On the Solvability of One Inverse Boundary Value Problem of Recovering Unknown coefficients and the Right-Hand Side in a Fourth-Order Pseudohyperbolic Equation

YASHAR T. MEHRALIYEV[1], AFAG HUSEYNOVA[1], ANAR MAMMEDOV[2]
[1]Department of Differential Equations and Control Systems Theory,
Baku State University,
Baku, AZ1148,
AZERBAIJAN

[2]Department of Information Communication Technologies,
Baku Slavic University, 33, S.Rustam str.,
Baku, AZ1014,
AZERBAIJAN

Abstract: - We study the classical solution of the nonlinear inverse boundary value problem for a pseudo-hyperbolic equation of the fourth order. The essence of the problem is that it is required, together with the solution, to determine the unknown coefficient. The problem is considered in a rectangular area. To solve the considered problem, the transition from the original inverse problem to an auxiliary inverse problem is carried out. The existence and uniqueness of a solution to the auxiliary problem are proved with the help of contracted mappings. Then the transition to the original inverse problem is made, and as a result, a conclusion is made about the solvability of the original inverse problem.

Key-Words: - Inverse boundary problem, pseudo-hyperbolic equation, Fourier method, classic solution, integral overdetermination condition, existence and uniqueness, nonlocal integral condition, functional analysis, differential equations, mathematical modelling.

Received: April 3, 2026. Revised: May 6, 2026. Accepted: June 11, 2026. Published: August 3, 2026.

1 Introduction

There are many cases where the needs of the practice bring about the problems of determining coefficients or the right-hand side of differential equations from some knowledge of their solutions. Such problems are called inverse boundary value problems of mathematical physics. Inverse boundary value problems arise in various areas of human activity, such as seismology, mineral exploration, biology, medicine, quality control in industry, etc., which makes them an active field of contemporary mathematics. Inverse problems for various types of PDEs have been studied in many papers. Among, them we should mention the papers of [1], [2], [3], [4] and their followers. For a comprehensive overview, the reader should see the monograph by [5]. Various inverse problems for certain types of partial differential equations have been studied in many works, [6], [7], [8]. In this paper, following, [9], [10], [11], [12], [13] we prove the existence and uniqueness of the solution to an inverse boundary value problem for a pseudo-hyperbolic equation of the fourth order.

2 Formulation of the Problem and Its Equivalent Form

Suppose that (x,y,t), $g(x,y,t)$, $\varphi(x,y)$, $\psi(x,y)$, $\omega_i(x,y)$ $(i=1,2,3)$, $h_i(t)$ $(i=1,2,3)$ are known functions and $D_T=Q_{xy}\times[0,T]$, $Q_{xy}\{(x,y)\colon\ 0\le x\le 1,\quad 0\le y\le 1\}$. Consider the following inverse problem: Now we study the problem: to determine four $\{u(x,t)\,,a(t),b(t),c(t)\}$ of the functions $u(x,y,t)$, $a(t)$, $b(t)$, $c(t)$ satisfying the equation:

$$\begin{aligned} u_{tt}(x,y,t)-\Delta u_{tt}(x,t)+\Delta^2 u(x,t)= \\ =a(t)u(x,y,t)+b(t)u_t(x,y,t)+ \\ +c(t)g(x,y,t)+f(x,y,t) \end{aligned} \tag{1}$$

with initial

$$u(x,y,0)=\varphi(x,y),\quad u_t(x,y,0)=\psi(x,y) \quad (0\le x\le 1, 0\le y\le 1), \tag{2}$$

and boundary conditions

$$u(0,y,t)=u_x(1,y,t)=u_{xx}(0,y,t)= u_{xxx}(1,y,t)=0 \quad (0\le y\le 1, 0\le t\le T), \tag{3}$$

$$u(x,0,t)=u_y(x,1,t)=u_{yy}(x,0,t)= u_{yyy}(1,t)=0 \quad (0\le y\le 1, 0\le t\le T \tag{4}$$

and with additional conditions

$$U_i(u)(t)\equiv u(x_i,y_i,t)+ + \int_0^1\int_0^1 \omega_i(x,y)u(x,y,t)dx\,dy=h_i(t) \quad (i=1,2.3; 0\le t\le T), \tag{5}$$

where

$$\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2},$$
$$\Delta^2=\frac{\partial^4}{\partial x^4}+2\frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial y^2}+\frac{\partial^4}{\partial y^4}.$$

