An Enhanced Decomposition Algorithm for Solving Bicriteria Multistage Pentagonal Neutrosophic Transportation Problems

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Abstract: - This paper studies a bicriteria multistage transportation problem under uncertainty, where transportation costs and deterioration are modeled using pentagonal neutrosophic numbers (NNs), while supply and demand are deterministic. The uncertain model is first converted into a crisp bicriteria form using a score function (SF). A parametric search-based approach is then developed to obtain nondominated extreme solutions in the criteria space. The proposed method combines classical transportation techniques with decomposition and bicriteria optimization strategies to handle large-scale problems efficiently. An illustration is provided to demonstrate the applicability and effectiveness of the approach.

Key-Words: - Bicriteria optimization, Multistage transportation problem, Pentagonal neutrosophic numbers, Score function, Decomposition, Objective space.

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1 Introduction

The classical transportation problem was first introduced by [1], focusing on a single objective of minimizing total transportation cost. A significant amount of research has been conducted on fuzzy and neutrosophic optimization models for transportation problems. Multiobjective programming under fuzzy and neutrosophic environments is studied by many authors, [2], [3], [3], [4], [5], [6], [7], [8], [9].

PN is an extension of neutrosophic set theory, designed to represent uncertainty, indeterminacy, and inconsistency in a more flexible and structured way. In addition, several researchers have focused on TPs under uncertainty. An early proposed solution approach for TP is given by [10]. [11] is developed a method for identifying nondominated extreme points in the objective space of multiobjective transportation problems. [12] is applied a compromise programming approach to neutrosophic MOTP. [13] is introduced a parametric approach for fuzzy multiobjective TPs, while, an interval approximation techniques are used in [14]. An intuitionistic fuzzy MOTP is developed in [15]. Under uncertainty, enormous authors are worked on TP, [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], [27], [28], [29], [30].

One of the most important methods for handling complex optimization problems is the decomposition method, which transforms a problem into smaller, manageable subproblems that can be solved independently to obtain the overall solution. This method is addressed by some of authors, [31], [32], [33].

Despite these advances, several gaps remain:
(i) Efficient solution methods for bicriteria multistage neutrosophic transportation problems,
(ii) Systematic generation of nondominated solutions in the objective space,
(iii) Integration of decomposition techniques with parametric strategies for large-scale models, and

(iv) Limited comparative analysis with existing bicriteria optimization methods.

Motivated by these issues, this paper proposes an enhanced decomposition-based framework for solving bicriteria multistage pentagonal neutrosophic transportation problems. The main contributions are: (i) Formulation of a new SVPNBMTP model, (ii) Development of a SF to convert the problem into a deterministic form, (iii) Use of a decomposition approach to generate nondominated solutions, (iv) Introduction of new measures in a pentagonal neutrosophic environment, (v) Objective-space-based search strategy, and (vi) capability to handle large-scale multistage transportation systems.

The remainder of this paper is organized as follows. Section 2 presents preliminaries. Section 3 formulates the model. Section 4 describes the solution methodology. Section 5 presents a numerical example. Section 6 provides a real-life application with advantages and limitations. Section 7 presents and interprets the obtained results, while Section 8 provides the concluding remarks along with possible avenues for future research.

2 Preliminaries

In this section, we present the fundamental concepts of fuzzy sets (FSs), NSs, single-valued NSs, and pentagonal neutrosophic numbers (NNs) along with their basic arithmetic representations.

Definition 1. ([34]). Let $\boldsymbol{R}$ be a universal set. A FS $\widetilde{\boldsymbol{\Upsilon}}$ in $\boldsymbol{R}$ is defined as $\widetilde{\boldsymbol{\Upsilon}} = \{(\boldsymbol{\zeta}, \boldsymbol{\eta}_{\widetilde{\Upsilon}}(\boldsymbol{\zeta})) \mid \boldsymbol{\zeta} \in \boldsymbol{R}\}$, where $\boldsymbol{\eta}_{\widetilde{\Upsilon}}: \boldsymbol{R} \to [\mathbf{0}, \mathbf{1}]$ is called the membership function (MF), which assigns each element $\boldsymbol{\zeta} \in \boldsymbol{R}$ its degree of membership in $\widetilde{\boldsymbol{\Upsilon}}$.

