Nonlinear Maligranda-Type and Dunkl-Williams Inequalities in Generalized Norm and Modular Structures

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Abstract: - In this paper, we introduce a class of generalized nonlinear norm-like functionals obtained by adding an increasing nonlinear perturbation to a norm. The principal example is $N(x) = \|x\| + \|x\|^2$, which is positive, definite, continuous, but non-homogeneous and hence is not a norm. We establish a distorted triangle inequality and a nonlinear Maligranda-type reverse-triangle estimate with explicit magnitude-dependent coefficients. The sharpness of the principal coefficient is demonstrated, and the exact modulus of continuity of N on bounded balls is determined. We further obtain perturbation estimates for the general form $N_\varphi(x) = \|x\| + \varphi(\|x\|)$, together with bounded-set Lipschitz and stability results. A nonlinear Dunkl-Williams-type inequality is also derived. Finally, the construction is extended to subadditive modular functionals, yielding corresponding modular-type inequalities.

Key-Words: -Norm inequality, Maligranda inequality, nonlinear functional, nonlinear perturbation, sharpness, modulus of continuity, Dunkl-Williams inequality, modular functional, modular spaces

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1 Introduction

Norm inequalities are central tools in functional analysis, approximation theory, optimization, nonlinear analysis, and operator theory. One of the most fundamental inequalities in every normed linear space is the triangle inequality

$$\|x + y\| \leq \|x\| + \|y\| \tag{1}$$

and its reverse form

$$|\|x\| - \|y\|| \leq \|x - y\|. \tag{2}$$

One of the classical inequalities in normed linear spaces is the Dunkl-Williams inequality, introduced by [1]. For nonzero vectors $x, y \in X$, the classical form is

$$\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\| \leq \frac{4\|x-y\|}{\|x\|+\|y\|}. \tag{3}$$

These inequalities provide elementary but powerful estimates for continuity, convergence, perturbation analysis, and stability. They also form the basis of many refined norm inequalities in Banach space theory.

Maligranda and subsequent authors studied simple but sharp refinements of classical norm inequalities, including estimates related to the Dunkl-Williams inequality and angular distance and stability under perturbations [2], [3], [4].

In parallel, generalized systems of calculus such as multiplicative calculus and Non-Newtonian calculus have been used to reformulate metric and norm-like structures through alternative operations. However, many of these approaches are obtained from logarithmic or exponential generators and therefore retain a strong connection with classical structures [5], [6], [7], [8], [9], [10], [11].

The present work adopts a different but more modest direction. Instead of claiming an entirely new calculus, we construct a nonlinear norm-like functional that preserves positivity and definiteness while losing homogeneity. This allows classical norm inequalities to be converted into nonlinear analogs of Maligranda-type inequalities with variable distortion coefficients.

The term non-isomorphic is used here in a precise and limited sense: the functional is not a norm under the usual linear homogeneity axiom, and it is not linearly equivalent to a classical norm. It is not used in the stronger sense that no scalar representation

$N(x) = g(\|x\|)$ exists, since the present example is generated by $g(t) = t + t^2$.

Motivated by these observations, we construct generalized nonlinear functionals and establish corresponding nonlinear Maligranda-type and Dunkl-Williams-type inequalities both in normed and modular settings.

The purpose of this paper is to construct a non-generator-based generalized norm-like functional, and derive a nonlinear Maligranda-type inequality that is not reducible to classical forms.

2 Main Results

In this section, we introduce the nonlinear functional N_φ and establish generalized nonlinear analogs of several classical norm inequalities.

Definition 1 (Generalized nonlinear functionals). Let X be a normed vector space. A mapping $N: X \to [0,\infty)$ is called a generalized norm-like functional if

1. $N(x) \geq 0$ for all $x \in X$,
2. $N(x) = 0$ iff $x = 0$,
3. N is continuous with respect to the norm topology of X.

No homogeneity or triangle inequality is assumed.

