

Integration of the Regularised Long Wave Equation via High Order Predictor-Corrector scheme along with Quartic B-spline Finite element

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Abstract: - In this study, an algorithm for integrating the Regularized Long Wave Equation (RLWE) is developed by combining a predictor–corrector scheme with the quartic B-spline Galerkin method. The algorithm is first formulated using the four-step implicit Adams–Bashforth method as the predictor and the four-step implicit Adams–Moulton method as the corrector for time integration of the RLWE. Subsequently, the quartic B-spline Galerkin method is applied to the semi-discrete form of the RLWE. The implementation of the algorithm yields a system of algebraic equations whose solutions are used to obtain both discrete and functional approximations of the RLWE. Highly accurate results are achieved through the application of higher-order schemes to test problems involving solitary wave propagation and undulation.

Key-Words: Multistep predictor-corrector method, Adams-Bashforth-Moulton method, Galerkin method, Quartic B-splines, RLWE equation

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1 Introduction

Since the introduction of the Regularised Long Wave Equation (RLWE) to model both behaviour of the undulation bore and small-amplitude long waves on surface water in the channel, many studies have appeared on finding solutions of the RLWE. The existence and uniqueness of the solution of the RLWE are studied in the paper [1]. The phenomena of the weak nonlinearity and dispersion waves are defined as solution of the RLWE [2]. Thus nonlinear transverse waves in shallow water, ion-acoustic and magneto hydrodynamic waves in plasma and phonon packets in nonlinear crystals are modelled through studying the RLWE. The general solution of The RLWE including nonlinear term does not exist for wide range of boundary and initial conditions. Therefore many wave phenomena are studied through numerical solution of the RLWE.

A large number of computational methods have been developed to obtain numerical solutions of the RLWE. The RLWE is mostly integrated by separating the time and space variables and handling them sequentially either by performing time integration first followed by space integration, or vice versa. One-step time integration methods are commonly employed to advance the solution in time. At each discrete time level, spatial integration of the semi-discrete RLWE provides the solution over the problem domain. B-spline-based finite element methods, in particular, yield smooth solutions, although their accuracy is sometimes limited. Recently, high-order time integration methods have been designed to achieve more

accurate solutions. The method of lines provides improved accuracy for solving the RLWE. In these methods, after performing space integration of the RLWE using finite element, meshless finite element, or differential quadrature methods, the resulting time-discrete system is typically solved using high-order Runge–Kutta methods. A method of lines approach to the numerical solution of the regularized long wave (RLW) is presented. To do that a higher order compact finite difference method in the space is applied and solution of the resulting system was obtained by applying fourth Runge-Kutta method in time discretization [3]. The meshless kernel based method of lines approach is implemented to integrate RLWE together with a variable order Adams-Bashforth-Moulton PECE solver [4]. The distributed approximating functional (DAF) method realized using the regularized Hermite local spectral kernel along with the classical fourth-order Runge-Kutta method, is one of the successful method of lines approaches for obtaining small errors in the solution of the RLWE [5]. High-order multistep methods combined with B-spline finite element techniques have been applied for the time integration of the RLWE. Consequently, high accuracy has been achieved in the numerical solutions of the RLWE. An algorithm based on the two- and three-step Adams–Moulton methods for time integration and a quadratic trigonometric B-spline Galerkin finite element method for spatial discretization has been developed to achieve minimal numerical error in the RLWE solutions [6]. Similar approach to

solving the RLWE was presented using a quartic trigonometric B-spline Galerkin method for space integration together with the one-step and two-step methods for time integration[7]. In another study, fast and accurate discrete Fourier transform with the explicit fourth-order time stepping methods are employed to integrate the RLWE [8]. Further studies have been carried out using predictor–corrector multistep methods for the time integration of the RLWE, which have proven to be very effective for solving this equation. Very recently, a numerical algorithm for solving the RLWE has been developed using a predictor–corrector method in which the fourth-order Adams–Moulton scheme is used as the predictor and the fourth-order Adams–Bashforth scheme as the corrector for time integration, combined with the quintic B-spline collocation finite element procedure in space[9]. A similar study is conducted in which a new numerical algorithm for solving the RLWE is proposed using high-order multistep Adams–Moulton and Adams–Bashforth schemes together with a quartic B-spline collocation method[10]. Consider RLWE:

$$u_t + u_x + \varepsilon uu_x - \mu u_{xxt} = 0, \quad (1)$$

$$x \in [a, b], \quad t \in (0, T]$$

subject to the initial condition (IC):

$$u(x, 0) = f(x), \quad (2)$$

and subject to the boundary conditions (BCs):

$$u(a, t) = \sigma_1, \quad u(b, t) = \sigma_2 \quad (3)$$

where u is the wave amplitude, ε and μ are positive dimensionless parameters.