Introduce the designation

$$\tilde{C}^{4,4,2}(D_T)=\{u(x,y,t): u(x,y,t)\in C^2(D_T), u_{ttxx}(x,t), u_{ttyy}(x,t), u_{xxyy}(x,t), u_{xxxx}(x,t), u_{xxxx}(x,t)\, u_{yyyy}(x,t\in C(D_T)\}.$$

Definition. *By the classical solution of the inverse boundary value problem (1)–(5), we mean the four* $\{u(x,y,t), a(t), b(t), c(t)\}$ *of the functions* $u(x,y,t)$, $a(t), b(t), c(t)$ *where* $u(x,y,t)\in \tilde{C}^{4,4,2}(D_T)$, $a(t)\in C[0,T]$, $b(t)\in C[0,T]$, $c(t)\in C[0,T]$, *and they satisfy the (1)–(5) in the ordinary sense.*

We prove the following:

Theorem 1. *Let* $f(x,y,t), g(x,y,t)\in C(D_T)$,

$$\omega_i(x,y)\in C(Q_{xy}) \ (i=1,2,3),$$
$$\varphi(x,y)\in C(Q_{xy}),\ \psi(x,y)\in C(Q_{xy}),$$
$$h_i(t)\in C^2[0,T] \ (i=1,2,3),$$

and the matching conditions

$$U_i(\phi)=h_i(0), U_i(\psi)=h'_i(0) \ (i=1,2,3)$$

are satisfied. Then the problem of finding a classical solution to problem (1)-(4) is equivalent to the problem of determining the functions $(x,y,t)\in \tilde{C}^{4,4,2}(D_T)$, $a(t)\in C[0,T]$, $b(t)\in C[0,T]$, *and* $c(t)\in C[0,T]$, *from (1)-(3) and*

$$h''_i(t)-U_i(\Delta u_{tt})+U_i(\Delta^2 u)= =a((t)h_i(t)+b(t)h'_i(t)+c(t)U_i(g)(t)+ U_i(f)(t) \ (i=1,2,3; 0\le t\le T), \tag{6}$$

$$h(t)\equiv\begin{vmatrix} h_1(t) & h'_1(t) & U_1(g)(t)\\ h_2(t) & h'_2(t) & U_2(g)(t)\\ h_3(t) & h'_3(t) & U_3(g)(t)\end{vmatrix}\neq 0 \quad (0\le t\le T).$$

Proof. Suppose that $\{u(x,t), a(t), b(t), c(t)\}$ is a classical solution to problem (1)-(5). From relation (5), we have:

$$U_i(u_t)(t)=h'_i(t),\ U_i(u_{tt})(t)=h''_i(t) \quad (i=1,2,3; 0\le t\le T). \tag{7}$$

From (1) we easily obtain:

$$U_i(u_{tt})(t)-U_i(u_{ttxx})(t)+U_i(u_{xxxx})(t)= =a(t)U_i(u)(t)+b(t)U_i(u_t)(t)+c(t)U_i(g)(t)+ U_i(f)(t) \quad (i=1,2,3; 0\le t\le T). \tag{8}$$

From this considering (8) and (7) we arrive at (6).

Now let's suppose that $\{u(x,y,t), a(t), b(t), c(t)\}$ is a solution of problem (1)-(4), (6).

Then from (6) and (8) we get:

$$\frac{d^2}{d^2}(U_i(t)(t)-h_i(t))-b(t)\frac{d}{d}(U_i(t)(t)- h_i(t))-a(t)(U_i(t)(t)-h_i(t))=0 \quad (i=1,2,3; 0\le t\le T). \tag{9}$$

Considering (2) and

$$U_i(\varphi)=h_i(0), U_i(\psi)=h'_i(0) \ (i=1,2,3)$$

we have

$$U_i(u)(0)-h_i(0)=U_i(\varphi)-h_i(0)=0,$$
$$U_i(u_t)(0)-h'_i(0)=U_i(\psi)-h'_i(0)=0 \,(i=1,2,3) \tag{10}$$

From (9), taking into account (10), it is clear that condition (5) is also satisfied. The theorem is proved.

