

Visualization of Solitary Waves Using a Multistep Predictor-Corrector Algorithm

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Abstract: - In this paper, we combine the cubic B-spline Galerkin method with the predictor-corrector method to get the numerical method for solving the RLW equation(RLWE). Thus fully-integration of the RLWE is managed by employing the time integrator of the four step explicit Adams-Bashforth as predictor and the four step implicit Adams-Moulton as corrector and the cubic B-spline Galerkin method for spatial discretization. The performance of this new method is carried out studying the single solitary wave, wave undulation and wave generation and findings of error norm L_∞ and numerical conservation laws are compared with exact solutions and those already published.

Key-Words:-Multistep predictor-corrector method, Adams-Bashforth-Moulton method, RLW equation, single solitary wave, wave undulation, wave generation

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1 Introduction

Scientists are striving to compose new hybrid numerical methods to get solutions of the nonlinear differential equations. Especially high accuracy numerical methods have been developed to understand behaviors of the nonlinear differential equation correctly. Traditional numerical techniques, such as finite difference, finite element methods, and others, have been adapted to achieve precise and reliable solutions for differential equations. The Compact finite difference method and Nonlinear finite difference methods are derived from the classical finite difference methods and are being showed to produce very accurate solutions of the differential equations, [1], [2]. Effective finite element method can be constructed with both suitable mesh generation and approximation function over the complex problem domain to solve the partial differential equations(PDE). Accurate solutions are obtained by using the B-spline functions to establish the approximate function to find numerical solutions of the PDE, [3]. Accuracy of solutions of the time dependent PDE is also increased by employing the high order time integrator together with an spatial integrator such as Fourier Tranform method, Finite difference and Finite element methods. Some variants of Runge-Kutta methods are implemented for time discretization to obtain the higher accurate solutions of the time dependent PDE. Accurate numerical algorithms have been provided to illustrate wave

behaviors of the RLWE, such as the propagation of solitary waves, undulation, and wave generation. An hybrit method is presented for the space discretization based on the Galerkin finite element method and the explicit multistep method for time discretization, [4], [5]. The discrete Fourier transform is used to approximate the spatial derivatives of the RLWE and then resulting system of time derivative equation is solved by using the exponential integrator of the Runge-Kutta types such as Runge-Kutta integrating factor (IFRK) and Runge-Kutta exponential time differencing (ETDRK), [6]. The other higher accurate algorithm has been developed for fuly-discretization of the Generalised RLWE using the Hermite distributed approximating functional(HDAF) method in spatial domain and fouth order Runge-kutta method in time, [7]. Both quadratic and quartic trigonometric B-spline Galerkin methods together with both the second and fourth order multistep time discretization method are employed to get fully-discretization of the RLWE so that well enough results are obtained in the studies, [8], [9]. Recently an numerical algorithm for solving the RLWE has been set up with help of the predictor-corrector method in time and quintic B-spline collocation procedure in space to have very accurate solutions in the work, [11]. The shallow water waves, plasma waves, pressure waves, longitudinal dispersive waves and etc are obtained by solutions of the RLWE. The RLWE provides solitary wave so-

lutions whose amplitudes subject to both phase shift and a change during interaction so that solution of the RLWE exhibits different forms of the waves. Due to lack of analytical solutions of the RLW, numerical studies are handy to exhibit some wave behaviors obtained by solutions of the RLWE. The higher order predictor-corrector method is a numerical technique used to solve ordinary differential equations with high accuracy by combining prediction and correction steps, [10]. In this approach, a predictor formula estimates the solution, and a corrector formula iteratively refines it using higher-order approximations to ensure both stability and precision. In this study, higher order predictor corrector method is carried out for time integration of the RLW equation, resulting time discrete RLW equation is fully integrated by using the cubic B-spline Galerkin method. Prosperity of the presented algorithm is exhibited by comparing maximum norm and quantities of RLW equation conserved by the solution by existing solutions in literature. Influence of the predictor-corrector method on time integration of RLW equation is pointed out:

$$u_t + u_x + \varepsilon u u_x - \mu u_{xxt} = 0, \quad x \in [a, b], \quad t \in (0, T], \quad (1)$$

subject to the initial condition (IC):

$$u(x, 0) = f(x), \quad (2)$$

and subject to the boundary conditions (BCs):

$$u(a, t) = \sigma_1, \quad u(b, t) = \sigma_2 \quad (3)$$

where u is the wave amplitude, ε and μ are positive, dimensionless parameters.

2 Numerical Method

A computational mesh is given by $[x_i, x_{i+1}]$ for $i = 0, \dots, n-1$ over the interval $[a = x_0, b = x_n]$ at the equally distributed mesh points $x_i, i = 0, \dots, n$. At these mesh points and additional mesh points $x_{-3}, x_{-2}, x_{-1}, x_{n+1}, x_{n+2}, x_{n+3}$ outside the region $[a, b]$, the cubic B-splines $Q_i, i = -1, \dots, n+1$ can be defined as:

$$Q_m(x) = \frac{1}{h^3} \begin{cases} (x - x_{m-2})^3, & \text{if } x \in [x_{m-2}, x_{m-1}] \\ h^3 + 3h^2(x - x_{m-1}) + 3h(x - x_{m-1})^2 & \text{if } x \in [x_{m-1}, x_m] \\ -3(x - x_{m-1})^3, & \text{if } x \in [x_m, x_{m+1}] \\ h^3 + 3h^2(x_{m-1} - x) + 3h(x_{m-1} - x)^2 & \text{if } x \in [x_m, x_{m+1}] \\ -3(x_{m-1} - x)^3, & \text{if } x \in [x_{m+1}, x_{m+2}] \\ (x_{m+2} - x)^3, & \text{if } x \in [x_{m+1}, x_{m+2}] \\ 0, & \text{otherwise.} \end{cases} \quad (4)$$

