\Gamma_n = \Gamma(G; H, K)$ is the subdivision of K_n: the duads are adjacent to the points they contain. Moreover:*

(a) $d_X = 2$, $d_Y = n-1$, $\operatorname{girth}\Gamma_n = 6$ *and* $\operatorname{diam}\Gamma_n = 4$.

(b) *The spectrum is*

$$\operatorname{Spec}(\Gamma_n) = \big\{\pm\sqrt{2(n-1)},\ (\pm\sqrt{n-2})^{(n-1)},\ 0^{(n(n-3)/2)}\big\}.$$

(c) $\operatorname{ht} I = n$, $\dim R/I = \binom{n}{2}$, $e(R/I) = 1$, *and* Γ_n *is neither unmixed nor SCM.*

(d) $\operatorname{reg} R/I(\Gamma_n) = \nu_{\mathrm{ind}}(\Gamma_n) = \mathrm{m}(\Gamma_n) = n-1$.

Proof. The identification follows from Proposition 3.3(b), since S_n is transitive on incident (duad, point) pairs. D fixes 1 and 2, so $d_X = 2$ and $d_Y = n-1$.

(a) Two duads share at most one point, so $HK \cap KH = H \cup K$ and the girth is at least 6; a triangle of K_n gives a 6-cycle. For the diameter, $\Pi_3(H,K) = G$, since every point lies on a duad meeting $\{1,2\}$, while $\Pi_2(H,K) \neq G$, since duads disjoint from $\{1,2\}$ exist for $n \geq 4$. Also $\Pi_2(K,H) = G$. Hence diam $= 1+3 = 4$.

(b) (S_n, H) is a Gelfand pair with $\mathbb{C}[G/H] = \chi^{(n)} \oplus \chi^{(n-1,1)} \oplus \chi^{(n-2,2)}$. By the branching rule, $\chi^{(n-2,2)}$ restricted to $K \cong S_{n-1}$ is $\chi^{(n-2,1)} + \chi^{(n-3,2)}$ (the second term is absent when $n = 4$), with no trivial constituent, so $\lambda_{(n-2,2)} = 0$ by Corollary 6.3(c). The trace identity $\operatorname{tr} M = \binom{n}{2} \cdot 2 = 2(n-1)+(n-1)\lambda_{(n-1,1)}$ gives $\lambda_{(n-1,1)} = n-2 > 0$. There are $\binom{n}{2}+n-2n = n(n-3)/2$ zero eigenvalues.

(c) $[G : H] = \binom{n}{2} > n = [G : K]$ and $|H| = 2(n-2)! \neq (n-1)! = |K|$, so this follows from Proposition 5.2 and Theorem 5.3.

(d) $|U| = 2(n-2)!+(n-1)!-(n-2)! = n\,(n-2)!$, so $|G|/|U| = n-1$. Let $Q \cong C_{n-1}$ be generated by an $(n-1)$-cycle on $\{1,3,4,\dots,n\}$, which fixes 2. The edges E_Q are $\{\{j,2\},j\}$ for $j \neq 2$, and these form an induced matching. Indeed, $\{j,2\}$ does not contain j' for $j' \neq j$, so $Q \cap S = \{e\}$. Theorem 5.6(d) applies. □

For $n = 3$ (where $\Gamma_3 = C_6$), 4 and 5, a computation with (2) gives $\operatorname{pd} R/I(\Gamma_n) = 4, 7, 11$, that is, $\binom{n}{2}+1$; see Section 8.

Example 7.5 (A non-Gelfand pair: the subdivided cube)**.** Let $G = S_4$, $H = \langle(1\,2)\rangle$ and $K = \langle(2\,3\,4)\rangle$. Then $D = 1$, $d_X = 2$, $d_Y = 3$, $|X| = 12$ and $|Y| = 8$. Identifying S_4 with the rotation group of the cube, with K a vertex stabilizer and H the stabilizer of an incident edge, Proposition 3.3(b) shows that Γ is the subdivision of the cube Q_3.