Definition 2. ([35]). A NS $\widetilde{\boldsymbol{\Upsilon}}^{\boldsymbol{NS}}$ is defined as:

$$\widetilde{\boldsymbol{\Upsilon}}^{\boldsymbol{NS}} = \left\langle \boldsymbol{\zeta}: \left(\boldsymbol{\vartheta}_{\widetilde{\Upsilon}^{NS}}(\boldsymbol{x}), \boldsymbol{\pi}_{\widetilde{\Upsilon}^{NS}}(\boldsymbol{x}), \boldsymbol{\xi}_{\widetilde{\Upsilon}^{NS}}(\boldsymbol{x})\right), \boldsymbol{x} \in \boldsymbol{\Omega} \right\rangle,$$

where $\boldsymbol{\Omega}$ denotes the universal set, and $\boldsymbol{\vartheta}_{\widetilde{\Upsilon}^{NS}}(\boldsymbol{x})$, $\boldsymbol{\pi}_{\widetilde{\Upsilon}^{NS}}(\boldsymbol{x})$, and $\boldsymbol{\xi}_{\widetilde{\Upsilon}^{NS}}(\boldsymbol{x})$ signify the truth, indeterminacy (hesitation), and falsity MF, respectively, such that $\mathbf{0}^- \leq \boldsymbol{\vartheta}_{\widetilde{\Upsilon}^{NS}}(\boldsymbol{x}) + \boldsymbol{\pi}_{\widetilde{\Upsilon}^{NS}}(\boldsymbol{x}) + \boldsymbol{\xi}_{\widetilde{\Upsilon}^{NS}}(\boldsymbol{x}) \leq \mathbf{3}^+$.

Definition 3. A single-valued NS (SVNS) is defined as

$$\widetilde{\boldsymbol{\Upsilon}}^{\boldsymbol{NS}} = \langle \boldsymbol{x}: (\boldsymbol{\vartheta}_{\widetilde{\Upsilon}^{NS}}(\boldsymbol{x}), \boldsymbol{\pi}_{\widetilde{\Upsilon}^{NS}}(\boldsymbol{x}), \boldsymbol{\xi}_{\widetilde{\Upsilon}^{NS}}(\boldsymbol{x})), \boldsymbol{x} \in \boldsymbol{\Omega} \rangle,$$

where $\boldsymbol{\vartheta}_{\widetilde{\Upsilon}^{NS}}(\boldsymbol{x})$, $\boldsymbol{\pi}_{\widetilde{\Upsilon}^{NS}}(\boldsymbol{x})$, and $\boldsymbol{\xi}_{\widetilde{\Upsilon}^{NS}}(\boldsymbol{x})$ denote the degrees of truth, indeterminacy, and falsity membership.

Definition 4. A single-valued pentagonal NNs (SVPNN) is defined as

$$\widetilde{\Upsilon}^{PN} = \langle \{(a_1, b_1, c_1, d_1, e_1); \varpi\}, \{(a_2, b_2, c_2, d_2, e_2); \pi\}, \{(a_3, b_3, c_3, d_3, e_3); \xi\} \rangle,$$

where $\varpi, \pi, \xi \in [0,1]$.

The truth, indeterminacy, and falsity MFs are defined on different sub-intervals as follows:

$$\varpi_{\widetilde{\Upsilon}^{PN}}(x) = \begin{cases} \varpi_{l1}(x), & a_1 \leq x \leq b_1, \\ \varpi_{l2}(x), & b_1 \leq x \leq c_1, \\ \varpi, & x = c_1, \\ \varpi_{u1}(x), & c_1 \leq x \leq d_1, \\ \varpi_{u2}(x), & d_1 \leq x \leq e_1, \\ 0, & \text{otherwise} \end{cases}$$

$$\pi_{\widetilde{\Upsilon}^{PN}}(x) = \begin{cases} \pi_{l1}(x), & a_2 \leq x \leq b_2, \\ \pi_{l2}(x), & b_2 \leq x \leq c_2, \\ \pi, & x = c_2, \\ \pi_{u1}(x), & c_2 \leq x \leq d_2, \\ \pi_{u2}(x), & d_2 \leq x \leq e_2, \\ 0, & \text{otherwise} \end{cases}$$

$$\xi_{\widetilde{\Upsilon}^{PN}}(x) = \begin{cases} \xi_{l1}(x), & a_3 \leq x \leq b_3, \\ \xi_{l2}(x), & b_3 \leq x \leq c_3, \\ \xi, & x = c_3, \\ \xi_{u1}(x), & c_3 \leq x \leq d_3, \\ \xi_{u2}(x), & d_3 \leq x \leq e_3, \\ 0, & \text{otherwise} \end{cases}$$