Each cubic B-spline is twice continuously differentiable and the set of cubic B-splines constitutes a basis for the cubic spline function space. The approximate solution is given by the combination of the the weighted the cubic B-splines $U(x, t)$ defined as follows:

$$U(x, t) = \sum_{m=-1}^{N+1} \delta_m(t) Q_m(x), \quad (5)$$

where $\delta_m(t)$ are time dependent unknown parameters to be determined from cubic B-spline Galerkin form of Eq. (1) and the cubic B-splines are forced to satisfy the boundary conditions. The nodal values given in (5), along with their first and second derivatives at the grid points, can be calculated as follows:

$$\begin{aligned} U_m &= \delta_{m-1} + 4\delta_m + \delta_{m+1}, \\ U'_m &= -\frac{3}{h} \delta_{m-1} + \frac{3}{h} \delta_{m+1}, \\ U''_m &= \frac{6}{h^2} \delta_{m-1} - \frac{12}{h^2} \delta_m + \frac{6}{h^2} \delta_{m+1}. \end{aligned} \quad (6)$$

Let us separate the time-derivative terms and the space-derivative terms of the RLWE.

$$(u - \mu u_{xx})_t = -u_x - \varepsilon u u_x. \quad (7)$$

The time integration of Eq. (7) is performed using a predictor-corrector multistep method. For this purpose, the following expansion is employed:

$$(u^{n+1})^* = p_1 u^n + p_2 u_t^n + p_3 u_t^{n-1} + p_4 u_t^{n-2} + p_5 u_t^{n-3} \quad (8)$$

is used with parameters

$$p_1 = 1, \quad p_2 = \frac{55 \Delta t}{24}, \quad p_3 = -\frac{59 \Delta t}{24}, \quad p_4 = \frac{37 \Delta t}{24}, \quad p_5 = -\frac{9 \Delta t}{24} \quad (9)$$

to yield the Adams-Bashforth four-step explicit method of order 4 as predictor equation. Similarly, discrete equation:

$$u^{n+1} = q_1 u^n + q_2 u_t^n + q_3 u_t^{n-1} + q_4 u_t^{n-2} + q_5 (u_t^{n+1})^* \quad (10)$$

together with parameters:

$$q_1 = 1, \quad q_2 = \frac{19 \Delta t}{24}, \quad q_3 = -\frac{5 \Delta t}{24}, \quad q_4 = \frac{\Delta t}{24}, \quad q_5 = \frac{9 \Delta t}{24} \quad (11)$$

leads to corrector equation known as Adams-Moulton three-step implicit method of order 4.

Eqs. (8) and (10) are organized using the RLWE (7) to obtain both the predictor and corrector equations given below:

$$\begin{aligned} (u^{n+1})^* - \mu (u_{xx}^{n+1})^* &= p_1 (u^n - \mu u_{xx}^n) - p_2 (u_x^n + \varepsilon u^n u_x^n) \\ &\quad - p_3 (u_x^{n-1} + \varepsilon u^{n-1} u_x^{n-1}) \\ &\quad - p_4 (u_x^{n-2} + \varepsilon u^{n-2} u_x^{n-2}) \\ &\quad - p_5 (u_x^{n-3} + \varepsilon u^{n-3} u_x^{n-3}), \end{aligned} \quad (12)$$

$$\begin{aligned} u^{n+1} - \mu u_{xx}^{n+1} &= q_1 (u^n - \mu u_{xx}^n) - q_2 (u_x^n + \varepsilon u^{n-1} u_x^{n-1}) \\ &\quad - q_3 (u_x^{n-1} + \varepsilon u^{n-1} u_x^{n-1}) \\ &\quad - q_4 (u_x^{n-2} + \varepsilon u^{n-2} u_x^{n-2}) \\ &\quad - q_5 ((u_x^{n+1})^* + \varepsilon (u^{n+1})^* (u_x^{n+1})^*). \end{aligned} \quad (13)$$

To follow the finite element procedure, the sample interval $[x_m, x_{m+1}]$ is mapped to the $[0, h]$ by using the mapping $\xi = x - x_m$. The branch of the cubic B-spline functions within the interval $[x_m, x_{m+1}]$ known as the B-spline shape functions are transformed by using the mapping below:

$$\begin{aligned} Q_{m-1} &= \left(1 - \frac{\xi}{h}\right)^3, \\ Q_m &= 4 - 3\left(\frac{\xi}{h}\right) + 3\left(1 - \frac{\xi}{h}\right)^2 - 3\left(1 - \frac{\xi}{h}\right)^3, \\ Q_{m+1} &= 1 + 3\left(\frac{\xi}{h}\right) + 3\left(\frac{\xi}{h}\right)^2 - 3\left(\frac{\xi}{h}\right)^3, \\ Q_{m+2} &= \left(\frac{\xi}{h}\right)^3. \end{aligned} \quad (14)$$