Definition 2 Let $(X, \|.\|)$ be a normed space, and let $\varphi: [0,\infty) \to [0,\infty)$ be increasing. Since $\|x\|$ is a nonnegative real number, the function φ acts on the scalar quantity $\|x\|$, rather than on the vector x. We define the associated nonlinear functional by

$$N_\varphi(x) = \|x\| + \varphi(\|x\|), \quad x \in X. \tag{4}$$

Thus, N_φ is obtained by adding a nonlinear perturbation $\varphi(\|x\|)$ to the original norm. The function φ controls the magnitude of this perturbation, while the underlying geometry of the space is still determined by the norm $\|.\|$. Different choices of φ produce different nonlinear functionals. Throughout this paper, the principal example is $\varphi(t) = t^2$, for which

$$N_\varphi(x) = \|x\| + \|x\|^2. \tag{5}$$

Proposition 1 Let

$$N(x) = \|x\| + \|x\|^2, \quad x \in X$$

Then N is positive, definite, and continuous. However, N is not homogeneous and therefore is not a norm.

Proof. Since $\|x\| \geq 0$, $N(x) = \|x\| + \|x\|^2 \geq 0$. Moreover, $N(x) = 0$ holds if and only if $\|x\| = 0$, which is equivalent to $x = 0$. Since both $t \to t$ and $t \to t^2$ are continuous on $[0,\infty)$, the mapping N is continuous with respect to the norm topology on X. Now let $\lambda \in \mathbb{R}$. Then

$$N(\lambda x) = |\lambda|\|x\| + |\lambda|^2\|x\|^2, \tag{6}$$

whereas $|\lambda|N(x) = |\lambda|\|x\| + |\lambda|\|x\|^2$. These expressions are generally different whenever $|\lambda| \neq 1$ and $|\lambda| \neq 0$. For example, if $\lambda = 2$, then

$$N(2x) = 2\|x\| + 4\|x\|^2,$$

while $2N(x) = 2\|x\| + 2\|x\|^2$. Hence

$$N(\lambda x) \neq |\lambda|N(x),$$

Therefore N fails the homogeneity axiom and hence is not a norm.

Remark 1 Although N admits the scalar representation

$$N(x) = g(\|x\|), g(t) = t + t^2,$$

the resulting functional is not a norm because it fails the homogeneity axiom. Consequently, the induced inequalities contain nonlinear magnitude-dependent coefficients that do not occur in the classical theory.

The following theorem provides a generalized nonlinear version of the triangle inequality (1).

Theorem 1 (**Triangle Inequality with Nonlinear Distortion**). For all x, y in X, the following distorted triangle inequality holds:

$$N(x + y) \leq N(x) + N(y) + 2\|x\|\|y\| \tag{7}$$

Proof. Using the usual triangle inequality,

$$\begin{aligned} N(x + y) &= \|x + y\| + \|x + y\|^2 \\ &\leq (\|x\| + \|y\|) + (\|x\| + \|y\|)^2 \\ &= \|x\| + \|y\| + \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| \\ &= N(x) + N(y) + 2\|x\|\|y\|. \end{aligned}$$

The inequality is sharp since equality holds whenever x and y are positively collinear. Thus, the coefficient 2 cannot be reduced.

The additional term $2\|x\|\|y\|$ measures the nonlinear distortion caused by the second-order perturbation. Without this perturbation, the estimate reduces to the ordinary triangle inequality.

We now establish the principal result of this paper.

Theorem 2 (**Main Nonlinear Maligranda-Type Inequality**). For all $x, y \in X$,

$$|N(x) - N(y)| \leq (1 + \|x\| + \|y\|)\|x - y\|. \quad (8)$$

Proof. By the definition of N, we have

$$\begin{aligned}|N(x) - N(y)| &= |(\|x\| - \|y\|) + (\|x\|^2 - \|y\|^2)| \\ &\leq |\|x\| - \|y\|| + |\|x\|^2 - \|y\|^2| \\ &= |\|x\| - \|y\||(1 + \|x\| + \|y\|).\end{aligned}$$

Applying the reverse triangle inequality

$$|N(x) - N(y)| \leq (1 + \|x\| + \|y\|)\|x - y\|$$

yields.

Proposition 2 (Sharpness of the coefficient). The coefficient $1 + \|x\| + \|y\|$ in inequality (8) is sharp.