2 Numerical Method

Consider the spatial domain $[a, b]$ and time interval $[0, T]$. The space-time domain on which the RLWE is posed is $[a, b] \times [0, T]$, partitioned at time grid points $0 = t^0 < t^1 < t^2 < \dots < t^n = T$ with $\Delta t = T/n$ and $t^i = i\Delta t$, and at space grid points $a = x_0 < x_1 < x_2 < \dots < x_m = b$ with $h = (b - a)/m$ and $x_j = x_0 + jh$ as well as fictitious points $x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_{m+1}, x_{m+2}, x_{m+3}, x_{m+4}$ outside the space domain $[a, b]$. The quartic B-splines $Q_i, i = -2, \dots, m + 1$ are

$$Q_i(x) = \frac{1}{h^4} \begin{cases} (x - x_{i-2})^4, & x \in [x_{i-2}, x_{i-1}] \\ (x - x_{i-2})^4 - 5(x - x_{i-1})^4, & x \in [x_{i-1}, x_i] \\ (x - x_{i-2})^4 - 5(x - x_{i-1})^4 + 10(x - x_i)^4, & x \in [x_i, x_{i+1}] \\ (x_{i+3} - x)^4 - (x_{i+2} - x)^4, & x \in [x_{i+1}, x_{i+2}] \\ (x_{i+3} - x)^4, & x \in [x_{i+2}, x_{i+3}] \\ 0, & \text{otherwise} \end{cases} \quad (4)$$

A global approximation $U(x, t)$ is constructed with help of the quartic B-splines:

$$U(x, t) = \sum_{i=-2}^{m+1} \delta_i(t) Q_i(x), \quad (5)$$

where $\delta_i(t)$ are time dependent unknown parameters determined through B-spline Galerkin finite element procedure.

The approximation $U(x, t)$ (5) and its derivatives of up to three degree can be found by using the quartic B-splines at the space grid points:

$$\begin{aligned} U_j &= \delta_{j-2} + 11\delta_{j-1} + 11\delta_j + \delta_{j+1}, \\ hU'_j &= 4(\delta_{j+1} + 3\delta_j - 3\delta_{j-1} - \delta_{j-2}), \\ h^2U''_j &= 12(\delta_{j-1} - \delta_j - \delta_{j+1} + \delta_{j+2}), \\ h^3U'''_j &= 24(\delta_{j+1} - 3\delta_j + 3\delta_{j-1} + \delta_{j-2}). \end{aligned} \quad (6)$$

where $'$ denotes differentiation with respect to the spatial variable x .

To integrate the Eq. (1) in time, the time derivative terms are separated from the space derivative term:

$$(u - \mu u_{xx})_t = -u_x - \varepsilon uu_x. \quad (7)$$

The multistep method

$$\begin{aligned} u^{n+1} &= \theta_1 u^n + \theta_2 u_t^n \\ &+ \theta_3 u_t^{n-1} + \theta_4 u_t^{n-2} \\ &+ \theta_5 u_t^{n-3} + \theta_6 u_t^{n-4} \end{aligned} \quad (8)$$

where $\theta_i = p_i$ and $\theta_i = q_i, 1 \leq i \leq 6$ turns into the prediction equation by choosing the parameters:

$$\begin{aligned} p_1 &= 1, & p_2 &= \frac{55\Delta t}{24}, \\ p_3 &= -\frac{59\Delta t}{24}, & p_4 &= \frac{37\Delta t}{24}, \\ p_5 &= -\frac{9\Delta t}{24}, & p_6 &= 0. \end{aligned} \quad (9)$$

which correspond to the Adams–Bashforth four-step explicit method of order four. The corrector equation is obtained by choosing the parameters in the Eq. (8):

$$\begin{aligned} q_1 &= 1, & q_2 &= \frac{646\Delta t}{720}, \\ q_3 &= -\frac{264\Delta t}{720}, & q_4 &= \frac{106\Delta t}{720}, \\ q_5 &= -\frac{19\Delta t}{720}, & q_6 &= \frac{251\Delta t}{720}. \end{aligned} \quad (10)$$

which correspond to Adams–Moulton four-step implicit method of order five.

We applied the Adams-Bashforth four-step explicit method and the Adams-Moulton four-step implicit method to the RLWE (7) in order to obtain the predictor and corrector equations given below:

$$\begin{aligned} (u^{n+1})^* - \mu(u_{xx}^{n+1})^* \\ = p_1 (u^n - \mu u_{xx}^n) \\ - p_2 (u_x^n + \varepsilon u^n u_x^n) \\ - p_3 (u_x^{n-1} + \varepsilon u^{n-1} u_x^{n-1}) \\ - p_4 (u_x^{n-2} + \varepsilon u^{n-2} u_x^{n-2}) \\ - p_5 (u_x^{n-3} + \varepsilon u^{n-3} u_x^{n-3}) \end{aligned} \quad (11)$$

$$\begin{aligned} u^{n+1} - \mu u_{xx}^{n+1} \\ = q_1 (u^n - \mu u_{xx}^n) \\ - q_2 (u_x^n + \varepsilon u^n u_x^n) \\ - q_3 (u_x^{n-1} + \varepsilon u^{n-1} u_x^{n-1}) \\ - q_4 (u_x^{n-2} + \varepsilon u^{n-2} u_x^{n-2}) \\ - q_5 (u_x^{n-3} + \varepsilon u^{n-3} u_x^{n-3}) \\ - q_6 ((u_x^{n+1})^* + \varepsilon (u^{n+1})^* (u_x^{n+1})^*) \end{aligned} \quad (12)$$