Distances. For $r = 0,\dots,5$ the products have sizes $|\Pi_r(H,K)| = 2, 6, 10, 18, 22, 24$ and $|\Pi_r(K,H)| = 3, 6, 12, 18, 21, 24$. These are $|F_r|$ times the numbers of vertices of the relevant type within distance r, which can be read off the subdivided cube. So $\operatorname{diam}\Gamma = 6$. The sphere sizes $|\mathcal{S}_r(eH)| = 2, 4, 4, 6$ for $r = 1,\dots,4$, compared with $t_r = 2, 4, 4, 8$, give $\operatorname{girth}\Gamma = 8$.

Spectrum. Here $\mathbb{C}[G/H]$ is not multiplicity-free, and Theorem 6.2 is needed. We have

$$HK = \{e,\ (2\,3\,4),\ (2\,4\,3),\ (1\,2),\ (1\,2\,3\,4),\ (1\,2\,4\,3)\}.$$

Writing $\chi_3 = \chi^{(3,1)}$, $\chi_3' = \chi^{(2,1,1)}$ and $\operatorname{av}(\chi) = \frac{1}{|HK|}\sum_{x\in HK}\chi(x)$, one finds:

χ	$\chi(1)$	m_χ	n_χ	$\operatorname{av}(\chi)$	$d_X d_Y c_j$
1	1	1	1	1	6
sgn	1	0	1	0	–
$\chi^{(2,2)}$	2	1	0	0	0
χ_3	3	2	1	1/3	2, 0
χ_3'	3	1	1	2/3	4

For χ_3 we have $m_\chi = 2 > n_\chi = 1$. One principal angle is forced to be $\pi/2$, and the other has $\cos^2\theta = 1/3$ by the trace formula. Hence $\operatorname{Spec}(M) = \{6, 4^{(3)}, 2^{(3)}, 0^{(5)}\}$ and

$$\operatorname{Spec}(\Gamma) = \big\{\pm\sqrt{6},\ (\pm 2)^{(3)},\ (\pm\sqrt{2})^{(3)},\ 0^{(6)}\big\},$$

which agrees with the subdivision formula $\pm\sqrt{3+\theta}$ for the cube eigenvalues $\theta \in \{3, 1, -1, -3\}$.

Edge ideal. $|U| = 4$ and $|G|/|U| = 6$. Let $Q = \operatorname{Stab}_{S_4}(2) \cong S_3$. Then

$$S = HK \cup KH = \{e,\ (2\,3\,4),\ (2\,4\,3),\ (1\,2),\ (1\,2\,3\,4),\ (1\,2\,4\,3),\ (1\,4\,3\,2),\ (1\,3\,4\,2)\},$$

and every nonidentity element of S moves 2, so $Q \cap S = \{e\}$. By Theorem 5.6(d), $\operatorname{reg} R/I(\Gamma) = 6$. Moreover $\operatorname{ht} I = 8$, $\dim R/I = 12$ and $e(R/I) = 1$.

Table 1. Graded Betti numbers $\beta_{i,i+r}$ of $R/I(\Gamma)$ for the Heawood graph (Example 7.2); zero entries are omitted. Rows indexed by i; the final column is the total.

$i\backslash r$	0	1	2	3	4	total
0	1					1
1		21				21
2		42	84			126
3		14	336	84		434
4			434	448		882
5			210	973		1183
6			28	994	21	1043
7				476	84	560
8				84	98	182
9					42	42
10					6	6

Table 2. Invariants of the examples. Here ν and ν_{ind} denote the matching and induced matching numbers, and reg is the regularity of $R/I(\Gamma)$. C_6.

Γ	C_6
$(G; H, K)$	$(S_3; C_2, C_2)$
(d_X, d_Y)	$(2, 2)$
girth	6
diam	3
nonzero spectrum	$\pm 2, (\pm 1)^{(2)}$
ν	3
ν_{ind}	2
reg	2

Table 2. Invariants of the examples. Here ν and ν_{ind} denote the matching and induced matching numbers, and reg is the regularity of $R/I(\Gamma)$. $K_{2,3}$.