Definition 5. ([36]). For a SVPNN $\widetilde{\boldsymbol{\Upsilon}}^{\boldsymbol{PN}} = (\boldsymbol{\nu_1}, \boldsymbol{\nu_2}, \boldsymbol{\nu_3}, \boldsymbol{\nu_4}, \boldsymbol{\nu_5}; \boldsymbol{\varpi}, \boldsymbol{\pi}, \boldsymbol{\xi})$, the score and accuracy functions are defined as:

$$SC(\widetilde{\Upsilon}^{PN}) = \frac{1}{15}(\nu_1 + \nu_2 + \nu_3 + \nu_4 + \nu_5)(2 + \varpi - \pi - \xi)$$

Accuracy function:

$$AC(\widetilde{\Upsilon}^{PN}) = \frac{1}{15}(\nu_1 + \nu_2 + \nu_3 + \nu_4 + \nu_5)(2 + \varpi - \pi)$$

Fgure 1 describes the graphical representation of the PN.

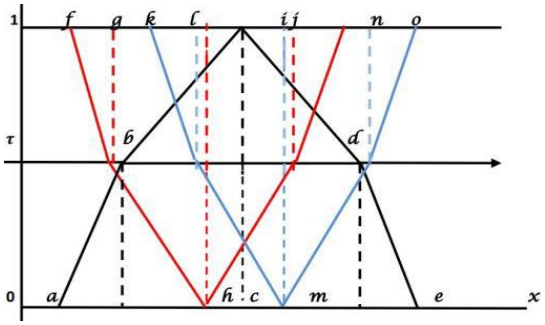


Fig. 1: Illustration of a linear PN structure
Source: [36]

3 Problem Formulation and Solution Concepts

3.1 Construction of the Model and Results

To construct the proposed mathematical model, we first introduce the required notation. The parameter $\tilde{c}_{ij}^{PN}$represents the unit transportation cost for shipping goods from the source $i$to destination j, while $\tilde{d}_{ij}^{PN}$ denotes the corresponding unit deterioration cost incurred during transportation. Further, $\varrho_i$and ς_j indicate the available supply at source $i$and the required demand at destination j, respectively. The decision variable x_{ij}defines the quantity of goods transported from $i$to j.

Based on these definitions, the bicriteria transportation model is formulated as:

$$\text{Min } \tilde{F}_1^{PN} = \sum_{i=1}^{m} \sum_{j=1}^{n} \tilde{c}_{ij}^{PN}\, x_{ij}$$

$$\text{Min } \tilde{F}_2^{PN} = \sum_{i=1}^{m} \sum_{j=1}^{n} \tilde{d}_{ij}^{PN}\, x_{ij}$$

Subject to:

$$\sum_{j=1}^{n} x_{ij} = \varrho_i,\ \sum_{i=1}^{m} x_{ij} = \varsigma_j, x_{ij} \geq 0 \qquad (1)$$

This formulation describes a bicriteria multistage transportation framework that simultaneously accounts for transportation cost and deterioration effects under uncertainty. For computational tractability, the model is further treated within a bicriteria optimization structure suitable for deriving efficient solution procedures.

3.2 Pentagonal Neutrosophic Bicriteria Multistage Transportation Problems of the Second Kind (PNBMTP 2)

In this model, the multistage transportation system is structured such that each stage operates independently, i.e., decisions taken at one stage do not influence the transportation activities at other stages. Accordingly, the overall problem is decomposed into M separate stages, where each stage is treated as an individual bicriteria transportation problem.

For the r^{th}stage, where$r = 1,2, \dots, M$, the following notations are introduced. The term $(\tilde{c}_{i_r j_r}^{r})^{PN}$ represents the unit transportation cost from the source $i_r$to destination j_r, while $(\tilde{d}_{i_r j_r}^{r})^{PN}$ denotes the corresponding unit deterioration cost during transportation. The available supply at source i_r is given by $\varrho_{i_r}^{r}$, and the demand at the destination j_r is represented by $\varsigma_{j_r}^{r}$. The decision variable $x_{i_r j_r}^{r}$ defines the quantity transported from $i_r$to $j_r$in stage r.