For the application of the Galerkin finite element method, parts of the quartic B-splines falling within the sample interval $[x_i, x_{i+1}]$ are expressed over the interval $[0, h]$ by employing the mapping: $\xi = x - x_i$. This leads to the quartic B-spline shape functions:

$$\begin{aligned} Q_{i-2} &= 1 - 4\frac{\xi}{h} + 6\left(\frac{\xi}{h}\right)^2 - 4\left(\frac{\xi}{h}\right)^3 + \left(\frac{\xi}{h}\right)^4, \\ Q_{i-1} &= 11 - 12\frac{\xi}{h} - 6\left(\frac{\xi}{h}\right)^2 + 12\left(\frac{\xi}{h}\right)^3 - 4\left(\frac{\xi}{h}\right)^4, \\ Q_i &= 11 + 12\frac{\xi}{h} - 6\left(\frac{\xi}{h}\right)^2 - 12\left(\frac{\xi}{h}\right)^3 + 6\left(\frac{\xi}{h}\right)^4, \\ Q_{i+1} &= 1 + 4\frac{\xi}{h} + 6\left(\frac{\xi}{h}\right)^2 + 4\left(\frac{\xi}{h}\right)^3 - \left(\frac{\xi}{h}\right)^4, \\ Q_{i+2} &= \left(\frac{\xi}{h}\right)^4. \end{aligned} \quad (13)$$

Approximate solution (5) over the sample interval $[0, h]$ is defined by the combination of the B-spline shape function as:

$$U^e = U(\xi, t) = \sum_{i=m-2}^{m+2} \delta_i(t) Q_i(\xi) \quad (14)$$

where δ_i are unknown which should be determined from the time-space discrete global equation of the RLW.

The mapping, $x = \xi + x_i, 0 \leq \xi \leq h$ is applied to the predictor-corrector equations (11,12) and then they are multiplied by the B-spline shape functions (13) and integrated from 0 to h to obtain the integral equation.

$$\begin{aligned} \int_0^h \left[(u^{n+1})^* - s_1 (u_{\xi\xi}^{n+1})^* - p_1 u^n \right. \\ + s_2 u_{\xi\xi}^n + s_3 u_{\xi}^n + \varepsilon s_3 u^n u_{\xi}^n \\ + s_4 u_{\xi}^{n-1} + \varepsilon s_4 u^{n-1} u_{\xi}^{n-1} \\ + s_5 u_{\xi}^{n-2} + \varepsilon s_5 u^{n-2} u_{\xi}^{n-2} \\ \left. + s_6 u_{\xi}^{n-3} + \varepsilon s_6 u^{n-3} u_{\xi}^{n-3} \right] W d\xi = 0. \end{aligned} \quad (15)$$

and

$$\begin{aligned} \int_0^h \left[u^{n+1} - \bar{s}_1 u_{\xi\xi}^{n+1} - q_1 u^n \right. \\ + \bar{s}_2 u_{\xi\xi}^n + \bar{s}_3 u_{\xi}^n + \varepsilon \bar{s}_3 u^n u_{\xi}^n \\ + \bar{s}_4 u_{\xi}^{n-1} + \varepsilon \bar{s}_4 u^{n-1} u_{\xi}^{n-1} \\ + \bar{s}_5 u_{\xi}^{n-2} + \varepsilon \bar{s}_5 u^{n-2} u_{\xi}^{n-2} \\ + \bar{s}_6 u_{\xi}^{n-3} + \varepsilon \bar{s}_6 u^{n-3} u_{\xi}^{n-3} \\ \left. + \bar{s}_7 (u_{\xi}^{n+1})^* + \varepsilon \bar{s}_7 (u^{n+1})^* (u_{\xi}^{n+1})^* \right] W d\xi = 0. \end{aligned} \quad (16)$$

$$\begin{aligned} s_1 &= \mu, & s_2 &= p_1 \mu, & s_3 &= p_2, \\ s_4 &= p_3, & s_5 &= p_4, & s_6 &= p_5, \\ \bar{s}_1 &= \mu, & \bar{s}_2 &= q_1 \mu, & \bar{s}_3 &= q_2, \\ \bar{s}_4 &= q_3, & \bar{s}_5 &= q_4, & \bar{s}_6 &= q_5, \\ \bar{s}_7 &= q_6. \end{aligned} \quad (17)$$