The approximate solution (5) can be defined by the weighted sum of the B-spline shape functions over the restricted sample interval $[0, h]$:

$$U^e = U(\xi, t) = \sum_{i=m-1}^{m+2} \delta_i(t) Q_i(\xi) \quad (15)$$

where δ_i are unknown parameters, which should be determined using collocation application to the RLWE. The weak formulation is obtained by multiplying the Eq. (12)-(13) with the weight functions $W = Q_i, i = m-1, \dots, m+2$ defined in Eq. (14), integrating these equations from 0 to h with respect to ξ and applying the transformation $\xi = x - x_m$ to these weak forms for the particular element $[x_m, x_{m+1}]$:

$$\begin{aligned} \int_0^h &\left[(u^{n+1})^* - s_1 (u_{\xi\xi}^{n+1})^* - p_1 u^n + s_2 u_{\xi\xi}^n + s_3 u_{\xi}^n + \varepsilon s_3 u^n u_{\xi}^n \right. \\ &\quad + s_4 u_{\xi}^{n-1} + \varepsilon s_4 u^{n-1} u_{\xi}^{n-1} + s_5 u_{\xi}^{n-2} + \varepsilon s_5 u^{n-2} u_{\xi}^{n-2} \\ &\quad \left. + s_6 u_{\xi}^{n-3} + \varepsilon s_6 u^{n-3} u_{\xi}^{n-3} \right] W d\xi = 0. \end{aligned} \quad (16)$$

$$\begin{aligned} \int_0^h &\left[u^{n+1} - \bar{s}_1 u_{\xi\xi}^{n+1} - q_1 u^n + \bar{s}_2 u_{\xi\xi}^n + \bar{s}_3 u_{\xi}^n + \varepsilon \bar{s}_3 u^n u_{\xi}^n \right. \\ &\quad + \bar{s}_4 u_{\xi}^{n-1} + \varepsilon \bar{s}_4 u^{n-1} u_{\xi}^{n-1} + \bar{s}_5 u_{\xi}^{n-2} + \varepsilon \bar{s}_5 u^{n-2} u_{\xi}^{n-2} \\ &\quad \left. + \bar{s}_6 (u_{\xi}^{n+1})^* + \varepsilon \bar{s}_6 (u^{n+1})^* (u_{\xi}^{n+1})^* \right] W d\xi = 0. \end{aligned} \quad (17)$$

where

$$s_1 = \mu, \quad s_2 = p_1 \mu, \quad s_3 = p_2, \quad s_4 = p_3, \quad s_5 = p_4, \quad s_6 = p_5, \quad (18)$$

$$\bar{s}_1 = \mu, \quad \bar{s}_2 = q_1 \mu, \quad \bar{s}_3 = q_2, \quad \bar{s}_4 = q_3, \quad \bar{s}_5 = q_4, \quad \bar{s}_6 = q_5. \quad (19)$$

Substituting the unknown u^n and its derivatives $u_{\xi}^n, u_{\xi\xi}^n$ with the approximate solution (15) and its derivatives yields the following integral equations.

$$\begin{aligned} \sum_{j=m-1}^{m+2} \int_0^h &\left[Q_j (\delta_j^{n+1})^* - s_1 Q_j'' (\delta_j^{n+1})^* - p_1 Q_j \delta_j^n + s_2 Q_j'' \delta_j^n + s_3 Q_j' \delta_j^n \right. \\ &\quad + \varepsilon s_3 Q_j' \bar{B}^n \delta_j^n + s_4 Q_j' \delta_j^{n-1} + \varepsilon s_4 Q_j' \bar{B}^{n-1} \delta_j^{n-1} \\ &\quad + s_5 Q_j' \delta_j^{n-2} + \varepsilon s_5 Q_j' \bar{B}^{n-2} \delta_j^{n-2} + s_6 Q_j' \delta_j^{n-3} \\ &\quad \left. + \varepsilon s_6 Q_j' \bar{B}^{n-3} \delta_j^{n-3} \right] Q_i d\xi = 0, \end{aligned} \quad (20)$$

$$\begin{aligned} \sum_{j=m-1}^{m+2} \int_0^h &\left[Q_j \delta_j^{n+1} - \bar{s}_1 Q_j'' \delta_j^{n+1} - q_1 Q_j \delta_j^n + \bar{s}_2 Q_j'' \delta_j^n + \bar{s}_3 Q_j' \delta_j^n \right. \\ &\quad + \varepsilon \bar{s}_3 Q_j' \bar{B}^n \delta_j^n + \bar{s}_4 Q_j' \delta_j^{n-1} + \varepsilon \bar{s}_4 Q_j' \bar{B}^{n-1} \delta_j^{n-1} \\ &\quad + \bar{s}_5 Q_j' \delta_j^{n-2} + \varepsilon \bar{s}_5 Q_j' \bar{B}^{n-2} \delta_j^{n-2} + \bar{s}_6 Q_j' (\delta_j^{n+1})^* \\ &\quad \left. + \varepsilon \bar{s}_6 Q_j' (\bar{B}^{n+1})^* (\delta_j^{n+1})^* \right] Q_i d\xi = 0, \end{aligned} \quad (21)$$

where

$$\begin{aligned} \bar{B}^n &= \sum_{k=m-2}^{m+2} Q_k(\xi) \delta_k^n, \\ (\bar{B}^n)^* &= \sum_{k=m-2}^{m+2} Q_k(\xi) (\delta_k^n)^*. \end{aligned} \quad (22)$$

indices i, j and k take values $m-1, m, m+1, m+2$, and $m = 0, 1, \dots, N-1$.

