On Pairwise Meta-P-Closed Spaces in Bitopological Spaces

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Abstract: - This study introduces the notion of pairwise meta-P-closed spaces, serving as a logical extension of pairwise P-closed spaces. It examines the inter connections among pairwise meta-P-closed, pairwise P-closed, and pairwise L-closed spaces, identifies their hereditary characteristics, and assesses their stability under pairwise homeomorphisms. Furthermore, we prove that every p.w.T_2, pairwise regular, pairwise meta-P-closed bitopological space is pairwise normal under weaker covering assumptions than full paracompactness. Several characterizations and product results are obtained.

Key-Words: - Pairwise meta-P-closed, pairwise P-closed, pairwise L-closed, pairwise-homeomorphism, pairwise-normality, meta-paracompactness.

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1 Introduction

Bitopological spaces was introduced by [1], provide a framework for studying two topologies on the same set. The notion of pairwise P-closed spaces was defined by [2], where ξ_1-paracompact subsets are ξ_2-closed and vice versa. In this paper, we introduce pairwise meta-P-closed spaces, a weaker condition that allows ξ_1-metacompact subsets (those admitting point finite open refinements) to play the role of paracompact ones.

We establish relationships among these spaces, prove hereditary properties, and demonstrate invariance under p-homeomorphisms. A key result is that p.w.T_2 + p-regular + p.w.meta-P-closed implies p-normal. Product theorems under meta-Lindel¨of density conditions are also provided.

2 Preliminaries

Definition 1 A bitopological space (X, ξ_1, ξ_2) is p.w.P-closed if every ξ_1- paracompact subset is ξ_2-closed and every ξ_2-paracompact subset is ξ_1-closed, [2].

Definition 2 The $\xi_1\xi_2$- open cover μ is a cover for the bitopological space (X, ξ_1, ξ_2) such that $\mu \subseteq \xi_1 \cup \xi_2$, while it is called a p-open cover if it has at least one nonempty ξ_1-open proper set and one nonempty ξ_2-open proper set of X, [3].

Definition 3 The bitopological space (X, ξ_1, ξ_2) is said to be pairwise T_0, for $m \neq h \in X$, there is either a ξ_1-open set containing m but not h, or a ξ_2-open set containing h but not m, [4].

Definition 4 The bitopological space (X, ξ_1, ξ_2) is said to be pairwise T_1, for $m \neq h \in X$, there is a ξ_1-open U with $m \in U, h \notin U$ **and** a ξ_2-open V with $h \in V$, $m \notin V$, [4].

Definition 5 The bitopological space (X, ξ_1, ξ_2) is said to be pairwise T_2 (p-Hausdorff) or briefly p.w.T_2 if every two distinct elements h, m in X have disjoint ξ_1-open neighborhood of h and ξ_2-open neighborhood of m, [5].

Definition 6 A bitopological space (X, ξ_1, ξ_2) is pairwise regular if for every point $x \in X$ and every ξ_i-closed set F with $x \notin F$ (i =1, 2), there exist disjoint open sets $U \in \xi_i$ and $V \in \xi_j$ ($j \neq i$) such that $x \in U$, $F \subseteq V$, [6].

Definition 7 If (X, ξ_1, ξ_2) and $(Y, \sigma 1, \sigma 2)$ are two bitopological spaces and
$f : (X, \xi_1, \xi_2) \to (Y, \sigma 1, \sigma 2)$ is a function, then f is said to be a p-homeomorphism provided that
it is a bijection and p-continuous
(ie $f_1 : (X, \xi_1) \to (Y, \sigma 1)$ and $f_2 : (X, \xi_2) \to (Y, \sigma 2)$ are continuous) as well as f^{-1}, [7].

Propostion 8 Let (X, ξ_1, ξ_2) be bitopological space such that every countable subset is closed,
then every countable subset is discrete and every compact subset is finite, [8].

Definition 9 A bitopological space (X, ξ_1, ξ_2) is pairwise paracompact if every ξi-open cover has a ξj-locally finite open refinement (for i ≠ j), [9].

Theorem 10 If (X, ξ_1,ξ_2) is pairwise paracompact and pairwise Hausdorff, then it is pairwise regular and pairwise normal, [10].

Definition 11 A bitopological space (X, ξ_1, ξ_2) is pairwise totally disconnected if the only nonempty subsets that are both ξ_1-connected and ξ_2-connected are singleton.

3 Pairwise Meta-P-Closed Spaces

Definition 12 A subset A ⊆ X is ξ_i-metacompact (i = 1, 2) if every ξ_i -open cover of A has a point-finite ξ_i -open refinement.

Definition 13 A bitopological space (X, ξ_1, ξ_2) is p.w.meta-P-closed if every ξ_1 -metacompact subset is ξ_2-closed and every ξ_2-metacompact subset is ξ_1-closed.

Example 14 Let (X, ξ_{dis}, ξ_{dis}) where ξ_{dis} is the discrete topology. Then X is p.w.meta-P-closed.

Example 15 Let (X, ξ_{coc}, ξ_u) on R. Then X is not p.w.meta-P-closed: R is ξ_{coc} -metacompact but not ξ_u-closed.