The unknown u^n and its derivatives in the integrands of the integral equation (15-16) and the weight function W are replaced by the quartic B-spline approximation (14), its derivatives and the quartic B-spline shape functions (13) to yield a system

of local discrete equations:

$$\sum_{j=m-2}^{m+2} \int_0^h \left[Q_j (\delta_j^{n+1})^* - s_1 Q_j'' (\delta_j^{n+1})^* \right. \\ \left. - p_1 Q_j \delta_j^n + s_2 Q_j'' \delta_j^n \right. \\ \left. + s_3 Q_j' \delta_j^n + \varepsilon s_3 Q_j' \bar{B}^n \delta_j^n \right. \\ \left. + s_4 Q_j' \delta_j^{n-1} + \varepsilon s_4 Q_j' \bar{B}^{n-1} \delta_j^{n-1} \right. \\ \left. + s_5 Q_j' \delta_j^{n-2} + \varepsilon s_5 Q_j' \bar{B}^{n-2} \delta_j^{n-2} \right. \\ \left. + s_6 Q_j' \delta_j^{n-3} \right. \\ \left. + \varepsilon s_6 Q_j' \bar{B}^{n-3} \delta_j^{n-3} \right] Q_i d\xi = 0. \quad (18)$$

$$\sum_{j=m-2}^{m+2} \int_0^h \left[Q_j \delta_j^{n+1} - \bar{s}_1 Q_j'' \delta_j^{n+1} \right. \\ \left. - q_1 Q_j \delta_j^n + \bar{s}_2 Q_j'' \delta_j^n \right. \\ \left. + \bar{s}_3 Q_j' \delta_j^n + \varepsilon \bar{s}_3 Q_j' \bar{B}^n \delta_j^n \right. \\ \left. + \bar{s}_4 Q_j' \delta_j^{n-1} + \varepsilon \bar{s}_4 Q_j' \bar{B}^{n-1} \delta_j^{n-1} \right. \\ \left. + \bar{s}_5 Q_j' \delta_j^{n-2} + \varepsilon \bar{s}_5 Q_j' \bar{B}^{n-2} \delta_j^{n-2} \right. \\ \left. + \bar{s}_6 Q_j' \delta_j^{n-3} + \varepsilon \bar{s}_6 Q_j' \bar{B}^{n-3} \delta_j^{n-3} \right. \\ \left. + \bar{s}_7 Q_j' (\delta_j^{n+1})^* \right. \\ \left. + \varepsilon \bar{s}_7 Q_j' (\bar{B}^{n+1})^* (\delta_j^{n+1})^* \right] Q_i d\xi = 0. \quad (19)$$

where

$$\bar{B}^n = \sum_{k=m-2}^{m+2} Q_k(\xi) \delta_k^n, \\ (\bar{B}^n)^* = \sum_{k=m-2}^{m+2} Q_k(\xi) (\delta_k^n)^*,$$

indices i, j and k take values $m-2, m-1, m, m+1, m+2$, and $m = 0, 1, \dots, N-1$. A matrix form of the system of Eqs. (18) is written below:

$$(A^e - s_1 C^e) (\delta^e)^{n+1})^* \\ = (p_1 A^e - s_2 C^e - s_3 B^e - \varepsilon s_3 D1^e) (\delta^e)^n \\ - (s_4 B^e + \varepsilon s_4 D2^e) (\delta^e)^{n-1} \\ - (s_5 B^e + \varepsilon s_5 D3^e) (\delta^e)^{n-2} \\ - (s_6 B^e + \varepsilon s_6 D4^e) (\delta^e)^{n-3}. \quad (20)$$

$$(A^e - \bar{s}_1 C^e) (\delta^e)^{n+1} \\ = (q_1 A^e - \bar{s}_2 C^e - \bar{s}_3 B^e - \varepsilon \bar{s}_3 D1^e) (\delta^e)^n \\ - (\bar{s}_4 B^e + \varepsilon \bar{s}_4 D2^e) (\delta^e)^{n-1} \\ - (\bar{s}_5 B^e + \varepsilon \bar{s}_5 D3^e) (\delta^e)^{n-2} \\ - (\bar{s}_6 B^e + \varepsilon \bar{s}_6 D4^e) (\delta^e)^{n-3} \\ - (\bar{s}_7 B^e + \varepsilon \bar{s}_7 D5^e) ((\delta^e)^{n+1})^*. \quad (21)$$

where $\delta^e = (\delta_{m-2}, \delta_{m-1}, \delta_m, \delta_{m+1}, \delta_{m+2})^T$. The matrices A^e, B^e and C^e have dimension 5×5 , and $Dp^e, p = 1, \dots, 5$ have dimension $5 \times 5 \times 5$:

$$A_{i,j}^e = \int_0^h Q_i Q_j d\xi, \\ B_{i,j}^e = \int_0^h Q_i Q_j' d\xi, \\ C_{i,j}^e = \int_0^h Q_i Q_j'' d\xi,$$

$$D1_{i,j,k}^e = \left(\int_0^h Q_i Q_j' Q_k d\xi \right) (\delta_k^e)^n, \\ D2_{i,j,k}^e = \left(\int_0^h Q_i Q_j' Q_k d\xi \right) (\delta_k^e)^{n-1}, \\ D3_{i,j,k}^e = \left(\int_0^h Q_i Q_j' Q_k d\xi \right) (\delta_k^e)^{n-2}, \\ D4_{i,j,k}^e = \left(\int_0^h Q_i Q_j' Q_k d\xi \right) (\delta_k^e)^{n-3}, \\ D5_{i,j,k}^e = \left(\int_0^h Q_i Q_j' Q_k d\xi \right) (\delta_k^e)^m.$$