3 Solvability of the Inverse Boundary Value Problem

We seek the component u(x,y,t) in the form:

$$u(x,y,t)=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}u_{k,n}(t)\sin\lambda_k x\sin\gamma_n x$$
$$\left(\lambda_k=\frac{\pi}{2}(2k-1), \gamma_n=\frac{\pi}{2}(2n-1)\right), \tag{11}$$

where

$$u_{k,n}(t) = 4\int_0^1\int_0^1 u(x,y,t)\sin\lambda_k x\sin\gamma_n x dxdy$$
$$(k = 1,2,\ldots, n = 1,2,\ldots).$$

Then applying the formal Fourier scheme, from (1) and (2) we obtain

$$\left(1+\mu_{k,n}^2\right)u''_{k,n}(t) + \mu_{k,n}^4 u_{k,n}(t) = F_{k,n}(t;u,a,b,c) \quad (0 \le t \le T;\ k = 1,2,\ldots) \tag{12}$$

$$u_{k,n}(0) = \varphi_{k,n},\ \ u'_{k,n}(0) = \psi_{k,n} \quad (k = 1,2,\ldots, n-1,2,\ldots), \tag{13}$$

where

$$\mu_{k,n}^2 = \lambda_k^2 + \gamma_n^2, F_{k,n}(t;u,a,b,c) = $$
$$= a(t)u_{k,n}(t) + b(t)u'_{k,n}(t) + $$
$$+c(t)g_{k,n}(t) + f_{k,n}(t) \quad ,$$

$$g_{k,n}(t) = 4\int_0^1\int_0^1 g(x,y,t)\sin\lambda_k x\sin\gamma_n x dxdy,$$

$$\varphi_{k,n} = 4\int_0^1\int_0^1 \varphi(x)\sin\lambda_k x\sin\gamma_n x dxdy,$$

$$f_{k,n}(t) = 4\int_0^1\int_0^1 f(x,y,t)\sin\lambda_k x\sin\gamma_n x dxdy,$$

$$\psi_{k,n} = 4\int_0^1\int_0^1 \psi(x)\sin\lambda_k x\sin\gamma_n x dxdy,$$
$$(k = 1,2,\ldots, n = 1,2,\ldots).$$

Solving problem (12),(13) we find

$$u_{k,n}(t) = \varphi_{k,n}\cos\beta_{k,n}t + \frac{1}{\beta_{k,n}}\psi_{k,n}sin\beta_{k,n}t + $$
$$+\frac{1}{\beta_{k,n}\left(1+\mu_{k,n}^2\right)}\int_0^t F_{k,n}(\tau;u,a,b,c) \times$$
$$\times \sin\beta_{k,n}(t-\tau)d\tau \ (k = 1,2,\ldots), \tag{14}$$

where

$$\beta_{k,n}^2 = \frac{\mu_{k,n}^4}{1+\mu_{k,n}^2}(k = 1,2,\ldots, n = 1,2,\ldots).$$

After substitution of the expression $u_{k,n}(t)$ $(k = 1,2,\ldots, n = 1,2,\ldots)$ into (11) for the determination of $u(x,t)$ we get:

$$u(x,t) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left\{\varphi_{k,n}\cos\beta_{k,n}t + \right.$$
$$+ \frac{1}{\beta_{k,n}}\psi_{k,n}sin\beta_{k,n}t \quad +$$
$$+\frac{1}{\beta_{k,n}\left(1+\mu_{k,n}^2\right)}\int_0^t F_{k,n}(\tau;u,a,b,c) \times$$
$$\left.\times \sin\beta_{k,n}(t-\tau)d\tau\right\}\sin\lambda_k x\sin\gamma_n x. \tag{15}$$

Now, from (6), taking into account (11), we have:

$$h''_i(t) + \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\mu_{k,n}^2 u''_k(t) + $$
$$\mu_{k,n}^4 u_k(t))\ U_i(y_k) = a(t)h_i(t) + b(t)h'_i(t) + $$
$$c(t)U_i(g)(t) + +U_i(f)(t)(i = 1,2,3;\ 0 \le t \le T)$$