Distributing the integral across the terms, evaluating the integrals appearing in Eqs. (16)–(17), and then organizing the result in matrix form lead to the systems of equations (20) and (21):

$$\begin{aligned} (A^e - s_1 C^e) ((\delta^e)^{n+1})^* &= (p_1 A^e - s_2 C^e - s_3 B^e - \varepsilon s_3 D1^e) (\delta^e)^n \\ &\quad - (s_4 B^e + \varepsilon s_4 D2^e) (\delta^e)^{n-1} \\ &\quad - (s_5 B^e + \varepsilon s_5 D3^e) (\delta^e)^{n-2} \\ &\quad - (s_6 B^e + \varepsilon s_6 D4^e) (\delta^e)^{n-3}. \end{aligned} \quad (23)$$

$$\begin{aligned} (A^e - \bar{s}_1 C^e) (\delta^e)^{n+1} &= (q_1 A^e - \bar{s}_2 C^e - \bar{s}_3 B^e - \varepsilon \bar{s}_3 D1^e) (\delta^e)^n \\ &\quad - (\bar{s}_4 B^e + \varepsilon \bar{s}_4 D2^e) (\delta^e)^{n-1} \\ &\quad - (\bar{s}_5 B^e + \varepsilon \bar{s}_5 D3^e) (\delta^e)^{n-2} \\ &\quad - (\bar{s}_6 B^e + \varepsilon \bar{s}_6 D5^e) ((\delta^e)^{n+1})^*. \end{aligned} \quad (24)$$

where $(\delta^e)^* = ((\delta_{m-1})^*, (\delta_m)^*, (\delta_{m+1})^*, (\delta_{m+2})^*)^T$, $\delta^e = (\delta_{m-1}, \delta_m, \delta_{m+1}, \delta_{m+2})^T$. The matrices A^e , B^e and C^e have dimension 4×4 :

$$\begin{aligned} A_{i,j}^e &= \int_0^h Q_i Q_j d\xi, \\ B_{i,j}^e &= \int_0^h Q_i Q_j' d\xi, \\ C_{i,j}^e &= \int_0^h Q_i Q_j'' d\xi \end{aligned} \quad (25)$$

and Dp^e , $p = 1, \dots, 5$ have dimension $4 \times 4 \times 4$:

$$\begin{aligned} D1_{i,j,k}^e &= \sum_{k=m-1}^{m+2} \left[\left(\int_0^h Q_i Q_j' Q_k d\xi \right) (\delta_k^e)^n \right], \\ D2_{i,j,k}^e &= \sum_{k=m-1}^{m+2} \left[\left(\int_0^h Q_i Q_j' Q_k d\xi \right) (\delta_k^e)^{n-1} \right], \\ D3_{i,j,k}^e &= \sum_{k=m-1}^{m+2} \left[\left(\int_0^h Q_i Q_j' Q_k d\xi \right) (\delta_k^e)^{n-2} \right], \\ D4_{i,j,k}^e &= \sum_{k=m-1}^{m+2} \left[\left(\int_0^h Q_i Q_j' Q_k d\xi \right) (\delta_k^e)^{n-3} \right], \\ D5_{i,j,k}^e &= \sum_{k=m-1}^{m+2} \left[\left(\int_0^h Q_i Q_j' Q_k d\xi \right) ((\delta_k^e)^{n+1})^* \right]. \end{aligned} \quad (26)$$

So the matrix Dp^e , $p = 1, \dots, 5$ are expressed on depending on the element parameters δ^e , $(\delta^e)^*$. We assembly all element contribution to obtain fully-discrete Galerkin formulation which is an system of am nonlinear algebraic equation.

$$\begin{aligned} (\mathbf{A} - s_1 \mathbf{C}) (\delta^{n+1})^* &= (p_1 \mathbf{A} - s_2 \mathbf{C} - s_3 \mathbf{B} - \varepsilon s_3 \mathbf{D1}) \delta^n \\ &\quad - (s_4 \mathbf{B} + \varepsilon s_4 \mathbf{D2}) \delta^{n-1} \\ &\quad - (s_5 \mathbf{B} + \varepsilon s_5 \mathbf{D3}) \delta^{n-2} \\ &\quad - (s_6 \mathbf{B} + \varepsilon s_6 \mathbf{D4}) \delta^{n-3}. \end{aligned} \quad (27)$$

