Convergence of Cauchy Sequences and Unique fixed points in MR-Metric Spaces

ALI JARADAT[1,*], ABED AL-RAHMAN M. MALKAWI[1], AYAT M. RABAIAH[1]
MOHAMMED M. M. JARADAT[2], ABDULLAH AL-JAWARNEH[1]

[1]Department of Mathematics
Amman Arab University
Amman 11953
JORDAN

[2]Department of Mathematics and Statistics
Qatar University
Doha 2713
QATAR

Corresponding Author

Abstract: This paper investigates various properties of self-maps on metric spaces, particularly in the context of MR-metric spaces. We provide a range of examples to demonstrate the behavior of maps within such spaces, with a focus on those defined by specific criteria involving the metric M and a parameter R. The findings show that under certain conditions, the iterated distances of these self-maps converge, leading to the identification of unique fixed points. Additionally, we prove that sequences of iterated maps converge and form M-Cauchy sequences, given the imposed conditions. The paper concludes with a theorem establishing the existence of a unique common fixed point for two partially commuting maps under appropriate assumptions. This research enhances the understanding of fixed point theory in metric spaces and offers potential applications in diverse areas of mathematics and computational science.

Key-Words: $MR-metric$ space, M-Convergent, M-Cauchy, fixed point theorems, self-mappings.

Received: May 24, 2026. Revised: June 26, 2026. Accepted: July 29, 2026. Published: October 30, 2026.

1 Introduction

In recent years, the study of metric spaces has witnessed significant advancements, particularly in the development of generalized metric spaces and their applications in fixed point theory. The introduction of MR-metric spaces by Malkawi et al. [1] has opened new avenues for research in this field, extending the classical notions of metric spaces and D-metric spaces. This generalization has proven particularly useful in establishing fixed point theorems under various contraction conditions. The foundational work on MR-metric spaces builds upon earlier developments in b-metric spaces [2], where researchers investigated weak contraction mappings and their fixed point properties. Recent studies have demonstrated the versatility of these spaces, with applications ranging from fractional calculus to graph theory and deep learning. Significant contributions have been made in the study of fixed point theorems in these generalized metric spaces. Qawasmeh [3], [4] explored interpolative contractions and simulation functions, while Bataihah et al. [5],[6],[7] investigated new types of distance spaces and fixed point results using Ω-distance mappings. The work of Shatanawi and collaborators [8],[9],[10],[11] has further enriched the theory through various contraction mappings in extended quasi b-metric spaces. Recent developments by Malkawi and colleagues [1],[2],[12] have established comprehensive results on the existence, uniqueness, and convergence of fixed points in MR-metric spaces. These include applications neutrosophic theory [12], measure theory and fractional calculus. The current paper continues this line of research by investigating the convergence of Cauchy sequences and the existence of unique fixed points in MR-metric spaces. The structure of this paper is as follows: Section 2 presents our main results, including several illustrative examples and theorems establishing the convergence properties of sequences in MR-metric spaces. We prove the existence of unique fixed points under various conditions and demonstrate applications of these results. For more information, see [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], [27], [28], [29].

Definition 1.1. *[1] Let $\mathbb{X} \neq \emptyset$ be a non-empty set and $R > 1$ be a real number. A function $M : \mathbb{X}\times\mathbb{X}\times\mathbb{X} \to [0, \infty)$ is referred to as an MR-metric if it satisfies the following conditions for all $\upsilon, \xi, \Im \in \mathbb{X}$:*

- $(M1) : M(\upsilon, \xi, \Im) \geq 0.$
- $(M2) : M(\upsilon, \xi, \Im) = 0$ *if and only if* $\upsilon = \xi = \Im.$
- $(M3) : M(\upsilon, \xi, \Im) = M(p(\upsilon, \xi, \Im))$, *for any permutation* $p(\upsilon, \xi, \Im)$ *of* $\upsilon, \xi, \Im.$
- $(M4) : M(\upsilon, \xi, \Im) \leq R\left[M(\upsilon, \xi, \ell_1) + M(\upsilon, \ell_1, \Im) + M(\ell_1, \xi, \Im)\right].$

A pair $(\mathbb{X}, M)$ *that satisfies these properties is called an* MR*-metric space.*

Definition 1.2. *Let* $\{\upsilon_{i_n}\}$ *be a sequence within an* MR*-metric space* $(\mathbb{Y}, M)$*. This sequence is called* MR*-convergent if there exists an element* $\upsilon_{i_1} \in \mathbb{Y}$ *such that for every* $\epsilon > 0$*, there exists a positive integer* N *where* $M(\upsilon_{i_n}, \upsilon_{i_m}, \upsilon_{i_1}) < \epsilon$ *for all* $m, n \geq N$*. In this case, we say the sequence* $\{\upsilon i_n\}$ MR*-converges to* υ_{i_1}*, and* υ_{i_1} *is the limit of the sequence.*

Definition 1.3. *A sequence* $\{\upsilon_{i_n}\}$ *in an* MR*-metric space* $(\mathbb{Y}, M)$ *is called* MR*-Cauchy if for every* $\epsilon > 0$*, there exists a positive integer* N *such that* $M(\upsilon i_n, \upsilon_{i_m}, \upsilon_{i_p}) < \epsilon$ *for all* $m, n, p \geq N.$

Definition 1.4. *An* MR*-metric space* (X, M) *is considered bounded if there exists a real number* $L > 0$ *such that* $M(\upsilon, \xi, \Im) \leq L$ *for all* $\upsilon, \xi, \Im \in \mathbb{X}$*. In this case,* M *is referred to as an* MR*-bound for the metric.*

Definition 1.5. *The subset* E *of an* MR*-metric space* $(\mathbb{X}, M)$ *is said to be* M*-bounded if there exists* $L > 0$ *such that* $M(\upsilon, \xi, \Im) \leq L$ *for all* $\upsilon, \xi, \Im \in E.$

This paper builds upon the foundational work of Al-Malkawi and colleagues in the realm of MR-metric spaces. It investigates the properties of self-maps defined within this framework, with particular attention to convergence and M-Cauchy sequences as established by Malkawi et al. Through illustrative examples, we explore the behavior of these maps under conditions involving the MR-metric M and parameter R. The results demonstrate that iterated distances of self-maps converge, leading to unique fixed points. Moreover, we prove that iterated maps form M-Cauchy sequences, provided

2 Main Results

We start has section by presents Examples.

Example 2.1. *Let* $\mathbb{X} = \{0, 1, \frac{1}{2}, \frac{1}{3}, \dots\}$ *and* $R > 1.$

Define a function $M : \mathbb{X} \times \mathbb{X} \times \mathbb{X} \to \mathbb{R}^+$ *as follows:*

$$M(\upsilon, \xi, \Im) = \begin{cases} \frac{1}{R}\big[& |\upsilon - \xi| + \\ & |\xi - \Im| + \\ & |\Im - \upsilon|\big], \\ & \upsilon, \xi, \Im \in \\ & \mathbb{X} \setminus \{1\}, \\ 0, & \upsilon = \xi \\ & = \Im = 1, \\ & \text{if 0 and 1} \\ & \text{appear among} \\ & \upsilon, \xi, \Im, \\ \frac{1}{R}, & \text{or if 1 appears} \\ & \text{exactly twice} \\ & \text{among } \upsilon, \\ & \xi, \Im, \\ & \big\{ \max\{\upsilon, \xi\}, \\ \frac{1}{R} \min & \max\{\xi, \Im\}, \\ & \max\{\Im, \upsilon\}\big\}, \\ & \text{if 1 appears} \\ & \text{exactly once} \\ & \text{and 0 does} \\ & \text{not appear} \\ & \text{among } \upsilon, \\ & \xi, \Im. \end{cases} \tag{2.1}$$

Then, the pair $(\mathbb{X}, M)$ *constitutes a complete* MR*-metric space. Moreover, the sequence* $\left\{\frac{1}{n}\right\}$ *demonstrates convergence to both* 0 *and* 1*. For any sequence* $\{\upsilon_n\}$ *within* $\mathbb{X}$ *that converges to a point* $\upsilon_0 \in \mathbb{X} \setminus \{0, 1\}$*, the limit* υ_0 *is uniquely defined. Therefore, every convergent sequence in* $\mathbb{X}$ *can have no more than two distinct limits.*

Proof. It is clear that M is symmetric in all three variables υ, ξ, and $\Im$, and that $M(\upsilon, \xi, \Im) = 0 \iff \upsilon = \xi = \Im$. Let υ, ξ, $\Im$, and κ be elements of $\mathbb{X}$.