Proof. The inequality (8) becomes an equality for $x = ae$, $y = be$, where e is a unit vector and $a, b \geq 0$. Indeed,

$$\|x - y\| = |a - b|,$$

while

$$\begin{aligned}|N(x) - N(y)| &= |(a + a^2) - (b + b^2)| \\ &= (1 + a + b)|a - b|.\end{aligned}$$

Therefore, the coefficient is sharp.

Corollary 1: On every bounded subset

$$B_r = \{x \in X: \|x\| \leq r\},$$

the functional N is Lipschitz continuous with constant $1 + 2r$.

Proof. If $x, y \in B_r$, then

$$1 + \|x\| + \|y\| \leq 1 + 2r.$$

Applying inequality (8) yields

$$|N(x) - N(y)| \leq (1 + 2r)\|x - y\|.$$

Hence the result follows.

To obtain a quantitative description of the continuity of the nonlinear functional N, we now determine its exact modulus of continuity on bounded subsets. This provides a sharper result than the bounded-set Lipschitz estimate established in Corollary 1 and answers the natural question of the optimal continuity behavior of N.

Definition 3. Exact modulus of continuity on B_r of N is defined by

$$\omega_N(r, \delta) = sup\{|N(x) - N(y)|: x, y \in B_r, \|x - y\| \leq \delta\}$$

for $0 \leq \delta \leq r$.

Proposition 3. For every $r > 0$,

$$\omega_N(r, \delta) = (1 + 2r - \delta)\delta, 0 \leq \delta \leq r.$$

Proof. Let $a = \|x\|$ and $b = \|y\|$. Then

$$0 \leq a, b \leq r.$$

Moreover, by the reverse triangle inequality,

$$|a - b|| \|x\| - \|y\|| \leq \|x - y\| \leq \delta.$$

Now define

$$d = |a - b|.$$

We can write

$$|N(x) - N(y)| = d(1 + a + b).$$

Now we maximize $d(1 + a + b)$ subject to $0 \leq a, b \leq r$, $d = |a - b|$. For a fixed d, suppose $a \geq b$ (the other case is symmetric). Then $a = b + d$. Since $a \leq r$,

$$b \leq r - d.$$

Therefore,

$$a + b = 2b + d \leq 2(r - d) + d = 2r - d.$$

Hence

$$\begin{aligned}|N(x) - N(y)| &= d(1 + a + b) \\ &\leq d(1 + 2r - d).\end{aligned}$$

Finally, for

$$f(d) = d(1 + a + b), \ 0 \leq d \leq \delta \leq \mathrm{r},$$

We have

$$f'(d) = 1 + a + b \geq 1 > 0,$$

thus f is increasing on $[0, \delta]$. Consequently,

$$|N(x) - N(y)| \leq \delta(1 + 2r - \delta).$$

Hence

$$\omega_N(r, \delta) \leq (1 + 2r - \delta)\delta.$$

To prove equality, let $x = re, y = (r - \delta)e$, where e is a unit vector. Then $\|x - y\| = \delta$, and

$$\begin{aligned}|N(x) - N(y)| &= r + r^2 - (r - \delta) - (r - \delta)^2 \\ &= ((1 + 2r - \delta)\delta.\end{aligned}$$

Hence equality holds.

Remark 2. Since the functional

$$\|x\| + \|x\|^2$$

is obtained by composing the norm with the scalar map

$$g(t) = t + t^2,$$

it does not change intrinsic geometric properties such as uniform convexity or uniform smoothness. Rather, it modifies quantitative estimates by introducing nonlinear magnitude-dependent coefficients. Thus, the nonlinear functional does not itself define a new linear norm geometry. In particular, geometric properties such as uniform convexity should be attributed to the underlying normed space rather than to N. The classical theory of uniformly convex spaces is represented, for example, by Clarkson's foundational work [12]. Consequently, the inequalities established here should be interpreted as nonlinear quantitative perturbations of classical norm inequalities rather than as new characterizations of Banach-space geometry.

Consequently, the inequalities established above should be viewed as nonlinear perturbations of classical geometric inequalities rather than as new characterizations of Banach space geometry.