So the matrix $Dp^e, p = 1, \dots, 5$ are expressed on depending on the element parameter δ^e . The element matrix equations obtained from the Galerkin procedure is combined to to have the global matrix system including all unknowns of the approximate solution in Eq. (5):

$$(\mathbf{A} - s_1 \mathbf{C}) (\delta^{n+1})^* \\ = (p_1 \mathbf{A} - s_2 \mathbf{C} - s_3 \mathbf{B} - \varepsilon s_3 \mathbf{D1}) \delta^n \\ - (s_4 \mathbf{B} + \varepsilon s_4 \mathbf{D2}) \delta^{n-1} \\ - (s_5 \mathbf{B} + \varepsilon s_5 \mathbf{D3}) \delta^{n-2} \\ - (s_6 \mathbf{B} + \varepsilon s_6 \mathbf{D4}) \delta^{n-3}. \quad (22)$$

$$\begin{aligned}
 & (\mathbf{A} - \bar{s}_1 \mathbf{C}) \delta^{n+1} \\
 &= (q_1 \mathbf{A} - \bar{s}_2 \mathbf{C} - \bar{s}_3 \mathbf{B} - \varepsilon \bar{s}_3 \mathbf{D1}) \delta^n \\
 &- (\bar{s}_4 \mathbf{B} + \varepsilon \bar{s}_4 \mathbf{D2}) \delta^{n-1} \\
 &- (\bar{s}_5 \mathbf{B} + \varepsilon \bar{s}_5 \mathbf{D3}) \delta^{n-2} \\
 &- (\bar{s}_6 \mathbf{B} + \varepsilon \bar{s}_6 \mathbf{D4}) \delta^{n-3} \\
 &- (\bar{s}_7 \mathbf{B} + \varepsilon \bar{s}_7 \mathbf{D5}) (\delta^{n+1})^*.
 \end{aligned} \quad (23)$$

where global element parameter:

$$\delta = (\delta_{-2}, \delta_{-1}, \dots, \delta_{N+1})^T \quad (24)$$

and $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{Dp}$, $p = 1, \dots, 5$ are derived from the corresponding element matrices A^e, B^e, C^e, Dp^e , $p = 1, \dots, 5$ respectively. The approximate solution at the time grid points can be computed using solution of the iterative system of equations. To begin the start of the recursive system, initial conditions $\delta^0, \delta^1, \delta^2$ and δ^3 must be calculated.

Step 1 Computation of δ^0 : Initial parameters of δ^0 can be obtained by interpolating with the initial conditions $u(x, 0) = f(x)$ at the time grid points along with its derivatives $u_x(a, t) = 0, u_x(b, t) = 0$ at the end boundaries.

Step 2 Computation of δ^1 : The Euler method as the predictor equation and the modified Euler method as the corrector method are employed to work out the second initial vector δ^1 . The parameters:

$$\begin{aligned}
 \theta_1 &= 1, \quad \theta_2 = \Delta t, \\
 \theta_3 &= 0, \quad \theta_4 = 0, \\
 \theta_5 &= 0, \quad \theta_6 = 0.
 \end{aligned}$$

are selected for the Euler method and the parameters:

$$\begin{aligned}
 \theta_1 &= 1, \quad \theta_2 = \frac{\Delta t}{2}, \\
 \theta_3 &= 0, \quad \theta_4 = 0, \\
 \theta_5 &= 0, \quad \theta_6 = \frac{\Delta t}{2}.
 \end{aligned}$$

are used for the modified Euler method, in the general multistep method.

Step 3 Computation of δ^2 : The two-step Adams–Bashforth method for prediction equation and the two-step Adams–Moulton method for correction equation are engaged to

determine the third initial condition δ^2 . The corresponding parameters in the multistep method

$$\begin{aligned}
 \theta_1 &= 1, \quad \theta_2 = \frac{3\Delta t}{2}, \\
 \theta_3 &= -\frac{\Delta t}{2}, \quad \theta_4 = 0, \\
 \theta_5 &= 0, \quad \theta_6 = 0.
 \end{aligned}$$

are chosen for the predictor equation and

$$\begin{aligned}
 \theta_1 &= 1, \quad \theta_2 = \frac{8\Delta t}{12}, \\
 \theta_3 &= -\frac{\Delta t}{12}, \quad \theta_4 = 0, \\
 \theta_5 &= 0, \quad \theta_6 = \frac{5\Delta t}{12}.
 \end{aligned}$$

for the corrector equation.