Case 1. Suppose $\upsilon, \xi, \Im \in \mathbb{X} \setminus \{1\}$.

- If $\kappa \in \mathbb{X} \setminus \{1\}$, then

$$\begin{aligned} M(\upsilon, \xi, \Im) &= \frac{1}{R}\left[|\upsilon - \xi| + |\xi - \Im| + |\Im - \upsilon|\right] \\ &\leq \frac{1}{R}[|\kappa - \xi| + |\xi - \Im| + |\Im - \kappa| \\ &+ |\upsilon - \kappa| + |\kappa - \Im| + |\Im - \upsilon| \\ &+ |\upsilon - \xi| + |\xi - \kappa| + |\kappa - \upsilon|] \\ &\leq R \qquad (1) \\ &[M(\kappa, \xi, \Im) + M(\upsilon, \kappa, \Im) + M(\upsilon, \xi, \kappa)] \qquad (2.2) \end{aligned}$$

- If $\kappa = 1$, then

$$\begin{aligned} M(\upsilon, \xi, \Im) &= \frac{1}{R}\left[|\upsilon - \xi| + |\xi - \Im| + |\Im - \upsilon|\right] \\ &\leq \frac{1}{R}\left[\max\{\xi, \Im\} + \max\{\Im, \upsilon\} + \max\{\upsilon, \xi\}\right] \\ &\leq R\left[M(\kappa, \xi, \Im) + M(\upsilon, \kappa, \Im) + M(\upsilon, \xi, \kappa)\right], \end{aligned} \tag{2.3}$$

since $M(\kappa,\xi,\Im) = M(1,\xi,\Im) = \frac{1}{R}$ or $\frac{1}{R}\max\{\xi,\Im\}$, and so on.

Case 2. If 1 appears among υ, ξ, and $\Im$.

- **Subcase 1.** If $\upsilon = \xi = \Im = 1$, then

$$M(\upsilon,\xi,\Im) = 0 \leq R\left[M(\kappa,\xi,\Im) + M(\upsilon,\kappa,\Im) + M(\upsilon,\xi,\kappa)\right]. \tag{2.4}$$

Subcase 2. If the value 1 appears exactly twice among υ, ξ, and $\Im$, To simplify the analysis, let us consider the case where $\upsilon = \xi = 1$. Under this assumption, it follows that $\Im \in \mathbb{X} \setminus \{1\}$.

- In the case where $\kappa = 1$, we have

$$\begin{aligned} M(\upsilon,\xi,\Im) &= M(1,1,\Im) = M(1,\xi,\Im) \\ &\leq R\left[M(\kappa,\xi,\Im) + M(\upsilon,\kappa,\Im) + M(\upsilon,\xi,\kappa)\right]. \end{aligned} \tag{2.5}$$

- If $\kappa \neq 1$, then

$$\begin{aligned} M(\upsilon,\xi,\Im) &= M(1,1,\Im) = \frac{1}{R} \\ &= M(1,1,\kappa) = M(\upsilon,\xi,\kappa) \\ &\leq R\left[M(\kappa,\xi,\Im) + M(\upsilon,\kappa,\Im) + M(\upsilon,\xi,\kappa)\right]. \end{aligned} \tag{2}$$

(2.6)

Subcase 3. If the value 1 appears exactly once among υ, ξ, and $\Im$, assume without loss of generality that $\upsilon = 1$, with $\xi, \Im \in \mathbb{X} \setminus \{1\}$. Further, assume that $\Im \leq \xi$. If $\kappa = 1$, then

$$\begin{aligned} M(\upsilon,\xi,\Im) &= M(1,\xi,\Im) = M(\kappa,\xi,\Im) \\ &\leq R\left[M(\kappa,\xi,\Im) + M(\upsilon,\kappa,\Im) + M(\upsilon,\xi,\kappa)\right]. \end{aligned} \tag{2.7}$$

If $\Im = 0$, then

$$\begin{aligned} M(\upsilon,\xi,\Im) &= M(1,\xi,0) = \frac{1}{R} \\ &= M(1,\kappa,0) = M(\upsilon,\kappa,\Im), \end{aligned} \tag{2.8}$$

and consequently,

$$M(\upsilon,\xi,\Im) \leq R\left[M(\kappa,\xi,\Im) + M(\upsilon,\kappa,\Im) + M(\upsilon,\xi,\kappa)\right]. \tag{2.9}$$

If $\kappa \neq 1$ and $\Im \neq 0$, then

$$\begin{aligned} M(\upsilon,\xi,\Im) = M(1,\xi,\Im) &= \frac{1}{R}\max\{\xi,\Im\} = \frac{\xi}{R} \\ &\leq \max\{\kappa,\xi\} \\ &= \frac{1}{R}\min\left\{\max\left\{\kappa,\xi\right\},\max\left\{1,\kappa\right\},\max\left\{1,\xi\right\}\right\} \\ &\leq R\left[M(\kappa,\xi,\Im) + M(\upsilon,\kappa,\Im) + M(\upsilon,\xi,\kappa)\right]. \end{aligned} \tag{2.10}$$

Thus, for all $\upsilon, \xi, \Im, \kappa \in \mathbb{X}$, we have

$$M(\upsilon,\xi,\Im) \leq M(\kappa,\xi,\Im) + M(\upsilon,\kappa,\Im) + M(\upsilon,\xi,\kappa). \tag{2.11}$$

Therefore, M is an MR-metric on $\mathbb{X}$.

Our objective is to demonstrate that $(\mathbb{X}, M)$ is a complete space. Consider a Cauchy sequence $\{\upsilon_n\}$ in $\mathbb{X}$.

Case 1. There exists some $N \in \mathbb{N}$ such that for all $n \geq N$, we have $\upsilon_n = \upsilon_N$.

In this situation, it is clear that the sequence $\{\upsilon_n\}$ converges to υ_N.

Case 2. For all $N \in \mathbb{N}$, there exists at least one pair of indices $i, j \in \mathbb{N}$ with $i > j > N$ such that $\upsilon_i \neq \upsilon_j$.

Within this case, we analyze the following subcase:

Subcase 1. There exists some $\gamma \in \mathbb{X}$ such that $\upsilon_n = \gamma$ for infinitely many values of n. In this scenario, for any $N \in \mathbb{N}$, there exist indices $i, j, k \in \mathbb{N}$ with $k > j > i > N$, satisfying $\upsilon_i = \upsilon_j = \gamma$ and $\upsilon_k \neq \gamma$. If $\gamma = 1$, then $M(\upsilon_i, \upsilon_j, \upsilon_k) = \frac{1}{R}$. However, since $\{\upsilon_n\}$ is a Cauchy sequence, it follows that $\gamma \neq 1$. Therefore, there exists $N_0 \in \mathbb{N}$ such that $\upsilon_n \in \mathbb{X} \setminus \{1\}$ for all $n \geq N_0$. Consequently, for all $i, j, k \geq N_0$, we have

$$M(\upsilon_i,\upsilon_j,\upsilon_k) = \frac{1}{R}\left[|\upsilon_i - \upsilon_j| + |\upsilon_j - \upsilon_k| + |\upsilon_k - \upsilon_i|\right].$$

Since $M(\upsilon_i,\upsilon_j,\upsilon_k) \leq \frac{\xi}{R} \Rightarrow |\upsilon_i - \upsilon_j| < \varepsilon \Rightarrow \{\upsilon_i\}$ is cauchy in $\mathbb{R}$, $\exists \xi \in \mathbb{R}$ such that $\upsilon_i \to \xi$. But $\xi = \gamma$, since $\{\gamma\}$ is a subsequence of $\{\upsilon_i\}$, so $\upsilon_i \to \gamma$.