The previous estimates can be extended to perturbations of the form $N_\varphi(x) = \|x\| + \varphi(\|x\|)$.

Theorem 3 Let φ be locally Lipschitz on bounded subintervals of $[0,\infty)$. For $x, y \in X$, let $L_\varphi(x,y)$ denote any Lipschitz constant of φ on the interval with endpoints $\|x\|$ and $\|y\|$. Then

$$\left|N_\varphi(x) - N_\varphi(y)\right| \le \left(1 + L_\varphi(x,y)\right)\|x - y\|. \quad (9)$$

Proof. We can write

$$\left|N_\varphi(x) - N_\varphi(y)\right| \le \left|\|x\| - \|y\|\right| + \left|\varphi(\|x\|) - \varphi(\|y\|)\right|.$$

By local Lipschitz continuity of φ, the second term is at most $L_\varphi(x,y)\left|\|x\| - \|y\|\right|$. Applying the reverse triangle inequality (2) yields the desired result.

Theorem 4 Assume that φ is differentiable and increasing on bounded subintervals of $[0,\infty)$. Then

$$\left|N_\varphi(x) - N_\varphi(y)\right| \le \left(1 + \sup_{t\in I}\varphi'(t)\right) . \|x - y\|, \quad (10)$$

where $I = [min\{\|x\|, \|y\|\}, max\{\|x\|, \|y\|\}]$.

Proof. By the mean value theorem,

$$|\varphi(\|x\|) - \varphi(\|y\|)| \le sup_{t\in I}\varphi'(t)\left|\|x\| - \|y\|\right|.$$

Combining this estimate with the reverse triangle inequality gives the desired result.

Corollary 2 If φ is differentiable and

$$sup_{0\le t\le r}|\varphi'(t)| \le M_r,$$

Then N_φ is Lipschitz continuous on B_r with constant $1 + M_r$.

The generalized nonlinear functional satisfies stability properties analogous to classical norm structures.

Corollary 3 (i) If $x_n \to x$ in norm, then $N(x_n) \to N(x)$.

(ii) If $x_n \to x$ and $y_n \to y$ in norm, then

$$|N(x_n) - N(y_n)| \to |N(x) - N(y)|$$

as $n \to \infty$.

Proof. Since $x_n \to x$ in norm, the sequence (x_n) is norm-bounded. Hence there exists $r > 0$ such that $x_n, x \in B_r$ for all sufficiently large n. Applying Corollary 1 gives

$$|N(x_n) - N(x)| \le (1 + 2r)\|x_n - x\|.$$

Since $\|x_n - x\| \to 0$, we conclude that $N(x_n) \to N(x)$, and thus (i) holds.

Assertion (ii) follows immediately from part (i) and the continuity of the absolute value function.

The classical Dunkl-Williams inequality indicates that for nonzero vectors x, y in a normed space,

$$\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\| \le 4\frac{\|x-y\|}{\|x\|+\|y\|}. \quad (11)$$

This inequality estimates angular distance and is a fundamental tool in the geometry of normed spaces [1], [3]. We now define the generalized angular functional associated with N.

Definition 4 For nonzero vectors $x, y \in X$, define the N-normalized difference by

$$A_N(x,y) = \left\|\frac{x}{N(x)} - \frac{y}{N(y)}\right\|. \quad (12)$$

Theorem 5 For all nonzero $x, y \in X$,

$$A_N(x,y) \le \left(\frac{N(x)+1+\|x\|+\|y\|}{N(x)N(y)}\right)\|x - y\|. \quad (13)$$

Proof. Since

$$\frac{x}{N(x)} - \frac{y}{N(y)} = \frac{xN(y)-yN(x)}{N(x)N(y)},$$

We obtain

$$A_N(x,y) = \frac{\|xN(y)-yN(x)\|}{N(x)N(y)}.$$

Now write

$$xN(y) - yN(x) = x\big(N(y) - N(x)\big) + N(x)(x - y).$$

Applying the triangle inequality,

$$\|xN(y) - yN(x)\| \le \|x\||N(y) - N(x)| + N(x)\|x - y\|.$$

By Theorem 2,

$$|N(y) - N(x)| \le (1 + \|x\| + \|y\|)\|x - y\|.$$

Hence

$$\|xN(y) - yN(x)\| \\ \leq [N(x) + \| x \| (1 + \| x \| + \| y \|)] \| x - y \|.$$

Dividing by $N(x)N(y)$ yields the inequality (13).