Step 4 Computation of δ^3 : Finally the fourth initial parameters δ^3 are found using a three-step Adams predictor method with the parameters in the multistep method:

$$\begin{aligned}
 \theta_1 &= 1, \quad \theta_2 = \frac{23\Delta t}{12}, \\
 \theta_3 &= -\frac{16\Delta t}{12}, \quad \theta_4 = \frac{5\Delta t}{12}, \\
 \theta_5 &= 0, \quad \theta_6 = 0.
 \end{aligned}$$

and a three-step Adams corrector method with the parameters in the multistep method:

$$\begin{aligned}
 \theta_1 &= 1, \quad \theta_2 = \frac{19\Delta t}{24}, \\
 \theta_3 &= -\frac{5\Delta t}{24}, \quad \theta_4 = \frac{\Delta t}{24}, \\
 \theta_5 &= 0, \quad \theta_6 = \frac{9\Delta t}{24}.
 \end{aligned}$$

Once the parameters $\delta^0, \delta^1, \delta^2$ and δ^3 are found, the iterative equations are started to run so that time-discrete solutions can be determined recursively at the space grid points.

3 Numerical Results

The proposed algorithm is tested on two problems to demonstrate its performance. First, an analytical solution describing solitary wave propagation is studied in order to compute the L_2 and L_∞ norms, enabling comparison with numerical solutions to assess accuracy. Second, the RLWE with an hyperbolic tangent type boundary condition is investigated to simulate wave undulation. For these

two classical problems, numerical invariants such as mass, momentum, and energy [11] are computed to illustrate that they remain constant throughout the execution of the algorithm:

$$\begin{aligned} I_1 &= \int_{-\infty}^{\infty} u dx \approx \int_a^b U dx, \\ I_2 &= \int_{-\infty}^{\infty} (u^2 + \xi(u_x)^2) dx \approx \int_a^b (U^2 + \xi(U_x)^2) dx, \\ I_3 &= \int_{-\infty}^{\infty} (u^3 + 3u^2) dx \approx \int_a^b (U^3 + 3U^2) dx. \end{aligned} \quad (25)$$

These invariants can be computed approximately using the trapezoidal integration formula.

3.1 Visualizations and results for Single Solitary Wave Modeling

Let $3c$, $v = 1 + \varepsilon c$ and $k = \frac{1}{2} \sqrt{\frac{\varepsilon c}{\mu(1+\varepsilon c)}}$ represent magnitude, velocity and width respectively. Solitary wave solution have the form for RLWE [12]:

$$U(x, t) = 3c \operatorname{sech}^2(k(x - x_0 - vt)). \quad (26)$$

Using the initial conditions $u(a, t) = 0$, $u(b, t) = 0$, the algorithm was run with parameters $\varepsilon = 1$, $\mu = 1$, $x_0 = 0$ and the amplitude $c = 0.1$, using space increments $h = 0.2$ and 0.1 and time increments $\Delta t = 0.02, 0.01$ and 0.001 up to the time $t = 20$, over the domain $-80 \leq x \leq 120$. Visual solutions are graphed at some times in Fig 1 and absolute maximum error distribution at time $t = 20$ is depicted in Fig 2. To reduce boundary effects, the space domain was extended to $-160 \leq x \leq 160$. Whenever the maximum error happens at the boundaries, the computational domain was further enlarged until the maximum error was located within the interior of the domain, thereby eliminating boundary-induced errors. Error norms are documented in Table 1-3 from which highly accurate result are obtained. The errors obtained by the proposed method can be compared with those reported for the Predictor-Corrector Quintic B-Spline Collocation method, the Hermite Distributed Functional method and the Sinc method. Furthermore, the numerical invariants remain constant for all space and time step sizes considered and can be compared with the approximate analytical values given below, referenced in Eq. (27). If analytical solution exists, the conserved invariants can be derived by putting analytical solution of amplitude

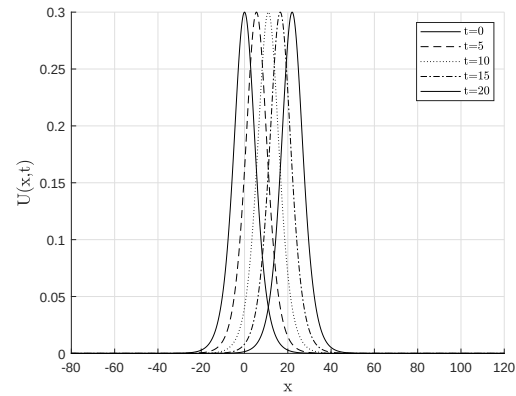


Figure 1: Evolution of the single solitary wave

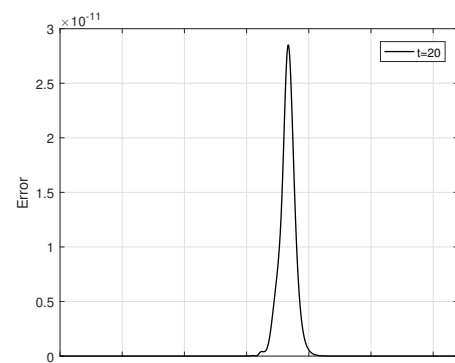


Figure 2: Maximum error norm at $t = 20$

0.3 at $t = 0$ in the Eqs. 25 as.