Subcase 2. For every $\gamma \in \mathbb{X}$, the equality $\upsilon_n = \gamma$ holds for only finitely many indices n.

Consider $\epsilon > 0$ and choose a positive integer L such that $\frac{1}{L} < \epsilon$. Define

$$N_1 = \sup\{n \in \mathbb{N} : \upsilon_n \in \{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{L}\}\},$$

where the supremum of the empty set is conventionally defined as zero. Since for any $k \in \mathbb{N}$, $1/k \in \mathbb{X}$ and $\upsilon_n = 1/k$ for only finitely many n, it follows that N_1 is a nonnegative integer. Let $N = N_1 + 1$. Then, for all $n \geq N$, we have

$$0 \leq \upsilon_n < \frac{1}{L} < \epsilon,$$

because $\mathbb{X} = \{0\} \cup \left\{\frac{1}{k} : k \in \mathbb{N}\right\}$. Consequently, the sequence $\{\upsilon_n\}$ converges to 0 in the usual sense.

For $n, m \geq N$, it follows that

$$\begin{aligned} M(0,\upsilon_n,\upsilon_m) &= \frac{1}{R}\left[|0 - \upsilon_n| + |\upsilon_n - \upsilon_m| + |\upsilon_m - 0|\right] \\ &\leq \frac{2}{R}|\upsilon_n| + \frac{2}{R}|\upsilon_m| \to 0 \quad \text{as} \quad n, m \to \infty. \end{aligned} \tag{2.12}$$

Thus, the sequence $\{\upsilon_n\}$ converges to 0 with respect to M. Consequently, in all cases, $\{\upsilon_n\}$ is a convergent sequence in $\mathbb{X}$ with respect to M. This establishes that $(\mathbb{X}, M)$ is a complete space.

It is evident that the sequence $\{1/n\}$ converges to both 0 and 1.

Now, consider a sequence $\{\upsilon_n\}$ in $\mathbb{X}$ such that it converges to an element $\upsilon_0 \in \mathbb{X} \setminus \{0,1\}$. Since $M(1,1,\upsilon_0) = \frac{1}{R}$ and $M(\upsilon_n,\upsilon_m,\upsilon_0) \to 0$ as $n,m \to \infty$, it follows that there exists $N_0 \in \mathbb{N}$ such that $\upsilon_n \in \mathbb{X} \setminus \{1\}$ for all $n \geq N_0$. For all $m,n \geq N_0$, we have

$$M(\upsilon_n,\upsilon_m,\upsilon_0) = \frac{1}{R}\left[|\upsilon_n - \upsilon_m| + |\upsilon_m - \upsilon_0| + |\upsilon_0 - \upsilon_n|\right]. \tag{2.13}$$

As $n,m \to \infty$, the condition $M(\upsilon_n,\upsilon_m,\upsilon_0) \to 0$ implies that $|\upsilon_n - \upsilon_0| \to 0$ as $n \to \infty$. Since υ_0 is strictly positive, there exists an integer $N_1 \in \mathbb{N}$ such that $\upsilon_n > 0$ for all $n \geq N_1$. For $n,m \geq \max\{N_0,N_1\}$, the following holds:

$$\begin{aligned} M(1,\upsilon_n,\upsilon_m) &= \frac{1}{R}\min\{\max\{1,\upsilon_n\},\max\{\upsilon_n,\upsilon_m\},\max\{\upsilon_m,1\}\} \\ &= \frac{1}{R}\max\{\upsilon_n,\upsilon_m\} \to \frac{\upsilon_0}{R} \quad \text{as} \quad n,m \to \infty. \end{aligned} \tag{2.14}$$

Hence, the sequence $\{\upsilon_n\}$ does not approach 1 as its limit. Consider $\xi \in \mathbb{X} \setminus \{1\}$. For all $m,n \geq N_0$, the following holds:

$$\begin{aligned} M(\xi,\upsilon_n,\upsilon_m) &= \frac{1}{R}\left[|\xi - \upsilon_n| + |\upsilon_n - \upsilon_m| + |\upsilon_m - \xi|\right] \\ &\to 2|\xi - \upsilon_0| \quad \text{as} \quad n,m \to \infty. \end{aligned} \tag{2.15}$$

$$\frac{2}{R}|\xi - \upsilon_0| = 0 \quad \text{if and only if} \quad \xi = \upsilon_0.$$

Thus, $\{\upsilon_n\}$ does not converge to ξ for any $\xi \in \mathbb{X} \setminus \{1,\upsilon_0\}$. Therefore, υ_0 is the only limit of $\{\upsilon_n\}$. □

Example 2.2. *Let the set $\mathbb{X}$ be defined as $\mathbb{X} = \{0,1,\frac{1}{2},\frac{1}{3},\dots\}$. Consider the function $M : \mathbb{X}\times\mathbb{X}\times\mathbb{X} \to \mathbb{R}^+$ defined by the following rule:*

$$M(\upsilon,\xi,\Im) = \begin{cases} \frac{1}{R}\big[& |\upsilon-\xi|+ \\ & |\xi-\Im|+ \\ & |\Im-\upsilon|\big], \\ & \upsilon,\xi,\Im \in \\ & \mathbb{X}\setminus\{1\}, \\ 0, & \upsilon = \xi \\ & = \Im = 1, \\ \frac{1}{R}, & \textit{both 0 and 1} \\ & \textit{appear among} \\ & \upsilon,\xi,\Im, \\ \frac{2}{R}\min & \big\{\max\{\upsilon,\xi\}, \\ & \max\{\xi,\Im\}, \\ & \max\{\Im,\upsilon\}\big\}, \\ & \textit{1 appears} \\ & \textit{exactly once} \\ & \textit{and 0 does not} \\ & \textit{appear among} \\ & \upsilon,\xi,\Im, \\ \frac{2}{R}\min & \{\upsilon,\xi,\Im\}, \\ & \textit{1 appears} \\ & \textit{exactly twice} \\ & \textit{among } \upsilon, \\ & \xi,\Im \textit{ and} \\ & \textit{0 does not appear}. \end{cases} \tag{2.16}$$

It follows that $(\mathbb{X},M)$ forms a complete MR-metric space. Additionally, the sequence $\{1/n\}$ is observed to converge to both 0 and 1.

If a sequence $\{\upsilon_n\}$ in $\mathbb{X}$ converges to some element $\upsilon_0 \in \mathbb{X}\setminus\{0,1\}$, then υ_0 serves as the unique limit of the sequence $\{\upsilon_n\}$. Consequently, any sequence in $\mathbb{X}$ that converges can possess no more than two distinct limits.

Proof. It is evident that M is symmetric in all three variables $\upsilon,\xi,\Im$, and $M(\upsilon,\xi,\Im) = 0$ if and only if $\upsilon = \xi = \Im$. Let $\upsilon,\xi,\Im,\kappa \in \mathbb{X}$.

Case 1. Suppose $\upsilon,\xi,\Im \in \mathbb{X}\setminus\{1\}$.