$$\begin{aligned} (\mathbf{A} - \bar{s}_1 \mathbf{C}) \delta^{n+1} &= (q_1 \mathbf{A} - \bar{s}_2 \mathbf{C} - \bar{s}_3 \mathbf{B} - \varepsilon \bar{s}_3 \mathbf{D1}) \delta^n \\ &\quad - (\bar{s}_4 \mathbf{B} + \varepsilon \bar{s}_4 \mathbf{D2}) \delta^{n-1} \\ &\quad - (\bar{s}_5 \mathbf{B} + \varepsilon \bar{s}_5 \mathbf{D3}) \delta^{n-2} \\ &\quad - (\bar{s}_6 \mathbf{B} + \varepsilon \bar{s}_6 \mathbf{D5}) (\delta^{n+1})^*. \end{aligned} \quad (28)$$

where global element parameter

$$\begin{aligned} \delta^* &= ((\delta_{-1})^*, (\delta_0)^*, \dots, (\delta_{N+1})^*)^T, \\ \delta &= (\delta_{-1}, \delta_0, \dots, \delta_{N+1})^T. \end{aligned} \quad (29)$$

and $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{Dp}$, $p = 1, \dots, 5$ are derived from the corresponding element matrices A^e , B^e , C^e , Dp^e , $p = 1, \dots, 5$ respectively. Before commencing the iteration procedure, the first four initial parameters $(\delta^0)^*$, $(\delta^1)^*$, $(\delta^2)^*$, $(\delta^3)^*$, δ^0 , δ^1 , δ^2 , δ^3 must be calculated.

2.1 Computation of Initial Vectors

Step 1: Computation of δ^0 : Initial parameter δ^0 at time $t = 0$ can be found using the IC and BCs given Eqs. (2)-(3) to have a system of parameters:

$$\begin{aligned} U_m' &= b_1 \delta_{m-1} + b_2 \delta_m + b_3 \delta_{m+1}, \\ U_m &= a_1 \delta_{m-1} + a_2 \delta_m + a_3 \delta_{m+1}, i = 0, \dots, n \\ U_m'' &= c_1 \delta_{m-1} + c_2 \delta_m + c_3 \delta_{m+1}. \end{aligned} \quad (30)$$

The initial vectors $(\delta^0)^*$, $(\delta^1)^*$, $(\delta^2)^*$, $(\delta^3)^*$, δ^1 , δ^2 , and δ^3 are computed using a sequence of predictor–corrector methods of increasing order.

Step 2: Computation of δ^1 : The first vector δ^1 is obtained by employing the Euler method as a predictor and the Modified Euler method as a corrector. For the Euler method, the parameters are chosen as

$$p_1 = 1, \quad p_2 = \Delta t, \quad p_3 = p_4 = p_5 = 0. \quad (31)$$

while for the Modified Euler method, the parameters are:

$$q_1 = 1, \quad q_2 = \frac{\Delta t}{2}, \quad q_3 = q_4 = 0, \quad q_5 = \frac{\Delta t}{2}. \quad (32)$$

Step 3: Computation of δ^2 : The second vector δ^2 is determined using the two-step Adams–Bashforth method (2ABM) for prediction

and the two-step Adams–Moulton method (2AMM) for correction. The corresponding parameters are predictor:

$$p_1 = 1, \quad p_2 = \frac{3\Delta t}{2}, \quad p_3 = -\frac{\Delta t}{2}, \quad p_4 = p_5 = 0. \quad (33)$$

Corrector:

$$q_1 = 1, \quad q_2 = \frac{8\Delta t}{12}, \quad q_3 = -\frac{\Delta t}{12}, \quad q_4 = 0, \quad q_5 = \frac{5\Delta t}{12}. \quad (34)$$

Step 4: Computation of δ^3 : The third vector, δ^3 , is computed using a three-step Adams–Bashforth–Moulton predictor–corrector method. For the predictor equation, the corresponding parameters are given by:

$$p_1 = 1, \quad p_2 = \frac{23\Delta t}{12}, \quad p_3 = -\frac{16\Delta t}{12}, \quad p_4 = \frac{5\Delta t}{12}, \quad p_5 = 0. \quad (35)$$

and for the corrector equation,

$$q_1 = 1, \quad q_2 = \frac{19\Delta t}{24}, \quad q_3 = -\frac{5\Delta t}{24}, \quad q_4 = \frac{\Delta t}{24}, \quad q_5 = \frac{9\Delta t}{24}. \quad (36)$$

The overall procedure is summarized as follows: Once δ^0 has been calculated using initial and boundary conditions. δ^1 , δ^2 , δ^3 must be computed in order described in the section on the computation of initial vectors. Then we begin determining the parameters $(\delta^{n+1})^*$, $n = 3, 4, 5, \dots, m$ from Eq. (27) using the predictor equation recursively and after obtaining these parameters, they are used in the predictor Eq. (28) to calculate the corrected parameters ref δ^{n+1} , $n = 3, 4, 5, \dots, m$ at the time grid points.

3 Numerical Results

Test problems are going to study to show the performance of the suggested scheme. First solitary wave propagation whose analytical solution exist is simulated and reliability of the method is shown by calculating the maximum error norm:

$$L_\infty = \max_{0 \leq m \leq N} |u_m - U_m| \quad (37)$$

where u is analytical solution and U is numerical solutions.