We now derive several consequences.

Corollary 4 (i) By continuity of the angular functional, if $x_n \to x$ and x is nonzero, then

$$A_N(x_n, x) \to 0$$

for all sufficiently large n.
(ii) (Generalized stability estimate) For nonzero vectors x, y;

$$A_N(x, y) = O(\|x - y\|).$$

(iii) (Classical limit case) Consider the family of functionals

$$N_\varepsilon(x) = \|x\| + \varepsilon \|x\|^2, \ \ \varepsilon \geq 0.$$

As $\varepsilon \to 0$, $N_\varepsilon(x) \to \|x\|$ for every $x \in X$, and the corresponding nonlinear estimate converges continuously to the classical normalization

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|$$

and inequality (13) reduces to the classical Dunkl–Williams estimate.

Proof. Since

$$N(x_n) \to N(x) \text{ as } n \to \infty.$$

Theorem 5 implies

$$A_N(x_n, x) \leq C\|x_n - x\| \to 0.$$

This nonlinear Dunkl-Williams type estimation differs from the classical angular inequality because the normalization denominator contains nonlinear perturbation terms. As a result, the coefficient depends not only on the relative direction but also on the vector magnitudes.

Example 1 (i) (One-dimensional example): Let $X = \mathbb{R}$ with the usual norm. Then

$$N(x) = |x| + |x|^2.$$

For $x = 1$, $y = 2$; we have $N(1) = 2$, $N(2) = 6$. Hence $|N(1) - N(2)| = 4$, which shows the equality in Theorem 2.
(ii) (Euclidean example) Let $X = \mathbb{R}^2$ with the Euclidean norm, and take $x = (1,0)$, $y = (0,1)$. Then $\|x\| = \|y\| = 1$ and $\|x - y\| = \sqrt{2}$. In this case, Theorem 2 gives

$$|N(x) - N(y)| \leq 3\sqrt{2}.$$

Since $N(x) = N(y) = 2$, the left-hand side equals zero and the inequality is strict.

Definition 4 (Modular space extensions) Let X_ϱ be a modular space generated by a modular functional

$$\varrho: X \to [0, \infty).$$

Define the nonlinear modular perturbation on X_ϱ by

$$M(x) = \varrho(x) + \left(\varrho(x)\right)^2.$$

Throughout this section we assume that the modular ϱ is subadditive, that is,

$$\varrho(x + y) \leq \varrho(x) + \varrho(y), x, y \in X_\varrho.$$

Theorem 6 For all $x, y \in X_\varrho$,

$$M(x + y) \leq M(x) + M(y) + 2\varrho(x)\varrho(y).$$

Proof. Using modular subadditivity

$$\varrho(x + y) \leq \varrho(x) + \varrho(y),$$

we obtain

$$M(x + y) = \varrho(x + y) + \left(\varrho(x + y)\right)^2 \\ \leq \left(\varrho(x) + \varrho(y)\right) + \left(\varrho(x) + \varrho(y)\right)^2.$$

Expanding the square terms gives

$$M(x + y) \leq M(x) + M(y) + 2\varrho(x)\varrho(y).$$

Theorem 7 For all $x, y \in X_\varrho$,

$$|M(x) - M(y)| \leq \left(1 + \varrho(x) + \varrho(y)\right) \\ . |\varrho(x) - \varrho(y)|.$$

Proof. Observe that

$$M(x) - M(y) = \left(\varrho(x) - \varrho(y)\right) \\ . \left(1 + \varrho(x) + \varrho(y)\right).$$

Taking absolute values, the desired result is achieved.