Subcase 1. $\kappa \in \mathbb{X}\setminus\{1\}$, then we have

$$\begin{aligned} M(\upsilon,\xi,\Im) &= \frac{1}{R}\left[|\upsilon-\xi| + |\xi-\Im| + |\Im-\upsilon|\right] \\ &\leq R\left[|\kappa-\xi| + |\xi-\Im| + |\Im-\kappa|\right] \\ &\quad + R\left[|\upsilon-\kappa| + |\kappa-\Im| + |\Im-\upsilon|\right. \\ &\quad \left.+|\upsilon-\xi| + |\xi-\kappa| + |\kappa-\upsilon|\right] \\ &= R\left[M(\kappa,\xi,\Im) + M(\upsilon,\kappa,\Im) + M(\upsilon,\xi,\kappa)\right]. \end{aligned} \tag{2.17}$$

Subcase 2. $\kappa = 1$, then

$$\begin{aligned} M(\upsilon,\xi,\Im) &= \frac{1}{R}\left[|\upsilon-\xi| + |\xi-\Im| + |\Im-\upsilon|\right] \\ &\leq \frac{1}{R}\left[\max\{\xi,\Im\} + \max\{\Im,\upsilon\} + \max\{\upsilon,\xi\}\right] \\ &\leq R\left[M(\kappa,\xi,\Im) + M(\upsilon,\kappa,\Im) + M(\upsilon,\xi,\kappa)\right], \end{aligned} \tag{2.18}$$

since $M(\kappa,\xi,\Im) = M(1,\xi,\Im) = \frac{1}{R}$ or $\frac{2}{R}\max\{\xi,\Im\}$, and so on.

Case 2. Suppose that 1 appears among $\upsilon,\xi,\Im$.

Subcase 1. Assume $\upsilon=\xi=\Im=1$. Then

$$M(\upsilon,\xi,\Im)=0\le R\left[M(\kappa,\xi,\Im)+M(\upsilon,\kappa,\Im)+M(\upsilon,\xi,\kappa)\right]. \quad (2.19)$$

Subcase 2. Suppose that the value 1 appears exactly twice among the elements $\upsilon,\xi,\Im$. To simplify the analysis, assume without loss of generality that $\upsilon=\xi=1$. This implies that $\Im\in\mathbb{X}\setminus\{1\}$.

If $\kappa=1$, then

$$\begin{aligned}M(\upsilon,\xi,\Im)&=M(1,\xi,\Im)=M(\kappa,\xi,\Im)\\&\le R\left[M(\kappa,\xi,\Im)+M(\upsilon,\kappa,\Im)+M(\upsilon,\xi,\kappa)\right]\end{aligned} \quad (2.20)$$

If $\Im=0$, then

$$\begin{aligned}M(\upsilon,\xi,\Im)&=M(1,1,0)=1=M(\kappa,1,0)=M(\kappa,\xi,\Im)\\&\le R\left[M(\kappa,\xi,\Im)+M(\upsilon,\kappa,\Im)+M(\upsilon,\xi,\kappa)\right]\end{aligned} \quad (2.21)$$

If $\Im\neq0$ and $\kappa\neq1$, then

$$\begin{aligned}M(\upsilon,\xi,\Im)&=M(1,1,\Im)=\frac{2}{R}\Im\\&\le\frac{2}{R}\max\{\kappa,\Im\}\\&\le M(1,\kappa,\Im)=M(\kappa,\xi,\Im)\\&\le R\left[M(\kappa,\xi,\Im)+M(\upsilon,\kappa,\Im)+M(\upsilon,\xi,\kappa)\right].\end{aligned} \quad (2.22)$$

Subcase 3. Assume that the value 1 appears only once among the elements $\upsilon,\xi,\Im$. Without loss of generality, let us take $\upsilon=1$. As a result, we have $\xi,\Im\in\mathbb{X}\setminus\{1\}$. Additionally, we assume that $\Im\le\xi$.

If $\kappa=1$, then

$$\begin{aligned}M(\upsilon,\xi,\Im)&=M(1,\xi,\Im)=M(\kappa,\xi,\Im)\\&\le R\left[M(\kappa,\xi,\Im)+M(\upsilon,\kappa,\Im)+M(\upsilon,\xi,\kappa)\right].\end{aligned} \quad (2.23)$$

If $\Im=0$, then

$$M(\upsilon,\xi,\Im)=M(1,\xi,0)=1=M(1,\kappa,0)=M(\upsilon,\kappa,\Im), \quad (2.24)$$

and consequently

$$M(\upsilon,\xi,\Im)\le R\left[M(\kappa,\xi,\Im)+M(\upsilon,\kappa,\Im)+M(\upsilon,\xi,\kappa)\right]. \quad (2.25)$$

If $\kappa\neq1$ and $\Im\neq0$, then

$$\begin{aligned}M(\upsilon,\xi,\Im)&=M(1,\xi,\Im)=\frac{2}{R}\max\{\xi,\Im\}=\frac{2}{R}\xi\\&\le\frac{2}{R}\max\{\kappa,\xi\}\\&\le M(1,\kappa,\xi)=M(\kappa,\xi,\Im)\\&\le R\left[M(\kappa,\xi,\Im)+M(\upsilon,\kappa,\Im)+M(\upsilon,\xi,\kappa)\right]\end{aligned} \quad (2.26)$$

Thus, for all $\upsilon,\xi,\Im,\kappa\in\mathbb{X}$, we obtain

$$M(\upsilon,\xi,\Im)\le R\left[M(\kappa,\xi,\Im)+M(\upsilon,\kappa,\Im)+M(\upsilon,\xi,\kappa)\right]. \quad (2.27)$$

Hence, M satisfies the conditions of an MR-metric on $\mathbb{X}$.

Our goal is to prove that the $MR-metric$ space $(\mathbb{X},M)$ is complete. Let $\{\upsilon_n\}$ be a Cauchy sequence in $\mathbb{X}$.

Case 1. Assume that $\upsilon_n=1$ for infinitely many terms of the sequence. For any $i\in\mathbb{N}$, there exist indices $j,k\in\mathbb{N}$ such that $k>j>i$ and $\upsilon_k=\upsilon_j=1$. In this case, the $MR-metric\ M$ satisfies the following equation:

$$M(\upsilon_i,\upsilon_j,\upsilon_k)=M(\upsilon_i,1,1)=\begin{cases}0 & \text{if } \upsilon_i=1,\\ \frac{1}{R} & \text{if } \upsilon_i=0,\\ \frac{2}{R}\upsilon_i & \text{if } \upsilon_i\in\mathbb{X}\setminus\{0,1\}.\end{cases} \quad (2.28)$$

Since $\{\upsilon_n\}$ is a Cauchy sequence, it follows that $M(\upsilon_p,\upsilon_q,\upsilon_r)\to0$ as $p,q,r\to\infty$. Consequently, $M(\upsilon_i,1,1)\to0$ as $i\to\infty$. This implies that $\upsilon_n=0$ for only finitely many values of n. Therefore, there exists an integer $N\in\mathbb{N}$ such that $\upsilon_n\neq0$ for all $n\ge N$.

If $\upsilon_n\neq1$ for infinitely many n, thus $\upsilon_n\to0$ in the usual sense.

For $m,n\ge N$, we have the following expression for $M(1,\upsilon_m,\upsilon_n)$:

$$M(1,\upsilon_m,\upsilon_n)=\begin{cases}0 & \text{if } \upsilon_m=\upsilon_n=1,\\ \frac{2}{R}\upsilon_m & \text{if } \upsilon_m=1,\upsilon_n\neq1,\\ \frac{2}{R}\upsilon_n & \text{if } \upsilon_m\neq1,\upsilon_n=1,\\ \frac{2}{R}\max\{\upsilon_m,\upsilon_n\} & \text{if } \upsilon_m,\upsilon_n\neq1.\end{cases} \quad (2.29)$$

Thus, $M(1,\upsilon_m,\upsilon_n)\to0$ as $m,n\to\infty$. Therefore, $\{\upsilon_n\}$ converges to 1 with respect to the metric M.

Case 2. Suppose that $\upsilon_n=1$ occurs for only finite number of terms of the sequence, and let there exist an element $\gamma\in\mathbb{X}\setminus\{1\}$ such that $\upsilon_n=\gamma$ for infinitely many terms.