Since the conserved invariants known as mass I_1 , momentum I_2 , energy I_3 defined below must be constant, we verify these quantities using numerical out-

come we have found, [12]:

$$\begin{aligned} I_1 &= \int_a^b U \, dx \\ I_2 &= \int_a^b (U^2 + \mu(U_x)^2) \, dx \\ I_3 &= \int_a^b (U^3 + 3U^2) \, dx \end{aligned} \quad (38)$$

Numerical conservation constants are computed with help of the rectangular integration formula to make comparison with the analytical ones computed using the initial condition.

3.1 Visualizations and results for Single Solitary Wave Modeling

The exact solitary wave solution can be expressed as:

$$U(x, t) = 3c \operatorname{sech}^2(k(x - x_0 - vt)), \quad (39)$$

where parameters $3c$, $v = \varepsilon c + 1$ and $k = \frac{1}{2} \sqrt{\left(\frac{\varepsilon c}{\mu(1+\varepsilon c)}\right)}$ correspond to magnitude, velocity and width respectively. Initial wave is extracted from the exact solitary wave solution with $t = 0$. Motion of the solitary wave located at x_0 initially is simulated over the domain $-80 \leq x \leq 120$ with parameters $\varepsilon = 1$, $\mu = 1$, $x_0 = 0$ and the amplitude $c = 0.1$. Using the initial condition, the conservation constants can be determined as follows:

$$I_1 = \frac{6c}{k}, \quad I_2 = \frac{12c^2}{k} + \frac{48kc^2\mu}{5}, \quad I_3 = \frac{36c^2}{k} \left(1 + \frac{4c}{5}\right). \quad (40)$$

Algorithm was run until time $t=20$ with some combinations of the time-space increments. The Figure 1 shows the solitary waves at times $t = 0, 5, 10, 15, 20$ and maximum error at time $t=20$. Visual solutions exhibit very smooth propagation in right direction. Tables 1-2 show computational errors in the maximum norm. Results of the very recent studies are added in Table 1 to make comparison error and conservation constants. The predictor–corrector quartic and quintic B-spline collocation schemes have been formulated to compute the solutions of the RLW equation, demonstrating excellent accuracy as shown in Table 1. In addition, the present results with best accurate ones existing in the literature can be compared in the Table 2.

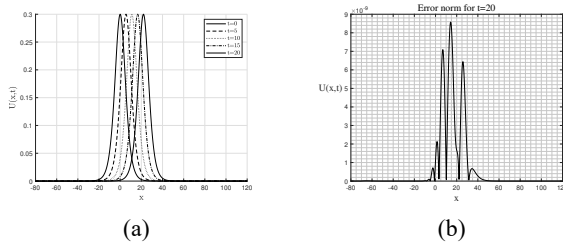


Fig. 1: Evolution of the single solitary wave at various times (a) Numerical simulations (b) Maximum error norm
Source: created by the authors.

Table 1. Results corresponding to a single solitary wave at $t = 20$ with $c = 0.1$.

Error measures				
h	Δt	L_∞	L_∞ [11]	
0.2	0.02	8.5791×10^{-9}	6.47×10^{-9}	
0.1	–	9.4295×10^{-9}	5.78×10^{-9}	
0.02	–	9.5558×10^{-9}	6.20×10^{-9}	
0.2	0.01	3.8551×10^{-9}	7.72×10^{-9}	
0.1	–	1.0817×10^{-9}	6.11×10^{-10}	
0.02	–	1.1332×10^{-9}	8.59×10^{-10}	
Conserved quantities				
h	Δt	I_1^{20}	I_2^{20}	I_3^{20}
0.2	0.02	3.97994975	0.81046249	2.57900744
0.1	–	3.97994975	0.81046249	2.57900743
0.02	–	3.97994975	0.81046249	2.57900744
0.2	0.01	3.97994975	0.81046249	2.57900743
0.1	–	3.97994975	0.81046249	2.57900744
0.02	–	3.97994975	0.81046249	2.57900743

Source: created by the authors; comparative results are taken from [11].

Table 2. Comparison of maximum errors with results from previous studies at $t = 20$ with $c = 0.1$

Method	L_∞	h	Δt	Interval
present	1.15×10^{-6}	0.12	0.1	$-120 \leq x \leq 120$
present	1.22×10^{-10}	0.05	0.01	$-120 \leq x \leq 120$
[11]	3.99×10^{-7}	0.12	0.1	$-120 \leq x \leq 120$
[11]	7.53×10^{-10}	0.05	0.01	$-120 \leq x \leq 120$
[6]	6.78×10^{-11}	$N = 512$	0.1	$-80 \leq x \leq 100$
[8] AM1	2.03×10^{-9}	0.01	0.01	$-80 \leq x \leq 100$
[9] M2	2.22×10^{-11}	0.02	0.02	$-80 \leq x \leq 100$
[1]	7.28×10^{-7}	0.1	0.1	$-40 \leq x \leq 60$
[3]	4.97×10^{-7}	0.78125	0.01	$-40 \leq x \leq 60$
[2]	7.71×10^{-7}	$N = 5000$	0.0001	$-100 \leq x \leq 100$

Source: created by the authors; comparative results are taken from [11], [6], [8], [9], [1], [3], [2].