Γ	$K_{2,3}$
$(G; H, K)$	$(\mathbb{Z}_6; C_3, C_2)$
(d_X, d_Y)	$(3, 2)$
girth	4
diam	2
nonzero spectrum	$\pm\sqrt{6}$
ν	2
ν_{ind}	1
reg	1

Table 2. Invariants of the examples. Here ν and ν_{ind} denote the matching and induced matching numbers, and reg is the regularity of $R/I(\Gamma)$. Heawood.

Γ	Heawood
$(G; H, K)$	$(\mathrm{GL}_3(2); S_4, S_4)$
(d_X, d_Y)	$(3, 3)$
girth	6
diam	3
nonzero spectrum	$\pm 3, (\pm\sqrt{2})^{(6)}$
ν	7
ν_{ind}	3
reg	4

Table 2. Invariants of the examples. Here ν and ν_{ind} denote the matching and induced matching numbers, and reg is the regularity of $R/I(\Gamma)$. Tutte–Coxeter.

Γ	Tutte–Coxeter
$(G; H, K)$	$(S_6; S_2 \times S_4, S_2 \wr S_3)$
(d_X, d_Y)	$(3, 3)$
girth	8
diam	4
nonzero spectrum	$\pm 3, (\pm 2)^{(9)}$
ν	15
ν_{ind}	9
reg	9

Table 2. Invariants of the examples. Here ν and ν_{ind} denote the matching and induced matching numbers, and reg is the regularity of $R/I(\Gamma)$. subdivided K_n.

Γ	subdivided K_n
$(G; H, K)$	$(S_n; S_2 \times S_{n-2}, S_{n-1})$
(d_X, d_Y)	$(2, n-1)$
girth	6
diam	4
nonzero spectrum	$\pm\sqrt{2(n-1)}, (\pm\sqrt{n-2})^{(n-1)}$
ν	n
ν_{ind}	$n-1$
reg	$n-1$

Table 2. Invariants of the examples. Here ν and ν_{ind} denote the matching and induced matching numbers, and reg is the regularity of $R/I(\Gamma)$. subdivided Q_3.

Γ	subdivided Q_3
$(G; H, K)$	$(S_4; C_2, C_3)$
(d_X, d_Y)	$(2, 3)$
girth	8
diam	6
nonzero spectrum	$\pm\sqrt{6}, (\pm 2)^{(3)}, (\pm\sqrt{2})^{(3)}$
ν	8
ν_{ind}	6
reg	6

8 Conclusion

We have shown that the coset incidence graph $\Gamma(G; H, K)$ is controlled, at the combinatorial, homological and spectral levels, by a small amount of data attached to the pair (H, K): the indices d_X and d_Y, the product sets HK and KH, and the alternating products $HKH\cdots$. Distances, the diameter and the girth are read off the sizes of these products. The Cohen–Macaulay, unmixed and sequentially Cohen–Macaulay properties reduce to the conditions $H = K$, $HK = KH$ with $|H| = |K|$, and $H \leq K$ or $K \leq H$. The edge ideal is described completely when H and K permute, and its regularity is determined in general whenever G contains a packing with respect to $HK \cup KH$ of size $|G|/|H \cup K|$. On the spectral side, viewing the biadjacency matrix as a Hecke operator reduces the spectrum to principal angles between fixed subspaces. For Gelfand pairs this gives explicit eigenvalues as character averages over the double coset HK, and in the 2-transitive case it recovers the eigenvalues of 2-designs and a Fisher-type inequality. The examples from the Fano plane, the generalized quadrangle $W(2)$ and the symmetric groups show that the girth, the diameter, the spectrum and, when a packing exists, the regularity can be determined by hand from the subgroups.

Two problems raised by these results remain open. First, the regularity of $R/I(\Gamma)$ is not determined when no packing as in Theorem 5.6(d) exists. The Heawood graph shows that it can then lie strictly between the induced matching number and the minimum maximal matching number ($3 < 4 < 5$). Second, outside the permutable case the projective dimension, the depth and the full Hilbert series of $R/I(\Gamma)$ are not known in terms of (H, K); Proposition 5.2 gives only the dimension and the multiplicity. For the subdivided complete graphs Γ_n of Proposition 7.4, our computations give $\operatorname{pd} R/I(\Gamma_n) = \binom{n}{2} + 1$ for $n = 3, 4, 5$, which suggests that this formula holds for all n.