Since $\upsilon_n=1$ happens finitely many times, there exists an integer $N_0\in\mathbb{N}$ such that $\upsilon_n\in\mathbb{X}\setminus\{1\}$ for all $n\ge N_0$. Consequently, for $i,j,k\ge N_0$, the following condition is satisfied:

$$M(\upsilon_i,\upsilon_j,\upsilon_k)=\frac{1}{R}\left[|\upsilon_i-\upsilon_j|+|\upsilon_j-\upsilon_k|+|\upsilon_k-\upsilon_i|\right].$$

Given that $\upsilon_n=\gamma$ for infinitely many terms in the sequence, for any $i\in\mathbb{N}$, there exist integers $j,k\in\mathbb{N}$ such that $k>j>i$ and $\upsilon_j=\upsilon_k=\gamma$.

Therefore, for any integer $i\ge N_0$, we can find $j,k\in\mathbb{N}$ such that $k>j>i$, and the following equation is satisfied:

$$\frac{2}{R}|\upsilon_i-\gamma|=M(\upsilon_i,\upsilon_j,\upsilon_k). \quad (2.30)$$

Given that $\{\upsilon_n\}$ forms a Cauchy sequence, we have $M(\upsilon_i,\upsilon_j,\upsilon_k)\to0$ as $i,j,k\to\infty$. Consequently, $|\upsilon_i-\gamma|\to0$ as $i\to\infty$.

Thus, for $n, m \geq N_0$, we have:

$$M(\upsilon_n, \upsilon_m, \gamma) = \frac{1}{R}\left[|\upsilon_n - \upsilon_m| + |\upsilon_m - \gamma| + |\gamma - \upsilon_n|\right]$$
$$\leq \frac{2}{R}|\upsilon_n - \gamma| + \frac{2}{R}|\upsilon_m - \gamma| \to 0 \quad \text{as } n, m \to \infty. \tag{2.31}$$

Therefore, the sequence $\{\upsilon_n\}$ converges to γ.

Case 3. Suppose $\gamma \in \mathbb{X}$. Assume that $\upsilon_n = \gamma$ occurs for only a finite number of indices n.

Let $\varepsilon > 0$ be given. Choose a positive integer L such that $\frac{1}{L} < \varepsilon$.

Define $N_1 = \max\{n \in \mathbb{N} : \upsilon_n \in \{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{L}\}\}$, where the supremum of an empty set is taken to be zero. Since $\frac{1}{k} \in \mathbb{X}$ for any $k \in \mathbb{N}$, and $\upsilon_n = \frac{1}{k}$ only for finitely many n, the value N_1 is well-defined as a nonnegative integer.

Let $N = N_1 + 1$. Since the set $\mathbb{X}$ is $\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$, it follows that $0 \leq \upsilon_n < \frac{1}{L} < \varepsilon$ for all $n \geq N$.

Hence, the sequence $\{\upsilon_n\}$ converges to 0 under the standard topology.

For $n, m \geq N$, we have:

$$M(0, \upsilon_n, \upsilon_m) = \frac{1}{R}\left[|0 - \upsilon_n| + |\upsilon_n - \upsilon_m| + |\upsilon_m - 0|\right]$$
$$\leq \frac{2}{R}|\upsilon_n| + \frac{2}{R}|\upsilon_m| \to 0 \quad \text{as } n, m \to \infty.$$

Thus, the sequence $\{\upsilon_n\}$ converges to 0 under the $MR - metric\ M$. Therefore, in all scenarios, the sequence $\{\upsilon_n\}$ is convergent in $\mathbb{X}$ with respect to M, establishing that $(\mathbb{X}, M)$ is a complete metric space.

Moreover, the sequence $\left\{\frac{1}{n}\right\}$ clearly converges to both 0 and 1 under M.

For any sequence $\{\upsilon_n\}$ in $\mathbb{X}$ that converges to a point $\upsilon_0 \in \mathbb{X} \setminus \{0, 1\}$, it can be shown, following the argument in Example 2.1, that υ_0 is the unique limit of $\{\upsilon_n\}$.

□

Example 2.3. *Let $(\mathbb{X}, M)$ be as in Example 2.2. Define the function $f : \mathbb{X} \to \mathbb{X}$ as*

$$f(\upsilon) = \begin{cases} 0 & \text{if } \upsilon = 1, \\ \frac{\upsilon}{3R} & \text{otherwise.} \end{cases} \tag{2.32}$$

Then, for all $\upsilon, \xi, \Im \in \mathbb{X}$, we have $M(f(\upsilon), f(\xi), f(\Im)) \leq \frac{2}{3R}M(\upsilon, \xi, \Im)$. Clearly, 0 is the unique fixed point of f.

Proof. Let $\upsilon, \xi, \Im \in \mathbb{X}$. Due to the symmetry of M, it is sufficient to consider the following five cases.

Case 1. $\upsilon = \xi = \Im = 1$.

Then,

$$M(f(\upsilon), f(y), f(\Im)) = M(f(1), f(1), f(1))$$
$$= M(0, 0, 0) = 0$$
$$\leq \frac{2}{3R}M(\upsilon, \xi, \Im). \tag{2.33}$$

Case 2. $\upsilon = \xi = 1$ and $\Im \neq 1$.

Then,

$$M(f(\upsilon), f(\xi), f(\Im)) = M(f(1), f(1), f(\Im))$$
$$= M\left(0, 0, \frac{\Im}{3R}\right) \leq \frac{2}{R}\frac{2\Im}{3R}$$
$$= \frac{2}{3R}\frac{2}{R}\min\{1, 1, \Im\}$$
$$\leq \frac{2}{3R}\frac{2}{R}\min\{\upsilon, \xi, \Im\}$$
$$\leq \frac{2}{3R}(\frac{2}{R}) = \frac{2}{3R}M(\upsilon, \xi, \Im). \tag{2.34}$$

Case 3. $\upsilon = 1$, $\xi = 0$ and $\Im \neq 1$.

Then,

$$M(f(\upsilon), f(\xi), f(\Im)) = M(f(1), f(0), f(\Im))$$
$$= M\left(0, 0, \frac{\Im}{3R}\right) = \frac{1}{R}\frac{2\Im}{3R}$$
$$\leq \frac{2}{3R}(\frac{1}{R}) = \frac{2}{3R}M(\upsilon, \xi, \Im). \tag{2.35}$$

Case 4. $\upsilon = 1$ and $0 < \Im \leq \xi < 1$.

Then,

$$M(f(\upsilon), f(\xi), f(\Im)) = M(f(1), f(\xi), f(\Im))$$
$$= M\left(0, \frac{\xi}{3R}, \frac{\Im}{3R}\right)$$
$$= \frac{1}{R}\left|\frac{\xi}{3R}\right| + \frac{1}{R}\left|\frac{\xi}{R} - \frac{\Im}{R}\right| + \frac{1}{R}\left|\frac{\Im}{3R}\right|$$
$$= \frac{1}{3R}\left[\frac{1}{R}\left[|\xi| + |\xi - \Im| + |\Im|\right]\right]$$
$$= \frac{1}{3R}M(\upsilon, \xi, \Im)$$
$$\leq \frac{2}{3R}M(\upsilon, \xi, \Im). \tag{2.36}$$

Case 5. $\Im \leq \xi \leq \upsilon < 1$.

$$M(f(\upsilon), f(\xi), f(\Im)) = M\left(\frac{\upsilon}{3R}, \frac{\xi}{3R}, \frac{\Im}{3R}\right)$$
$$= \frac{1}{R}\left[\left|\frac{\upsilon}{3R} - \frac{\xi}{3R}\right| + \left|\frac{\xi}{3R} - \frac{\Im}{3R}\right| + \left|\frac{\Im}{3R} - \frac{\upsilon}{3R}\right|\right]$$
$$= \frac{2}{3R}(\upsilon - \Im) \leq \frac{2}{3R}M(\upsilon, \xi, \Im). \tag{2.37}$$

Thus, $M(f(\upsilon), f(\xi), f(\Im)) \leq \frac{2}{3R}M(\upsilon, \xi, \Im)$ for all $\upsilon, \xi, \Im \in \mathbb{X}$.