$$\begin{aligned} I_1 &= \frac{6c}{k} = 3.97995, \\ I_2 &= \frac{12c^2}{k} + \frac{48kc^2\mu}{5} = 0.81046, \\ I_3 &= \frac{36c^2}{k} \left(1 + \frac{4c}{5}\right) = 2.57901. \end{aligned} \quad (27)$$

3.2 Visualizations and results for Wave Undulation Modeling

When an initially steep disturbance propagates in a shallow-water medium, it evolves into a train of smooth oscillatory waves consisting of a leading solitary wave followed by a sequence of decreasing-amplitude waves. Thus the second test problem concerns undular bore produced by a hyperbolic tangent-type boundary forcing profile to obtain the undular bore solution of the RLWE:

$$U(x, 0) = 0.5 \left(1 - \tanh\left(\frac{x - x_c}{d}\right)\right) U_0, \quad (28)$$

Table 1: Error comparison and conserved quantities for $\Delta t = 0.02$

h	L_∞	$L_\infty[9]$	I_1^{20}	I_2^{20}	I_3^{20}
0.2	6.15×10^{-9}	6.47×10^{-9}	3.97994975	0.81046249	2.57900744
0.1	6.16×10^{-9}	5.78×10^{-9}	3.97994975	0.81046249	2.57900744

Table 2: Error comparison and conserved quantities for $\Delta t = 0.01$

h	L_∞	$L_\infty[9]$	I_1^{20}	I_2^{20}	I_3^{20}
0.2	7.65×10^{-10}	7.72×10^{-9}	3.97994975	0.81046249	2.57900743
0.1	7.70×10^{-10}	6.11×10^{-10}	3.97994975	0.81046249	2.57900744
0.7825	(HDAF)[5]	2.3382×10^{-12}	3.979949	0.810462	2.579007
0.7825	(Sin-Met)[5]	2.7039×10^{-12}	3.979949	0.810462	2.579007

Table 3: Error comparison and conserved quantities for $h = 0.1$ and $\Delta t = 0.001$ on $-160 \leq x \leq 160$.

L_∞	I_1^{20}	I_2^{20}	I_3^{20}
2.85×10^{-11}	3.97994975	0.81046249	2.57900744

$$\begin{aligned}\sigma_1 &= U(a, t) = U_0, \\ \sigma_2 &= U(b, t) = 0, \\ U_x(a, t) &= 0.\end{aligned}\quad (29)$$

U_0 is the forcing magnitude and the parameter d represents the steepness of the transition and x_c is center point of the transition. The right boundary, $U(b, t) = 0$, is imposed sufficiently far from the left boundary to minimize boundary effect. Since the boundary forcing is applied in time, RLWE generates a sequence of wave. The number of generated waves depends not only on the magnitude of the forcing at the left boundary but also on the duration for which the forcing is applied. Wave production continues as long as the forcing persists, and the waves propagate to the right with certain amplitudes. The experiment is conducted up to time $t = 300$ in the space interval $-36 \leq x \leq 300$. The algorithm is implemented using parameters $\varepsilon = 1.5$, $\mu = 1/6$, $U_0 = 0.1$, $x_c = 0$, $t_0 = 0$ and space increment $h = 0.24$. Two different slope values, $d = 2$ (steeper slope) and $d = 5$, are considered in the simulations. Conserved invariants are reported at selected time levels, along with results of the study for the parameters $d = 2$ and $d = 5$ in Table 4. Undular bore behavior is depicted for $d = 2$ at times $t = 150$ and $t = 250$ in Figs. 3-4, where more oscillatory waves are observed. Similar profiles are shown for a slightly steeper parameter, $d = 5$, in Figs 5 and 6.

4 Conclusion

A numerical algorithm is constructed by integrating the multistep methods with the quartic B-spline Galerkin finite element method to obtain approximate solutions of the RLWE. The effectiveness and robustness of the proposed algorithm are demonstrated through two test problems. Higher

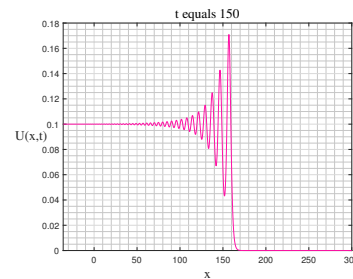


Figure 3: Undulation profiles corresponding to $d = 2$ and $t = 150$

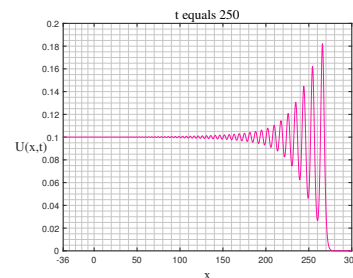


Figure 4: Undulation profiles corresponding to $d = 2$ and $t = 250$

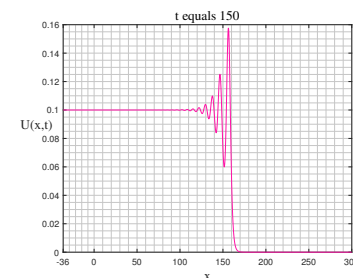


Figure 5: Undulation profiles corresponding to $d = 5$ and $t = 150$

Table 4: Conservation invariants for wave undulation cases $d = 2$ and $d = 5$

Case $d = 2$			
t	I_1^t	I_2^t	I_3^t
0	3.61200000	0.35147778	1.08822000
50	8.98700000	0.90147795	2.79324665
100	14.36200000	1.45147854	4.49803222
150	19.73700000	2.00147956	6.20270511
200	25.11200000	2.55148103	7.90733727
250	30.48699991	3.10148296	9.61195450