Based on these definitions, each stage can be formulated as a bicriteria transportation problem:

$$\min (\tilde{F}_1^{r})^{PN} = \sum_{i_r=1}^{m_r} \sum_{j_r=1}^{n_r} (\tilde{c}_{i_r j_r}^{r})^{PN}\, x_{i_r j_r}^{r}$$

$$\min (\tilde{F}_2^{r})^{PN} = \sum_{i_r=1}^{m_r} \sum_{j_r=1}^{n_r} (\tilde{d}_{i_r j_r}^{r})^{PN}\, x_{i_r j_r}^{r}$$

Subject to:

$$\sum_{j_r=1}^{n_r} x_{i_r j_r}^{r} = \varrho_{i_r}^{r},\ \sum_{i_r=1}^{m_r} x_{i_r j_r}^{r} = \varsigma_{j_r}^{r},$$
$$x_{i_r j_r}^{r} \geq 0 \qquad (2)$$

Using Definition 5, the neutrosophic model can be transformed into an equivalent deterministic form. Accordingly, the objective functions become:

$$\min F_1^{r} = \sum_{i_r=1}^{m_r} \sum_{j_r=1}^{n_r} c_{i_r j_r}^{r}\, x_{i_r j_r}^{r}$$

$$\min F_2^{r} = \sum_{i_r=1}^{m_r} \sum_{j_r=1}^{n_r} d_{i_r j_r}^{r}\, x_{i_r j_r}^{r} \qquad (3)$$

Subject to constraints in (2)

Thus, the original multistage problem is decomposed into M independent bicriteria transportation problems. The overall nondominated solution is obtained by aggregating the optimal solutions of all individual stages. Finally, the total minimum pentagonal neutrosophic transportation

cost is expressed as:$\tilde{F}_1^{PN*} = \sum_{r=1}^{M} (\tilde{F}_1^r)^{PN}$. Similarly, the total minimum pentagonal neutrosophic deterioration is:
$\tilde{F}_2^{PN*} = \sum_{r=1}^{M} (\tilde{F}_2^r)^{PN}$.

3.3 Pentagonal Neutrosophic Bicriteria Multistage Transportation Problems of the Third Kind (PNBMTP 3)

This class of problems extends the bicriteria multistage transportation framework by incorporating additional operational constraints at intermediate stages. These supplementary restrictions do not modify the basic structure of the bicriteria transportation problem at each individual stage, but they impose further feasibility conditions across stages.

The proposed mathematical formulation is given as follows:

$$\min \tilde{F}_1^{PN} = \sum_{r=1}^{M} \sum_{i_r=1}^{m_r} \sum_{j_r=1}^{n_r} (\tilde{c}_{i_r j_r}^{r})^{PN} x_{i_r j_r}^{r}$$

$$\min \tilde{F}_2^{PN} = \sum_{r=1}^{M} \sum_{i_r=1}^{m_r} \sum_{j_r=1}^{n_r} (\tilde{d}_{i_r j_r}^{r})^{PN} x_{i_r j_r}^{r}$$

Subject to:

$$\sum_{j_r=1}^{n_r} x_{i_r j_r}^{r} = \varrho_{i_r}^{r}, \sum_{i_r=1}^{m_r} x_{i_r j_r}^{r} = \varsigma_{j_r}^{r}$$

$$Z_{k_r}\left(x_{i_{(r-1)}^{j}(r-1)}^{(r-1)}, x_{i_r j_r}^{r}, x_{i_{(r+1)}^{j}(r+1)}^{(r+1)}\right) = b^{o}$$

$$x_{i_r j_r}^{r} \geq 0 \qquad (4)$$

Here, the function:

$$Z_{k_r}\left(x_{i_{(r-1)}^{j}(r-1)}^{(r-1)}, x_{i_r j_r}^{r}, x_{i_{(r+1)}^{j}(r+1)}^{(r+1)}\right) = b^{o}, r = 1,2,\dots,M,$$

Represents the additional linear restrictions imposed at each stage. The index $k_r$denotes the number of such constraints at the r^{th} stage. This extended formulation captures inter-stage operational limitations while preserving the bicriteria structure within each stage. Due to the decomposable nature of transportation problems, the model can be efficiently solved using a decomposition-based approach, which facilitates the determination of nondominated solutions in the objective space.