□

Example 2.4. *Let $(\mathbb{X}, M)$ be as in Example 2.2. Define $f : \mathbb{X} \to \mathbb{X}$ and $g : \mathbb{X} \to \mathbb{X}$ as follows:*

$$f(\upsilon) = \begin{cases} 0 & \text{if } \upsilon = 1, \\ \frac{\upsilon}{3R} & \text{otherwise,} \end{cases} \quad g(\upsilon) = \begin{cases} 1 & \text{if } \upsilon \in \{0, 1\}, \\ \frac{\upsilon}{3R} & \text{otherwise.} \end{cases} \tag{2.38}$$

It can be verified that

$$M(f(\upsilon), f(\xi), f(\Im)) \leq \frac{1}{3R} M(\upsilon, \xi, \Im), \\ M(g(\upsilon), g(\xi), g(\Im)) \leq \frac{1}{3R} M(\upsilon, \xi, \Im) \tag{3.39}$$

for all $\upsilon, \xi, \Im \in \mathbb{X}$.

For any $\upsilon \in \mathbb{X} \setminus \{0, 1\}$*, the sequences* $\{f^n(\upsilon)\}$ *and* $\{g^n(\upsilon)\}$ *are equivalent and have limit points at* 0 *and* 1 *exclusively. In this context,* 0 *represents the fixed point of* f*, while* 1 *serves as the fixed point of* g *within* $\mathbb{X}$.

In the following definition, Shatanawi *[source key 5 pending]* defined a $(c)-$comparison function with base s.

Definition 2.1. *[source key 100 pending] Let* $s \in [1, \infty)$*. A map* $\Psi : [0, +\infty) \to [0, +\infty)$ *is called a* $(c) - comparison$ *function with base* s *if* Ψ *satisfies the following:*

(i) Ψ *is monotone increasing,*

(ii) $\sum_{n=0}^{\infty} s^n \Psi^n(st)$ *converges for all* $t \geq 0$.

If ψ *is a (c)-comparison function, then for all* $t > 0$ *we have* $\psi(t) < t$ *and* $\psi(0) = 0$.

Theorem 2.1. *Let* $(\mathbb{X}, M)$ *be a space equipped with an* $M\mathbb{R}$*-metric. Consider* ψ *as a mapping from* $\mathbb{X}$ *to itself, and let* $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ *be a* $(c)-$*comparison function. Assume that*

$$M(\psi\upsilon, \psi\xi, \psi\Im) \leq \frac{1}{R} \varphi(M(\upsilon, \xi, \Im)) \tag{2.40}$$

For all $\upsilon, \xi, \Im \in \mathbb{X}$*, assume there exists an element* $\upsilon_0 \in \mathbb{X}$ *such that the set* $\{M(\upsilon_0, \psi^i(\upsilon_0), \psi^j(\upsilon_0)) : i, j \in \{0\} \cup \mathbb{N}\}$ *is bounded. In this case, the sequence* $\{\psi^n(\upsilon_0)\}$ *forms an* M*-Cauchy sequence. If* $\{\psi^n(\upsilon_0)\}$ *converges and has a finite number of limit points, then* ψ *admits a unique fixed point in* $\mathbb{X}$*, which is the limit of* $\{\psi^n(\upsilon_0)\}$.

Proof. Define $\upsilon_n = \psi^n(\upsilon_0)$ for every $n \in \mathbb{N}$.

For any nonnegative integers r and T, and for $n \in \mathbb{N}$, utilizing (2.40) and the fact that φ is a non-decreasing function, it follows that

$$\begin{aligned} M(\upsilon_n, \upsilon_{n+r}, \upsilon_{n+s}) &= M(\psi^n\upsilon_0, \psi^{n+r}\upsilon_r, \psi^{n+s}\upsilon_s) \\ &\leq \frac{1}{R}\varphi\left(\frac{1}{R} M(\psi^{n-1}\upsilon_0, \psi^{n-1+r}\upsilon_r, \right. \\ &\quad \left. \psi^{n-1+s}\upsilon_s)\right) \\ &\leq \frac{1}{R}\varphi\left(\varphi\left(M(\psi^{n-2}\upsilon_0, \psi^{n-2+r}\upsilon_r, \right.\right. \\ &\quad \left.\left. \psi^{n-2+s}\upsilon_s)\right)\right) \\ &\vdots \\ &\leq \varphi^n\left(M(\upsilon_0, \upsilon_r, \upsilon_s)\right) \end{aligned} \tag{2.41}$$

Since the set $\{M(\upsilon_0, \psi^i(\upsilon_0), \psi^j(\upsilon_0)) : i, j \in \{0\} \cup \mathbb{N}\}$ is bounded, there exists a constant $L > 0$ such that $M(\upsilon_0, \psi^i(\upsilon_0), \psi^j(\upsilon_0)) \leq L$ for all $i, j \in \{0\} \cup \mathbb{N}$. Using inequality (2.41) along with the non-decreasing property of φ, we find that $M(\upsilon_n, \upsilon_{n+r}, \upsilon_{n+s}) \leq \varphi^n(L)$. Since $\varphi^n(L) \to 0$ as $n \to \infty$, it follows that the sequence $\{\upsilon_n\}$ is M-Cauchy.

Now, assume that $\{\psi^n(\upsilon_0)\}$ converges and possesses only finitely many limit points. Denote by T the set of all such limits. Then T is a finite, non-empty subset of $\mathbb{X}$. Given that φ is a (c)-comparison function.

Let $\Im \in T$. Then, it follows that

$$\begin{aligned} M(\psi^{m+1}\upsilon_0, \psi^{m+1}\upsilon_0, \psi\Im) &\leq \frac{1}{R}\varphi(M(\psi^n\upsilon_0, \psi^m\upsilon_0, \Im)) \\ &\leq M(\psi^n\upsilon_0, \psi^m\upsilon_0, \Im) \to 0 \qquad (3) \\ &\text{as} \quad m, n \to \infty. \end{aligned} \tag{2.42}$$

Thus, $\{\psi^n\upsilon_0\}$ also converges to $\psi\Im$.

Therefore, $\psi\Im \in T$. This shows that $\psi(T) \subseteq T$. For any positive integer m, we have $\psi^m(T) \subseteq \psi^{m-1}(T)$, where $\psi^0 = I$, the identity map on $\mathbb{X}$.

Suppose that m_0 is a positive integer such that $\psi^{m_0-1}(T)$ contains at least two distinct elements. Let ϑ denote the infimum of the set

$$\{M(\upsilon, \xi, \Im) : \upsilon, \xi, \Im \in \psi^{m_0-1}(S) \text{ and at least two of } \upsilon, \xi, \Im \text{ are distinct}\}.$$

Since $\psi^{m_0-1}(T)$ is a finite set with at least two elements, the set $\{M(\upsilon, \xi, \Im) : \upsilon, \xi, \Im \in \psi^{m_0-1}(T)$ and at least two of $\upsilon, \xi, \Im$ are distinct$\}$ is a nonempty finite subset of $(0, \infty)$. Therefore, there exist $\kappa, \upsilon, \Im \in \psi^{m_0-1}(T)$ such that at least two of them are distinct and

$$M(\kappa, \upsilon, \Im) = \vartheta.$$

Now, we have the following inequality:

$$\begin{aligned} M(\psi\kappa, \psi\upsilon, \psi\Im) &\leq \frac{1}{R}\varphi(M(\kappa, \upsilon, \Im)) \\ &\leq \varphi(M(\kappa, \upsilon, \Im)) \\ &= \varphi(\vartheta) \\ &< \vartheta, \end{aligned}$$

Since ϑ is a positive real number and $\varphi(t) < t$ for all $t > 0$, the inequality holds.