Case $d = 5$			
t	I_1^t	I_2^t	I_3^t
0	3.61200013	0.33631114	1.04097008
50	8.98700014	0.88631114	2.74645725
100	14.36200014	1.43631125	4.45166072
150	19.73700014	1.98631163	6.15653826
200	25.11200014	2.53631241	7.86124330
250	30.48700013	3.08631364	9.56588695

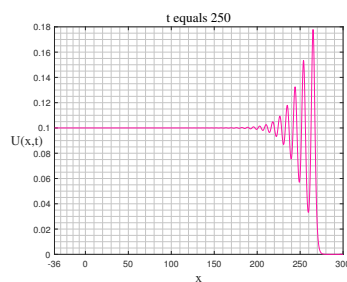


Figure 6: Undulation profiles corresponding to $d = 5$ and $t = 250$

accuracy is achieved when compared with existing well-established numerical schemes, such as the fourth-order central difference scheme, the quartic and quintic B-spline collocation methods combined with a predictor–corrector scheme [10, 9, 13], the quartic B-spline Galerkin method with a fourth-order multistep method, and the distributed approximating functional method [5].

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References:

- [1] J. L. Bona, W. G. Pritchard, and L. R. Scott, (1983). A comparison of solutions of two model equations for long waves. MRC Technical Report 2477.
- [2] J. P. Albert & J. L. Bona, (1991). Comparisons between Model Equations for Long Waves. J. Nonlinear Sci. 1, 345-374.
- [3] J. Lin, Z. Xie, & J. Zhou, (2007). High-order compact difference scheme for the regularized long wave equation. *Communications in Numerical Methods in Engineering*, 23(2), 135-156. Doi.org/10.1002/cnm.892
- [4] Y. Dereli, (2012). Numerical solutions of the MRLW Equation using meshless kernel based method of lines. *International Journal of Nonlinear Science*, 13(1), 28-38. DOI.org/10.1088/1742-6596/766/1/012030.
- [5] E. Pindza, & E. Mare, (2014). Solving the Generalized Regularized Long Wave Equation using a Distributed Approximating Functional Method. *International Journal of Computational Mathematics*, 178024. Doi.org/10.1155/2014/178024.
- [6] D. Irk, & P. Keskin, (2017). Quadratic trigonometric B-spline Galerkin methods for the Regularised Long Wave Equation. *Journal of Applied Analysis and Computation*, 7(2), 617-631. Doi.org/10.11948/2017038.
- [7] D. Irk, P. K. Yildiz, & M. Z Gorgulu, (2019). Quartic trigonometric B-spline algorithm for numerical solution of the regularized long wave equation. *Turkish Journal of Mathematics*, 43(1), 112–125. Doi.org/10.3906/mat-1804-55.
- [8] A. Mohebbi, (2012). Solitary wave solutions of the nonlinear generalized Pochhammer-Chree and Regularized Long Wave equations. *Nonlinear Dynamics*, 70, 2463-2474. Doi.org/10.1007/s11071-012-0634-5.
- [9] B. Saka, I. Dağ, & O. E. Hepson, (2023). Integration of the RLW equation using higher-order predictor–corrector scheme and quintic B-spline collocation method. *Mathematical Sciences*, 17, 491-502. Doi.org/10.1007/s40096-022-00475-z.

- [10] I. Dag, O. E. Hepson, & B. Saka, (2022). A higher-order efficient approach to numerical simulations of the RLW equation. *Pramana Journal of Physics*, 96(30), 1-11. Doi.org/10.1007/s12043-021-02256-0.
- [11] P. J. Olver, (1979). Euler operators and conservation laws of the BBM equation. *Mathematical Proceedings of the Cambridge Philosophical Society*, 85, 143–160. 10.1017/S0305004100055572
- [12] D. H. Peregrine, (1966). Calculations of the development of an undular bore. *Journal of Fluid Mechanics*, 25(2), 321-330. Doi.org/10.1017/S0022112066001678.
- [13] S. Bu, & Y.J eon, (2025). Higher-order predictor–corrector methods for fractional Benjamin–Bona–Mahony–Burgers equations. *Journal of Applied Mathematics and Computing*, 71, 1–30. Doi.org/10.1007/s12190-024-02223-z

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The authors equally contributed in the present research, at all stages from the formulation of the problem to the final findings and solution.

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Conflicts of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.

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