3.2 Visualizations and results for Wave Undulation Modeling

Use of the following IC:

$$U(x, 0) = 0.5 \left(1 - \tanh \left(\frac{x - x_c}{d} \right) \right) U_0, \quad (41)$$

$$\sigma_1 = U(a, t) = U_0, \quad \sigma_2 = U(b, t) = 0, \quad U_x(a, t) = 0. \quad (42)$$

trigger development of the undulation for the RLW equation. A series of waves known as undulation forms when deeper water flow into the area of still water. The total number of waves in the undulation is influenced by the gradient of the deeper water. $U(x, 0)$ is the height of the water surface at $t = 0$ over the space interval $-36 \leq x \leq 300$. U_0 is the height of the deeper stream water flowing from the the left. The parameters $\varepsilon = 1.5$, $\mu = 1/6$, $U_0 = 0.1$, $x_c = 0$ and time increment $-36 \leq x \leq 300$, slope $d = 5$, steeper slope $d = 2$, space increment $h = 0.24$ are chosen to compare with solution profiles of earlier studies. Simulations are conducted up time $t = 300$ and conserved invariants together with those obtained with the method of predictor-corrector collocation method using quintic B-splines are documented for the given slopes in Table 3 and Table 4 to verify the true solution of the suggested algorithm. Amplitude profiles of the undulations at $t = 250$ are documented in the Table 5 for both $d = 2$ and $d = 5$. The undulation profiles corresponding to $d = 2$ and $d = 5$ at (a) $t = 150$ and (b) $t = 250$ are shown in Figure 2 and Figure 3, respectively.

Table 3. Conservation invariants corresponding to wave undulation case $d = 2$

Present method			
t	I_1^t	I_2^t	I_3^t
0	3.61200000	0.35147778	1.08822000
50	8.98700000	0.90147795	2.79324665
100	14.36200000	1.45147854	4.49803222
150	19.73700000	2.00147956	6.20270511
200	25.11200000	2.55148103	7.90733727
250	30.48699991	3.10148296	9.61195450
Reference [11]			
t	I_1^t	I_2^t	I_3^t
0	3.61200000	0.35147777	1.08822000
50	8.98700000	0.90147790	2.79324643
100	14.36200000	1.45147823	4.49803122
150	19.73700000	2.00147884	6.20270286
200	25.11200000	2.55147973	7.90733324
250	30.48699968	3.10148085	9.61194801

Source: created by the authors; comparative results are taken from [11].

Table 4. Conservation invariants corresponding to wave undulation case $d = 5$

<i>Present method</i>			
t	I_1^t	I_2^t	I_3^t
0	3.61200013	0.33631114	1.04097008
50	8.98700014	0.88631114	2.74645725
100	14.36200014	1.43631125	4.45166072
150	19.73700014	1.98631163	6.15653826
200	25.11200014	2.53631241	7.86124330
250	30.48700013	3.08631364	9.56588695

<i>Reference [11]</i>			
t	I_1^t	I_2^t	I_3^t
0	3.61200013	0.33631114	1.04097008
50	8.98700021	0.88631116	2.74645729
100	14.36200028	1.43631122	4.45166063
150	19.73700036	1.98631144	6.15653770
200	25.11200043	2.53631191	7.86124179
250	30.48700050	3.08631267	9.56588396

Source: created by the authors; comparative results are taken from [11].

Table 5. Amplitude distributions of the wave undulations at time $t = 250$

	Position ($d = 2$)	Amplitude ($d = 2$)	Position ($d = 5$)	Amplitude ($d = 5$)
Undulation-1	265.92	0.1821	264.96	0.1779
Undulation-2	254.40	0.1622	253.92	0.1534
Undulation-3	244.08	0.1446	244.08	0.1324
Undulation-4	234.72	0.1307	234.96	0.1178

Source: created by the authors.

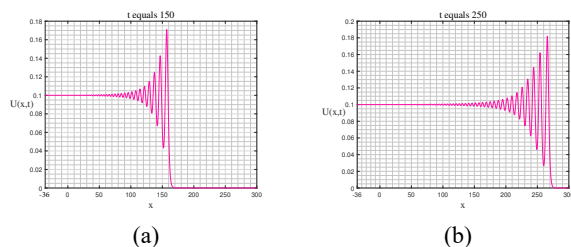


Fig. 2: Undulation profiles corresponding to $d = 2$
(a) $t = 150$ (b) $t = 250$

Source: created by the authors.

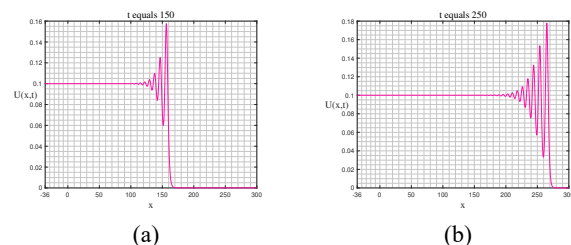


Fig. 3: Undulation profiles corresponding to $d = 5$
(a) $t = 150$ (b) $t = 250$

Source: created by the authors.