Given that $\kappa, \upsilon, \Im \in \psi^{m_0-1}(T)$, it follows that $\psi\kappa, \psi\upsilon, \psi\Im \in \psi^{m_0}(T) \subseteq \psi^{m_0-1}(T)$. By the definition of ϑ, this implies $\psi\kappa = \psi\upsilon = \psi\Im$. Since $\psi^{m_0-1}(T)$ is a finite set, and at least two of $\kappa, \upsilon, \Im$ are distinct while satisfying $\psi\kappa = \psi\upsilon = \psi\Im$, it follows that $\psi^{m_0}(T)$ is a strict subset of $\psi^{m_0-1}(T)$. As T is finite, this leads to the conclusion that $\psi^{n_0-1}(T)$ becomes a singleton for some $n_0 \in \mathbb{N}$.

Let $\psi^{n_0-1}(T) = \{w\}$. Since $\psi^n(T) \subseteq \psi^{n_0-1}(T)$ for all $n \geq n_0$, we conclude that $\psi w = w$.

□

Remark 2.1. *Example 2.3 and Example 2.4 illustrate Theorem 2.1.*

Theorem 2.2. *Let $(\mathbb{X}, M)$ be a bounded, complete MR-metric space, and let ψ and T be self-maps defined on $\mathbb{X}$. Suppose that $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a (c)-comparison function. Furthermore, assume that ψ and T partially commute, meaning $\psi(\mathbb{X}) \subseteq T(\mathbb{X})$, and that for any $\upsilon, \xi, \Im \in \mathbb{X}$, the following inequality holds:*

$$M(\psi\upsilon, \psi\xi, \psi\Im) \leq \frac{1}{R}\varphi(M(T\upsilon, Ty, Tz)) \quad (2.43)$$

Furthermore, assume that every M-Cauchy sequence within $\psi(\mathbb{X})$ converges in $T(\mathbb{X})$ and has a unique limit in $T(\mathbb{X})$. Under these conditions, ψ and T possess a unique shared fixed point in $\mathbb{X}$.

Proof. Let $\upsilon_0 \in \mathbb{X}$. Given that $\psi(\mathbb{X}) \subseteq T(\mathbb{X})$, there exists a sequence $\{\upsilon_n\}$ in $\mathbb{X}$ such that for each $n = 0, 1, 2, \ldots$, the relationship $\psi\upsilon_n = T\upsilon_{n+1}$ holds.

Define $\xi_n = \psi\upsilon_n$. Since $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a non-decreasing function on $\mathbb{R}^+$, and $\{\varphi^n(t)\} \to 0$ as $n \to \infty$ for every $t \in (0, \infty)$, it follows that $\varphi(t) < t$ for all $t \in (0, \infty)$. Therefore, $\varphi(0^+) = 0$.

For positive integers m, n, p, applying inequality (2.43) yields

$$\begin{aligned} M(\xi_n, \xi_m, \xi_p) &= M(\psi\upsilon_n, \psi\upsilon_m, \psi\upsilon_p) \\ &\leq \frac{1}{R}\varphi(M(T\upsilon_n, T\upsilon_m, T\upsilon_p)) \\ &\leq \varphi(M(T\upsilon_n, T\upsilon_m, T\upsilon_p)) \\ &= \varphi(M(\xi_{n-1}, \xi_{m-1}, \xi_{p-1})) \quad (2.44) \end{aligned}$$

which simplifies to

$$M(\xi_n, \xi_m, \xi_p) \leq \varphi(M(\xi_{n-1}, \xi_{m-1}, \xi_{p-1})).$$

Thus, for $n \in \mathbb{N}$ and nonnegative integers i and j, by the monotonicity of φ, we get

$$M(\xi_n, \xi_{n+i}, \xi_{n+j}) \leq \varphi^n(M(\xi_0, \xi_i, \xi_j)). \quad (2.45)$$

Let $L = \sup\{M(\xi_0, \xi_i, \xi_j) : i, j \in \mathbb{N} \cup \{0\}\}$. Since $(\mathbb{X}, M)$ is bounded, we have $L < \infty$. From inequality (2.45), it follows that

$$M(\xi_n, \xi_{n+i}, \xi_{n+j}) \leq \varphi^n(L) \quad (2.46)$$

for all $n \in \mathbb{N}$ and $i, j \in \mathbb{N} \cup \{0\}$.

Since $\varphi^n(L) \to 0$ as $n \to \infty$, the previous inequality implies that $\{\xi_n\}$ forms an M-Cauchy sequence. Given that $\{\xi_n\} = \{\psi\upsilon_n\} \subseteq \psi(\mathbb{X})$, and knowing that every M-Cauchy sequence in $\psi(\mathbb{X})$ converges with a unique limit in $T(\mathbb{X})$, there exists a unique element $\upsilon \in T(\mathbb{X})$ such that $\{\xi_n\}$ converges to υ. Moreover, since $\upsilon \in T(\mathbb{X})$, there exists $\kappa \in \mathbb{X}$ such that $T\kappa = \upsilon$.

For positive integers n, m, we have

$$\begin{aligned} M(\xi_n, \xi_m, \psi\kappa) &= M(\psi\upsilon_n, \psi\upsilon_m, \psi\kappa) \\ &\leq \frac{1}{R}\varphi(M(T\upsilon_n, T\upsilon_m, T\kappa)) \\ &\leq \varphi(M(T\upsilon_n, T\upsilon_m, T\kappa)) \\ &= \varphi(M(\xi_{n-1}, \xi_{m-1}, \upsilon)) \quad (2.47) \end{aligned}$$

Given that $\{\xi_n\}$ converges to υ and $\varphi(0^+) = 0$, the previous inequality implies that $\{\xi_n\}$ also converges to $\psi\kappa$. Since υ is the sole limit of $\{\xi_n\}$ in $T(\mathbb{X})$, and $\psi\kappa \in \psi(\mathbb{X}) \subseteq T(\mathbb{X})$, it must be that $\psi\kappa = \upsilon$. Therefore, $\psi\kappa = T\kappa$. Given that ψ and T partially commute and $\psi\kappa = T\kappa$, it follows that $\psi(\psi\kappa) = T(\mathbb{X}\psi\kappa)$.

By setting $\upsilon = \xi = \kappa$ and $\Im = \psi\kappa$ in inequality (2.43), we obtain

$$M(\psi\kappa, \psi\kappa, \psi^2\kappa) \leq \frac{1}{R}\varphi(M(\psi\kappa, \psi\kappa, \psi^2\kappa)) \leq \varphi(M(\psi\kappa, \psi\kappa, \psi^2\kappa)).$$

Since $\varphi(t) < t$ for all $t \in (0, \infty)$, it follows that $M(\psi\kappa, \psi\kappa, \psi^2\kappa) = 0$. Consequently, $\psi^2\kappa = \psi\kappa$, which leads to $T(\psi\kappa) = \psi\kappa$. Hence, $\psi\kappa$ serves as a common fixed point for both ψ and T.

From inequality (2.43), it is evident that ψ and T cannot have more than one common fixed point.

Declaration of Generative AI and AI-assisted Technologies in the Writing Process

During the preparation of this work, the author(s) utilized artificial intelligence (AI) technologies solely for the purpose of linguistic refinement and grammatical correction. Following the application of these tools, the author(s) comprehensively reviewed, verified, and edited the resulting text. The author(s) maintain full accountability and responsibility for the accuracy, integrity, and final content of the publication.

References:

[1] A. Malkawi, A. Rabaiah, W. Shatanawi and A. Talafhah, (2025), MR-metric spaces and an Application, Journal of Interdisciplinary Mathematics, acceptance.

[2] Malkawi, A. A. R. M. (2025). Existence and Uniqueness of Fixed Points in MR-Metric Spaces and Their Applications. EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS, Vol. 18, No. 2, Article Number 6077.