3.4 Pentagonal Neutrosophic Bicriteria Multistage Transportation Problems of the Fourth Kind (PNBMTP 4)

This formulation considers a more advanced multistage transportation structure in which the difference between incoming and outgoing commodity flows at each source of intermediate stages is explicitly known. In addition, the model incorporates inter-stage transportation restrictions similar to previous cases; however, these constraints do not alter the bicriteria transportation formulation at each individual stage.

The mathematical formulation of the problem is given as:

$$\min \tilde{F}_1^{PN} = \sum_{r=1}^{M} \sum_{i_r=1}^{m_r} \sum_{j_r=1}^{n_r} (\tilde{c}_{i_r j_r}^{r})^{PN} x_{i_r j_r}^{r}$$

$$\min \tilde{F}_2^{PN} = \sum_{r=1}^{M} \sum_{i_r=1}^{m_r} \sum_{j_r=1}^{n_r} (\tilde{d}_{i_r j_r}^{r})^{PN} x_{i_r j_r}^{r}$$

Subject to:

$$\sum_{j_r=1}^{n_r} x_{i_r j_r}^{r} = \varrho_{i_1}^{1},$$

$$\sum_{i_{r-1}=1}^{m_{r-1}} x_{i_{r-1} j_{r-1}}^{(r-1)} - \sum_{j_r=1}^{n_r} x_{i_r j_r}^{r} = \varsigma_{j_{r-1}}^{(r-1)},$$

$$\sum_{j_M=1}^{m_M} x_{i_M j_M}^{M} = \varsigma_{j_M}^{N}, x_{i_r j_r}^{r} \geq 0. \qquad (5)$$

Problem (5) can be solved using standard solution procedures for multiobjective linear programming models.

To facilitate further analysis, we now express the model in matrix form.

Let E^rdenote the technological matrix of the r^{th}stage, where $E^r$is an $(m_r + n_r) \times (m_r + n_r)$matrix. Here, M is the total number of stages, m_r represents the number of sources, and n_r represents the number of destinations at stage r.

Let b^rbe the resource vector corresponds to the r^{th}subproblem, where $b^r$is an $(m_r + n_r) \times 1$ column vector. Then the independent constraints can be written compactly as:

$$E^r x^r = b^r, r = 1,2,\dots,M$$

where $x^r$is the decision vector of size $(m_r \times n_r) \times 1$.

For coupling (column) constraints, let A^rdenote the technological matrix of the r^{th}stage, where $A^r$is an $m_0 \times (m_r \times n_r)$matrix, and m_0 represents the number of common constraints. Let b^obe the

corresponding common resource vector of dimension $m_0 \times 1$. Then the inter-stage relationship is expressed as:

$$A^1x^1 + A^2x^2 + \cdots + A^rx^r + \cdots + A^Mx^M = b^o$$

Objective Function Representation
For each stage, define:

- c^r: coefficient vector of the first objective, of size $1 \times (m_r \times n_r)$
- d^r: coefficient vector of the second objective, of size $1 \times (m_r \times n_r)$

Master Problem Structure
Let:

- B: basis matrix corresponding to the current basic solution, of size $(m_0 \times M) \times (m_0 + M)$
- C_B: row vector of basic cost coefficients in the objective function, of size $1 \times (m_0 + M)$
- R_0: matrix of size $(m_0 + M) \times m_0$, consisting of the first m_0 columns of B^{-1}
- u_j: the $(m_0+j)^{th}$ column of matrix D^{-1}

This formulation provides a structured representation of PNBMTP4, incorporating both stage-wise independence and inter-stage coupling effects. The resulting model can be efficiently handled using decomposition-based approaches, which allow systematic generation of nondominated solutions in the objective space.

4 Solution Procedure for PNBMTP 3

To identify all nondominated extreme points in the objective space, a two-phase solution approach is employed. The correctness of the procedure is supported by the following theorem.

Theorem 1 ([37]). A point $F^{(p)} = (F_1^{(p)}, F_2^{(p)})$ is a nondominated extreme point in the objective space if and only if it is generated by the proposed algorithm.