Theorem 8 (Generalized Modular Dunkl-Williams Inequality) For all nonzero $x, y \in X_\varrho$,

$$A_M(x,y) \leq \frac{(2+\varrho(x)+\varrho(y))\|x-y\|}{M(x)+M(y)},$$

where

$$A_M(x,y) = \left\|\frac{x}{M(x)} - \frac{y}{M(y)}\right\|.$$

Proof. The proof follows similarly to the generalized Dunkl–Williams inequality by using the modular perturbation estimate obtained in Theorem 7 together with the triangle inequality.

The estimates developed above can be used in nonlinear stability analysis, perturbation theory, generalized approximation problems, nonlinear metric geometry, and functional inequalities with growth-dependent coefficients. Bounded-set Lipschitz estimates are particularly useful when nonlinear functionals are applied to convergent sequences or iterative schemes.

The nonlinear functional

$$\|x\| + \|x\|^2$$

naturally combines linear and quadratic growth. Functionals of this form arise naturally in simplified models of nonlinear regularization, trust-region methods, and composite optimization. The explicit continuity estimates established in this paper provide quantitative perturbation bounds for iterative optimization algorithms.

In machine learning, nonlinear regularization terms combining ℓ_2- and quadratic-type penalties are frequently employed to improve numerical stability and robustness. The bounded-set Lipschitz estimates obtained here guarantee controlled sensitivity of the functional to data perturbations. Consequently, the nonlinear inequalities developed in this paper may be useful in the stability analysis of optimization algorithms involving non-homogeneous objective functions.

3 Conclusion

We introduced a nonlinear perturbation of the norm, established sharp Maligranda-type inequalities, obtained the exact modulus of continuity, proved Lipschitz continuity on bounded sets, derived nonlinear Dunkl-Williams-type estimates, and extended the framework to modular spaces. These results provide a systematic nonlinear extension of several classical norm inequalities while preserving explicit quantitative estimates.

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References:

[1] C. F. Dunkl and K. S. Williams, A Simple Norm Inequality, *The American Mathematical Monthly*, 71(1), 1964, 53-54. DOI: 10.2307/2311304.

[2] L. Maligranda, Simple Norm Inequalities, *The American Mathematical Monthly*, 113(3), 2006, 256-260. DOI 10.2307/27641893.

[3] J. Pecaric and R. Rajic, The Dunkl-Williams Inequality with n Elements in Normed Linear Spaces, *Mathematical Inequalities and Applications*, 10(2), 2007, 461-470. DOI 10.7153/mia-10-44.

[4] S. S. Dragomir, *Advances in Inequalities of the Schwarz, Triangle and Heisenberg Type in Inner Product Spaces*, Nova Science Publishers, 2007. ISBN: 978-1-59454-903-8.

[5] M. Grossman and R. Katz, *Non-Newtonian Calculus*, Lee Press, 1972. ISBN: 0-912938-01-3.

[6] A. E. Bashirov, E. M. Kurpınar and A. Özyapıcı, Multiplicative Calculus and Its Applications, *Journal of Mathematical Analysis and Applications*, 337(1), 2008, 36-48. DOI: 10.1016/j.jmaa.2007.03.081.

[7] C. Duyar, O. Oğur, Some basic topological properties on non-Newtonian real line, *BJMCS*, 2015, 300-307. Doi: 10.9734/BJMCS/2015/17941

[8] C. Duyar, O. Oğur, A note on topology of Non-Newtonian real numbers, *IOSR J. Math.*, 2017, 13(6), 11-14.

[9] N. Değirmen, C. Duyar, A new perspective on Fibonacci and Lucas numbers, *Filomat*, 2023, 37(28), 9561-9574. DOI: 10.2298/FIL2328561D.

[10] N. Güngör, Some geometric properties of the non-newtonian sequence $\ell_p(N)$, *Math. Slovaca*, 2020, 70, 689-696. DOI: 10.1515/ms-2017-0382.

[11] İ. Eryılmaz, Non-Newtonian Lebesgue spaces with their basic features, *Journal of Mathematical Extension*, 2024, 18(7), 1-23. DOI 10.30495/JME.2024.3005.

[12] J. A. Clarkson, Uniformly Convex Spaces, *Transactions of the American Mathematical Society*, 40(3), 1936, 396-414.

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