[3] T. Qawasmeh, (H, Ω_b)- Interpolative Contractions in Ω_b- Distance Mappings with Applications, European Journal of Pure and Applied Mathematics, 16(3), 1717-1730, 2023. DOI: https://doi.org/10.29020/nybg.ejpam.v16i3.4819

[4] T. Qawasmeh, H-Simulation functions and Ω_b-distance mappings in the setting of G_b-metric spaces and application. Nonlinear Functional Analysis and Applications,28 (2), 557-570, 2023. DOI: 10.22771/nfaa.2023.28.02.14

[5] Wasfi Shatanawi, Georgeta Maniu, Anwar Bataihah, F.B Ahmad, (2017). Common fixed points for mappings of cyclic form satisfying linear contractive conditions with omega-distance. UPB Scientific Bulletin, Series A: Applied Mathematics and Physics. 79. 11-20.

[6] Anwar Bataihah, Tariq Qawasmeh, and Mutaz Shatnawi, (2022). DISCUSSION ON b-METRIC SPACES AND RELATED RESULTS IN METRIC AND G-METRIC SPACES. 27. 233-247. 10.22771/nfaa.2022.27.02.02.

[7] Qawasmeh, T., Shatanawi, W., Bataihah, A. and Tallafha, A., 2019. Common Fixed Point Results for Rational $(\alpha, \beta)_\phi$-mω Contractions in Complete Quasi Metric Spaces. Mathematics, 7(5), p.392.

[8] Abu-Irwaq, I., Shatanawi, W., Bataihah, A. and Nuseir, I., 2019. Fixed point results for nonlinear contractions with generalized Ω-distance mappings. UPB Sci. Bull. Ser. A, 81(1), pp.57-64.

[9] Bataihah, A., Tallafha, A. and Shatanawi, W., 2020. Fixed point results with Ω-distance by utilizing simulation functions. Italian J. Pure and Appl. Math, 43, pp.185-196.

[10] Wasfi Shatanawi, Anwar Bataihah, Ariana Pitea, (2016). Fixed and common fixed point results for cyclic mappings of Ω-distance. Journal of Nonlinear Science and Applications. 9. 727-735. 10.22436/jnsa.009.03.02

[11] A. Rabaiah, A. Tallafha and W. Shatanawi, *Common fixed point results for mappings under nonlinear contraction of cyclic form in b-Metric Spaces, Advances in mathematics scientific journal, 2021, 26(2), pp. 289–301.*

[12] Abed Al-Rahman M. Malkawi, Fixed Point Theorems for Fuzzy Mappings in Neutrosophic MR-Metric Spaces, Journal of Nonlinear Modeling and Analysis, 2025, acceptance.

[13] Bataihah, A., Qawasmeh, T., Batiha, I. M., Jebril, I. H., and Abdeljawad, T. (2024). GAMMA DISTANCE MAPPINGS WITH APPLICATION TO FRACTIONAL BOUNDARY DIFFERENTIAL EQUATION. Journal of Mathematical Analysis, 15(5), 99-106. https://doi.org/10.54379/jma-2024-5-7.

[14] Hazaymeh, Ayman.A and Anwar Bataihah. ”Neutrosophic fuzzy metric spaces and fixed points for contractions of nonlinear type.” Neutrosophic Sets and Systems 77, 1 (2025). https://digitalrepository.unm.edu/nssjournal/vol77/iss1/6

[15] Malkawi, A. A. R. M., Mahmoud, D., Rabaiah, A. M., Al-Deiakeh, R., and Shatanawi, W. (2024). ON FIXED POINT THEOREMS IN MR-METRIC SPACES. Nonlinear Functional Analysis and Applications, 1125-1136.

[16] Bataihah, A. and Hazaymeh, A., 2025. Quasi Contractions and Fixed Point Theorems in the Context of Neutrosophic Fuzzy Metric Spaces. European Journal of Pure and Applied Mathematics, 18(1), pp.5785-5785.

[17] I. A. Bakhtin, ”The contraction mapping principle in almost metric spaces,” Funct. Anal., vol. 30, pp. 26–37, 1989.

[18] S. Czerwik, ”Contraction mappings in b-metric spaces,” Acta Math. Inform. Univ. Ostrav., vol. 1, pp. 5–11, 1993.

[19] Bataihah, A. and Qawasmeh, T., 2024. A new type of distance spaces and fixed point Results. Journal of Mathematical Analysis, 15(4), pp.81-90.

[20] Bataihah, A., 2024. Some fixed point results with application to fractional differential equation via new type of distance spaces. Results in Nonlinear Analysis, 7, p.202208.

[21] Ganie, A. H., Gheith, N. E. M., Al-Qudah, Y., Ganie, A. H., Sharma, S. K., Aqlan, A. M., & Khalaf, M. M. (2024). An innovative Fermatean fuzzy distance metric with its application in classification and bidirectional approximate reasoning. IEEE Access, 12, 4780-4791. DOI: https://doi.org/10.1109/ACCESS.2023.3348780

[22] Hassan, N., Al-Qudah, Y. Fuzzy parameterized complex multi-fuzzy soft set. Journal of Physics: Conference Series, 2019, 1212(1), 012016 DOI:10.1088/1742-6596/1212/1/012016

[23] Al-Qudah, Y., Jaradat, A., Sharma, S. K., & Bhat, V. K. Mathematical analysis of the structure of one-heptagonal carbon nanocone in terms of its basis and dimension. Physica Scripta, 2024, 99(5), 055252. DOI: https://doi.org/10.1088/1402-4896/ad3add

[24] Al-Qudah, Y., Hamadameen, A.O. Kh, N.A. and Al-Sharqi, F., 2025. A new generalization of interval-valued Q-neutrosophic soft matrix and its applications. International Journal of Neutrosophic Science (IJNS), 25(3), 242-257. DOI: https://doi.org/10.54216/IJNS.250322

[25] Al-Qudah, Y., Al-Sharqi, F., Mishlish, M., & Rasheed, M. M. Hybrid integrated decision-making algorithm based on AO of possibility interval-valued neutrosophic soft settings. International Journal of Neutrosophic Science,2023, 22(3), 84-98. DOI: https://doi.org/10.54216/IJNS.220306

[26] Al-Qudah, Y., Alhazaymeh, K., Hassan, N., Almousa, M., Alaroud, M. Transitive Closure of Vague Soft Set Relations and its Operators. International Journal of Fuzzy Logic and Intelligent Systems, 2022, 22(1), pp. 59–68. DOI: https://doi.org/10.5391/IJFIS.2022.22.1.59

[27] Qoqazeh, H., Al-Qudah, Y., Almousa, M., Jaradat, A. On D-Compact topological spaces. Journal of Applied Mathematics and Informatics, 2021, 39(5-6), pp. 883–894 DOI: https://doi.org/10.14317/jami.2021.883

[28] Al-Qudah Y. A robust framework for the decision-making based on single-valued neutrosophic fuzzy soft expert setting. International Journal of Neutrosophic Science, 2024,23(2): 195-210. Doi: https://doi.org/10.54216/IJNS.230216

[29] Al-qudah, Y., Alaroud, M., Qoqazeh, H.,Alhazmi, S.E., Al-omari, S. Approximate Analytic–Numeric Fuzzy Solutions of Fuzzy Fractional Equations Using a Residual Power Series Approach. Symmetry, 2022, 14(4), 804 DOI:https://doi.org/10.3390/sym14040804

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The authors equally contributed in the present research, at all stages from the formulation of the problem to the final findings and solution.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

No funding was received for conducting this study.

Conflict of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.

Creative Commons Attribution License 4.0 (Attribution 4.0 International, CC BY 4.0)

This article is published under the terms of the Creative Commons Attribution License 4.0
https://creativecommons.org/licenses/by/4.0/deed.en_US