Proposition 16 In bitopological space (X, ξ_1, ξ_2), every pairwise P-closed space is pairwise meta-P-closed, but the converse is not true.

Proof 17 If A ⊆X is ξi-paracompact (i.e, every ξ_i -open cover of A has a ξ_i -locally finite ξ_i -open refinement), then it is ξ_i -metacompact (i.e., every ξ_i -open cover of A has a point-finite ξ_i -open refinement)
(i = 1,2). Thus every locally finite collection is point finite.

Counter example 18 X = Q ∪ {√2 }, ξ_1 usual on Q with isolated √2, ξ_2 discrete.

Then Q is ξ_1-metacompact but not ξ_2-closed, so not p.w.P-closed, but every ξ_1-metacompact subset is countable, hence ξ_2-closed.

Proposition 19 Every p.w.meta-P-closed space is p.w.L-closed.

Proof 20 Lindelof implies metacompact. By [2, Proposition 60], the result follows from Proposition 9.

Theorem 21 p.w.P-closed p.w.meta-P-closed p.w.L-closed.

Proof 22 From Propositions 16 and 19. Strictness via examples and [2, Example 22].

Proposition 23 The p.w.meta-P-closed property is hereditary.

Proof 24 Let Y ⊆ X, F ⊆ Y , ξ_{Y1} -metacompact. Then F is ξ_1 -metacompact in X, hence ξ_2 -closed in X, so ξ_{Y2} -closed in Y.

Proposition 25 p.w.meta-P-closedness is invariant under p-homeomorphism.

Proof 26 Let f : X → Y be p-homeomorphism, B ⊆ Y σ1-metacompact. Then f^{-1}(B) is ξ_1-metacompact, hence ξ2-closed, so B is σ2-closed.

4 Characterizations and Normality

Definition 27 A subset A ⊆ X has a dense ξi-meta-Lindelof subspace if ∃ B ⊆A, B ξi-Lindelof, Cl_{ξ_i}(B) = A.

Theorem 28 Let (X, ξ_1, ξ_2) be p.w.L-closed. If every ξi-metacompact subset has a dense ξi-meta-Lindelof subspace, then X is p.w.meta-P-closed.

Proof 29 Let W be ξ1-metacompact, V ⊆ W dense ξ1-meta-Lindelof. Then V is ξ1- Lindelof, hence ξ2-closed. Thus W is subset of Clξ1 (V) ⊆ Clξ2 (V) = V ⊆ W, therefore W = V, it is mean A is ξ2-closed.

Theorem 30 Let (X, ξ_1, ξ_2) is pairwise L-closed bitopological space. If for each (i =1, 2), every ξi-metacompact subset of X has a dense ξi-meta-Lindelöf subspace, then X is pairwise meta-P-closed.

Proof 31 Let K is ξ1-metacompact. By hypothesis, there exists M ⊂ K that is ξ1-meta-Lindelöf (i.e., Lindelöf in ξ1) and dense in K such that (Clξ1(M) = K). Since X is pairwise L-closed, every ξ1-Lindelöf subset (like M) is ξ2-closed.
Now, k ⊆ Clξ1(M) ⊆ Clξ2(M) . Since M is ξ2-closed, such that Clξ2(M) = M.
Thus, K ⊆ M ⊆ K, so K = M.
Therefore, K is ξ2-closed, i.e., Clξ2(K) = K (as M is). Symmetrically for ξ 2-metacompact subsets. The condition holds, making X p.w.meta-P-closed.

5 Product of Pairwise Meta-P-Closed Spaces

Theorem 32 Let (Xi, ξ_{i1}, ξ_{i2}), (i = 1, 2) is p-T_1, p-regular and p.w.meta-P-closed. Assume every (ξ11 ×ξ21)-metacompact subset of X1 ×X2 has a dense (ξ11 ×ξ21)- meta-Lindelof subset. Then the product is p.w.meta-P-closed.

Proof 33 Let P be (ξ11 ×ξ21) -metacompact, Q ⊆ P is dense meta-Lindelof. Let (x_0, y_0) ∉ P . Then Q_{x0} is ξ21-meta-Lindelof, hence ξ22-closed. ∃ V ∈ ξ22, y0 ∈ V , V ∩ Q_{x0} = ∅ . Let R = Q ∩ (X1 × V), S = π_{X1} (R). Then S is ξ11-meta-Lindelof, hence ξ12-closed. ∃ U ∈ ξ12, x_0 ∈ U , U ∩ S = ∅ . Then U × V avoids Q, hence P .

Corollary 34 Finite products are p.w.meta-P-closed.

6 Conclusion and p-Normality

Theorem 35 Let (X, ξ1, ξ2) be p.w.T_2, p-regular, p.w.meta-P-closed. Then X is p-normal.

Proof 36 Let F_1 is ξ_1-closed, F_2 is ξ_2-closed, disjoint. By p.w.T_2 and p-regularity, construct p-open cover. F1 is ξ_2-metacompact. By [1, Theorem 66], refine to point-finite ω. Let G_1 ={W ∈ ω : W ⊆ Ux}.
Then $Cl\xi_2$ (G_1) ∩ F_2 = ∅. Similarly G_2.

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