Phase I: Generation of Nondominated Solutions
Step 1: Solve the problem sequentially to obtain extreme solutions. First compute: $F_1^{(1)} = \min(F_1 : x \in S)$ then evaluate $F_2^{(1)} = min(F_2 \mid F_1 = F_1^{(1)} : x \in S)$ Record $\left(F_1^{(1)}, F_2^{(1)}\right)$ and set $p = 1$. Next, compute: $F_2^{(2)} = \min(F_2 : x \in S)$ and $F_1^{(2)} = \min(F_1 \mid F_2 = F_2^{(2)} : x \in S)$. If $\left(F_1^{(2)}, F_2^{(2)}\right) = \left(F_1^{(1)}, F_2^{(1)}\right)$, stop; otherwise, record it and set $p = 2$. Initialize the sets:

$$U = \{(1,2)\}, \Delta = \emptyset$$

Step 2: Select a pair $(\omega, \psi) \in U$and define weighting parameters: $\upsilon_1^{(\omega,\psi)} = \mid F_2^{(\psi)} - F_2^{(\omega)} \mid$, $\upsilon_2^{(\omega,\psi)} = \mid F_1^{(\psi)} - F_1^{(\omega)} \mid$. Solve the weighted single objective problem:

$$\sum_{r=1}^{M} \sum_{i_r j_r} \left(\upsilon_1^{(\omega,\psi)} c_{i_r j_r}^r + \upsilon_2^{(\omega,\psi)} d_{i_r j_r}^r\right) x_{i_r j_r}^r.$$

If multiple optimal solutions exist, select one that minimizes: $\sum_{r=1}^{M} \sum_{i_r j_r} c_{i_r j_r}^r \, x_{i_r j_r}^r$. Let the corresponding objective values be: $\hat{F}_1, \hat{F}_2$ If $\left(\hat{F}_1, \hat{F}_2\right)$ coincides with either $\left(F_1^{(\omega)}, F_2^{(\omega)}\right)$or $\left(F_1^{(\psi)}, F_2^{(\psi)}\right)$, update Γand proceed. Otherwise, store the new solution and update the set U.
Step 3: Update the pair set U. If $U = \emptyset$, terminate; otherwise return to Step 2.

Phase II: Decomposition-Based Optimization
Step 1: Transform the original model into a reduced form using new variables ϕ^r.
Step 2: Determine an initial basic feasible solution for the transformed system.
Step 3: Solve subproblems of the form:

$$\Psi^r = (c^r - C_B R_0 A^r) x^r$$

Subject to: $E^r x^r = b^r, x^r \geq 0$. Let $\bar{\Psi}^r$ be the optimal value of the subproblem r.
Step 4: Compute: $\xi^r = \bar{\Psi}^r - C_B u^r$and define $\xi = \min \xi^r$. If $\xi \geq 0$, the current solution is optimal, and the procedure terminates with:

$$x^r = \sum_U \phi_U^r \, x_U^r, r = 1,2, \ldots, M$$

Otherwise, proceed to the next step.
Step 5: Introduce the entering variable corresponding to ξ into the basis, determine the leaving variable using feasibility conditions, and update the basis using the revised simplex method. Return to Step 3.

5 Numerical Example

Consider the following PNBMTP 1 with data inserted in Table 1, Table 2, Table 3 and Table 4.

Table 1. PN data of the first stage problem

	D_1^1	D_2^1	Supply
S_1^1	$\tilde{c}_{11}^{1\,PN}$	$\tilde{c}_{12}^{1\,PN}$	
	$\tilde{c}_{11}^{2\,PN}$	$\tilde{c}_{12}^{2\,PN}$	a_1^1
S_2^1	$\tilde{c}_{21}^{1\,PN}$	$\tilde{c}_{22}^{1\,PN}$	
	$\tilde{c}_{21}^{2\,PN}$	$\tilde{c}_{22}^{2\,PN}$	a_2^1
S_3^1	$\tilde{c}_{31}^{1\,PN}$	$\tilde{c}_{32}^{1\,PN}$	
	$\tilde{c}_{31}^{2\,PN}$	$\tilde{c}_{32}^{2\,PN}$	a_3^1
Demand	b_1^1	b_2^1	