Declaration of Generative AI and AI-assisted Technologies in the Writing Process

During the preparation of this work, the author(s) utilized artificial intelligence (AI) technologies solely for the purpose of linguistic refinement and grammatical correction. Following the application of these tools, the author(s) comprehensively reviewed, verified, and edited the resulting text. The author(s) maintain full accountability and responsibility for the accuracy, integrity, and final content of the publication.

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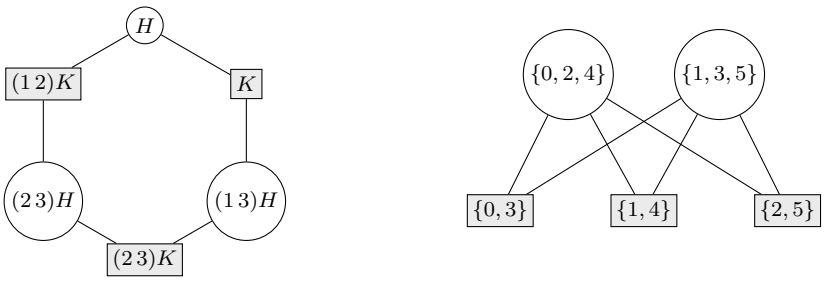
Conflicts of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.

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APPENDIX



(a) $\Gamma(S_3; \langle(1\,2)\rangle, \langle(1\,3)\rangle) \cong C_6$ (b) $\Gamma(\mathbb{Z}_6; \langle 2\rangle, \langle 3\rangle) \cong K_{2,3}$

Fig. 1. Two small coset incidence graphs (Example 7.1); circles are cosets of H, squares are cosets of K. In (a), H and K do not permute, and $HK \cap KH = H \cup K$, so the girth is 6. In (b), $HK = G$, so the graph is complete bipartite (Theorem 3.5) and the girth is 4.

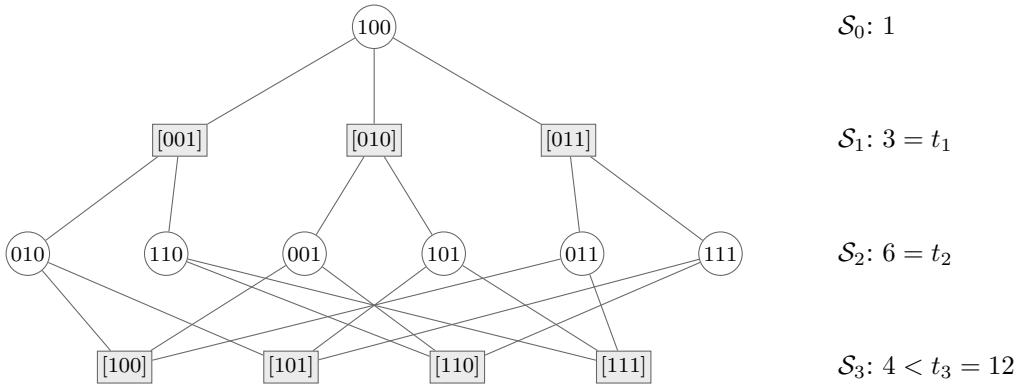


Fig. 2. The Heawood graph $\Gamma(\mathrm{GL}_3(2); H, K)$ of Example 7.2, drawn by distance from $eH = p_0 = 100$. Points are the nonzero vectors of $\mathbb{F}_2^3$, and $[abc]$ is the line $ax + by + cz = 0$. The layers are the spheres $\mathcal{S}_r(eH)$ of Theorem 4.2(b), of sizes $|\Pi_r \setminus \Pi_{r-2}|/|F_r| = 1, 3, 6, 4$. Each layer is a single H-orbit, so equality holds in Corollary 4.3. The first deficit $|\mathcal{S}_3| < t_3$ occurs at $r = 3$, so the girth is 6 (Theorem 4.4).