3.3 Visualizations and results for Wave Generation Modeling

Generation of the solitary waves is managed by applying an initial function at $x = 0$ within the region $0 \leq x \leq x_R$:

$$U(a, t) = \sigma_1 = \begin{cases} U_0 \frac{t}{\tau}, & 0 \leq t \leq \tau, \\ U_0, & \tau \leq t \leq t_0 - \tau, \\ U_0 \frac{t_0 - t}{\tau}, & t_0 - \tau \leq t \leq t_0, \\ 0, & \text{otherwise.} \end{cases} \quad (43)$$

and right boundary point x_R is taken far from the left boundary point $x_L = 0$. Thus effect of the right BCs $U(b, t) = 0$ and $U_x(a, t) = 0$ are reduced for the wave generation. The boundary condition is an external forcing having trapezoid form at left boundary point $x = 0$ to produce solitary waves. We take the parameters of initial condition as boundary forcing value $U_0 = 2$, forcing effect value $\tau = 0.3$, forcing period $t_0 = 20$. The program is run with time increment $\Delta = 0.1$, space increment $h = 0.4$, RLW equation's constants $\varepsilon = \mu = 1$ over the space interval $0 \leq x \leq 400$ up to time $t = 170$. Values of both the conserved quantities and solitary waves amplitude are recorded seeing Table 6. During forcing period, four solitary waves are observed to be generated and with no forcing, waves propagates in the positive direction to the right with the velocities depending upon their magnitudes. Conserved quantities remain close to constant when the forcing is off. Their profiles at some times are drawn in Figure 4.

Table 6. Amplitude of the solitary wave during the forcing period $0 \leq t_0 \leq 20$.

t	5	4	3	2	1	I_1^t	I_2^t	I_3^t
2.5	—	—	—	—	2.3028	9.8724	18.3656	86.0210
5	—	—	—	—	2.8077	20.2611	43.2346	216.1353
7.5	—	—	—	2.2617	3.0826	30.2470	66.5130	336.0614
10	—	—	2.0229	2.5445	3.2574	40.1146	89.3190	452.9999
15	—	2.0684	2.4464	2.9158	3.4371	60.1524	136.1399	694.0322
20	1.9765	2.3909	2.7512	3.1499	3.5410	77.7083	178.7591	896.8864
40	1.1874	2.4051	3.1151	3.4819	3.6897	79.6000	177.8586	901.9347
60	1.1829	2.3859	3.0899	3.5235	3.7343	79.6013	177.9140	902.2967
80	1.1842	2.3871	3.0908	3.5273	3.7548	79.5987	177.9713	902.6719
100	1.1840	2.3850	3.0796	3.5278	3.7446	79.6008	178.0291	903.0504
130	1.1845	2.3797	3.0843	3.5250	3.7577	79.5995	178.1161	903.6206
150	1.1862	2.3858	3.0894	3.5277	3.7520	79.6003	178.1743	904.0016
170	1.1863	2.3879	3.0859	3.5238	3.7638	79.5997	178.2325	904.3833

Source: created by the authors.

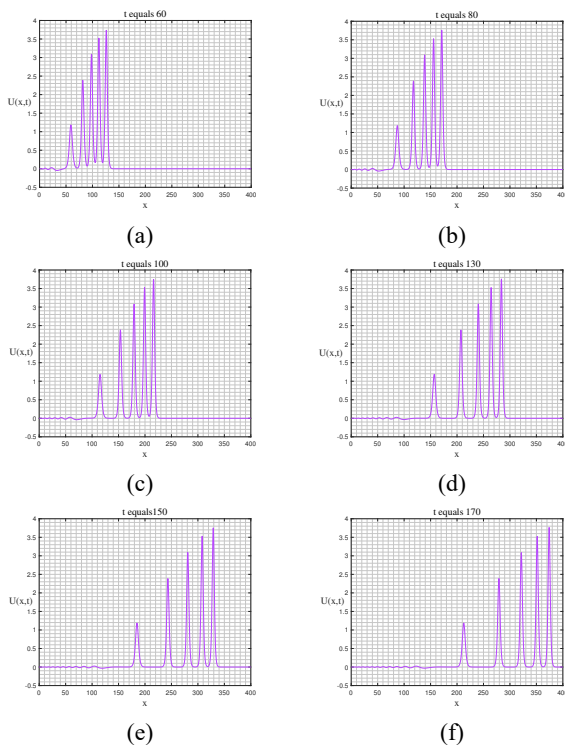


Fig. 4: Wave generation (a) $t = 60$ (b) $t = 80$ (c) $t = 100$ (d) $t = 130$ (e) $t = 150$ (f) $t = 170$
Source: created by the authors.

4 Conclusion

A new algorithm is established for getting numerical solution of the RLW equation. An system of algebraic equations is obtained by use of the predictor-corrector time integration and cubic B-spline Galerkin method space discretization. We have seen that adaptation of the high order predictor-corrector time integration to the the cubic B-spline Galerkin method provides very low accuracy in solving the RLWE. Thus the the algorithm becomes one of the efficient algorithm. To support the claim, The maximum error norms of solitary wave motion show almost same accuracy with the best earlier methods tabulated in Table. In addition, conservation quantities of undulation and wave generation remain constant when the forcing and initial effect are canceled.

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Declaration of Generative AI and AI-assisted Technologies in the Writing Process

During the preparation of this work, the author(s) uti-

lized artificial intelligence (AI) technologies solely for the purpose of linguistic refinement and grammatical correction. Following the application of these tools, the author(s) comprehensively reviewed, verified, and edited the resulting text. The author(s) maintain full accountability and responsibility for the accuracy, integrity, and final content of the publication.

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Conflict of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.

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