Source: created by the authors

Table 2. PN data of the second stage problem

	D_1^2	D_2^2	D_3^2	Supply
S_1^2	$\tilde{d}_{11}^{1\,PN}$	$\tilde{d}_{12}^{1\,PN}$	$\tilde{d}_{13}^{1\,PN}$	
	$\tilde{d}_{11}^{2\,PN}$	$\tilde{d}_{12}^{2\,PN}$	$\tilde{d}_{12}^{2\,PN}$	a_1^2
S_2^2	$\tilde{d}_{21}^{1\,PN}$	$\tilde{d}_{22}^{1\,PN}$	$\tilde{d}_{23}^{1\,PN}$	
	$\tilde{d}_{21}^{2\,PN}$	$\tilde{d}_{22}^{2\,PN}$	$\tilde{d}_{22}^{2\,PN}$	a_2^2
Demand	b_1^2	b_2^2	b_3^2	

Source: created by the authors

where,
$\tilde{c}_{11}^{1\,PN} = (0.5, 1, 1.5, 3, 4;\ 1,0,0\ )$,
$\tilde{c}_{12}^{1\,PN} = (0, 0.5, 1, 1.5,2, 1;\ 1,0,0\ ) = \tilde{c}_{31}^{1\,PN}$,
$\tilde{c}_{21}^{1\,PN} = (1, 2, ,3, 4, 5, 1;\ 1,0,0\ )$,
$\tilde{c}_{22}^{1\,PN} = (2, 3, 4, 5, 6, 1;\ 1,0,0\ ) = \tilde{c}_{32}^{1\,PN}$,
$\tilde{c}_{11}^{2\,PN} = (0, 0.5, 1, 1.5,2, 1;\ 1,0,0\ ) = \tilde{c}_{32}^{2\,PN}$,
$\tilde{c}_{12}^{2\,PN} = (1, 2, ,3, 4, 5, 1;\ 1,0,0\ ) = \tilde{c}_{21}^{2\,PN}$,
$\tilde{c}_{22}^{2\,PN} = (2, 3, 4, 5, 6, 1;\ 1,0,0\ )$
$\tilde{c}_{31}^{2\,PN} = (0.5, 1, 1.5, 3, 4;\ 1,0,0\ )$,
$\tilde{c}_{11}^{1\,PN} = (0.5, 1, 1.5, 3, 4;\ 1,0,0\ )$,
$\tilde{c}_{12}^{1\,PN} = (0, 0.5, 1, 1.5,2, 1;\ 1,0,0\ ) = \tilde{c}_{31}^{1\,PN}$,
$\tilde{c}_{21}^{1\,PN} = (1, 2, ,3, 4, 5, 1;\ 1,0,0\ )$,
$\tilde{c}_{22}^{1\,PN} = (2, 3, 4, 5, 6, 1;\ 1,0,0\ ) = \tilde{c}_{32}^{1\,PN}$
By applying the SFs to the data, we have

Table 3. PN data of the first stage problem

	D_1^1	D_2^1	Supply
S_1^1	2	1	
	1	3	10
S_2^1	3	4	
	3	4	5
S_3^1	1	4	
	2	1	3
Demand	12	6	18

Source: created by the authors

Table 4. Pentagonal neutrosophic data of the second-stage problem

	D_1^2	D_2^2	D_3^2	Supply
S_1^2	1	4	2	
	2	1	3	12
S_2^2	3	3	1	
	4	3	4	6
Demand	7	3	8	18

Source: created by the authors

It is required that the quantity shipped from source 1 to destination 1 in the first stage is equal to the quantity shipped from source 1 to destination 1 in the second stage.

The mathematical model of the problem is:

$$\text{Min } F_1 = 2x_{11}^1 + x_{12}^1 + 3x_{21}^1 + 4x_{22}^1 + x_{31}^1 + 4x_{32}^1 + x_{11}^2 + 4x_{12}^2 + 3x_{21}^2 + 3x_{22}^2 + 2x_{13}^2 + 4x_{23}^2$$

$$\text{Min } F_2 = x_{11}^1 + 3x_{12}^1 + 3x_{21}^1 + 4x_{22}^1 + 2x_{31}^1 + x_{32}^1 + 2x_{11}^2 + x_{12}^{12} + 4x_{21}^2 + 3x_{22}^2 + 3x_{13}^2 + 4x_{23}^2$$

Subject to

$$x_{11}^1 + x_{11}^2 = 0,$$
$$x_{11}^1 + x_{12}^1 = 10,$$
$$x_{21}^1 + x_{22}^1 = 5,$$
$$x_{31}^1 + x_{32}^1 = 3,$$
$$x_{11}^1 + x_{21}^1 + x_{31}^1 = 12,$$
$$x_{12}^1 + x_{22}^1 + x_{32}^1 = 6,$$
$$x_{i_1 j_1}^1 \geq 0; \forall\, i_1 = 1,2,3;\ j_1 = 1,2,$$
$$x_{i_2 j_2}^2 \geq 0; \forall\, i_2 = 1,2;\ j_2 = 1,2,3.$$

The problem can be decomposed into two subproblems:

The first subproblem:

$$x^1 = [x_{11}^1\ \ x_{12}^1\ \ x_{21}^1\ \ x_{22}^1\ \ x_{31}^1\ \ x_{32}^1]^T,$$

$$c^1 = [2\ \ 1\ \ 3\ \ 4\ \ 1\ \ 4], d^1 = [1\ \ 3\ \ 3\ \ 4\ \ 2\ \ 1]$$

$$E^1 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix},$$

$$A^1 = [1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0],$$

$$b^1 = \begin{bmatrix} 10 \\ 5 \\ 3 \\ 12 \\ 6 \end{bmatrix}.$$

The second subproblem

$x^2 = [x_{11}^2 \;\; x_{21}^2 \;\; x_{12}^2 \;\; x_{22}^2 \;\; x_{31}^2 \;\; x_{32}^2]^T$,

$c^2 = [1 \;\; 3 \;\; 4 \;\; 3 \;\; 2 \;\; 1]$,

$d^2 = [2 \;\; 4 \;\; 1 \;\; 3 \;\; 3 \;\; 4]$

$$\mathrm{E}^2 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix},$$

$$A^2 = [-1 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0],$$

$$b^2 = \begin{bmatrix} 7 \\ 3 \\ 8 \\ 12 \\ 6 \end{bmatrix}.$$

Then

$\mathrm{F}_1^{(1)} = 67,\ \mathrm{F}_2^{(1)} = 87, \mathrm{F}_1^{(2)} = 73,\ \mathrm{F}_2^{(2)} = 81$.

With a^1 and a^2 , we have $\check{\mathrm{F}}_1 = 73, \check{\mathrm{F}}_2 = 81$. Which is better than the one obtained by ([11]).

The resulted data generated from the solution approach including the non-dominated extreme points are investigated in the Table 5 (Appendix).

6 Conclusions and Future Works

In this study, a pentagonal neutrosophic bicriteria multistage transportation problem is examined, where costs and deterioration are modeled using pentagonal NNs, while supply and demand are treated as deterministic. An SF is used to manage uncertainty and convert the model into a tractable form. A new solution framework is proposed to generate nondominated extreme solutions in the objective space by integrating classical transportation techniques, decomposition strategies, and bicriteria optimization methods. A numerical example confirms that the method is efficient, systematic, and capable of producing meaningful trade-offs between objectives for multistage decision problems under neutrosophic uncertainty. Future research may extend this work to fully neutrosophic multistage models, problems with more than two objectives, and hybrid or metaheuristic solution techniques. Further extensions may also include stochastic or dynamic frameworks and applications to real-world logistics systems.

Declaration of Generative AI and AI-assisted Technologies in the Writing Process

During the preparation of this work, the author(s) utilized artificial intelligence (AI) technologies solely for the purpose of linguistic refinement and grammatical correction. Following the application of these tools, the author(s) comprehensively reviewed, verified, and edited the resulting text. The author(s) maintain full accountability and responsibility for the accuracy, integrity, and final content of the publication.

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APPENDIX

Table 5. Phase II of the solution procedure

Iteration	x_{basic}	b	B^{-1}	C_B	Ψ^r	x^r	ξ^r, ξ
5^{th}	$\begin{bmatrix} B_4^1 \\ B_3^1 \\ B_2^2 \end{bmatrix}$	$\begin{bmatrix} 3 \\ \overline{5} \\ 2 \\ \overline{5} \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1/5 & -4/5 & 7/5 \\ -1/5 & 9/5 & -7/5 \\ 0 & 0 & 1 \end{bmatrix}$	$[42 \quad 32 \quad 29]$	$\Psi^1 = 24$ $\Psi^2 = 43$	9 1 0 5 3 0 7 3 2 0 0 6	$\xi^1 = 0$ $\xi^2 = 0$ $\xi = 0$

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