Let's assume that

$$h(t) \equiv \begin{vmatrix} h_1(t) & h'_1(t) & U_1(g)(t) \\ h_2(t) & h'_2(t) & U_2(g)(t) \\ h_3(t) & h'_3(t) & U_3(g)(t) \end{vmatrix} \ne 0$$
$$(0 \le t \le T).$$

Then from the last relations we have:

$$a(t) = [h(t)]^{-1}\left\{q_1(t) + \right.$$
$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\mu_{k,n}^2 u''_k(t) + $$
$$\left.+\ \mu_{k,n}^4 u_k(t))q_{1k,n}(t)\ \right\}, \tag{16}$$

$$b(t) = [h(t)]^{-1}\left\{q_2(t) + \right.$$
$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\mu_{k,n}^2 u''_k(t) + $$
$$\left.+\ \mu_{k,n}^4 u_k(t))q_{2k,n}(t)\ \right\}, \tag{17}$$

$$c(t) = $$
$$= [h(t)]^{-1}\left\{q_3(t) + \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\mu_{k,n}^2 u''_k(t) + \right.$$
$$\left.+\mu_{k,n}^4 u_k(t))q_{3k,n}(t)\ \right\}, \tag{18}$$

where

$$y_{k,n} \equiv y_{k,n}(x,y) = \sin\lambda_k x\sin\gamma_n y,$$
$$q_1(t) = (h_1''(t) - U_1(f)(t))(h_2'(t)U_3(g)(t) - h_3'(t)U_2(g)(t)) -$$
$$-(h_2''(t) - U_2(f)(t))(h_1'(t)\ U_3(g)(t) - h_3'(t)\ U_1(g)(t)) +$$
$$+(h_3''(t) - U_3(f)(t))(h_1'(t)\ U_2(g)(t) - h_2'(t)\,U_1(g)(t))),$$
$$q_{1k,n}(t) = U_1(y_{k,n})(t)(h_2'(t)U_3(g)(t) - h_3'(t)U_2(g)(t)) -$$

$$-U_2(y_{k,n})(h'_1(t)U_3(g)(t)-h'_3(t)U_1(g)(t))+$$
$$+U_3(y_{k,n})(h'_1(t)U_2(g)(t)-h'_2(t)U_1(g)(t))$$
$$q_2(t)=$$
$$=-(h''_1(t)-U_1(f)(t))(h_2(t)U_3(g)(t)-h_3(t)U_2(g)(t))+$$
$$+(h''_2(t)-U_2(f)(t))(h_1(t)\ U_3(g)(t)-h_3(t)\ U_1(g)(t))-$$
$$-(h''_3(t)-U_3(f)(t))(h_1(t)\,U_2(g)(t)-h_2(t)\ U_1(g)(t)),$$
$$q_{2k,n}(t)=-U_1(y_{k,n})(h_2(t)\,U_3(g)(t)-h_3(t)\,U_2(g)(t))+$$
$$+U_2(y_{k,n})(h_1(t)\ U_3(g)(t)-h_3(t)\ U_1(g)(t))-$$
$$-U_3(y_{k,n})(h_1(t)\,U_2(g)(t)-h_2(t)\ U_1(g)(t)),$$
$$q_3(t)=(h''_1(t)-U_1(f)(t))(h_2(t)h'_3,t)-h_3(t)h'_2(t))-$$
$$-(h''_2(t)-U_2(f)(t))(h_1(t)h'_3(t)-h_3(t)h'_1(t))+$$
$$+(h''_3(t)-U_3(f)(t))(h_1(t)h'_2(t)-h_2(t)h'_1(t))$$
$$q_{3k,n}(t)=U_1(y_{k,n})(h_2(t)h'_3,t)-h_3(t)h'_2(t))-$$
$$+U_2(y_{k,n})(h_1(t)h'_3(t)-h_3(t)h'_1(t))+$$
$$+U_3(y_{k,n})(h_1(t)h'_2(t)-h_2(t)h'_1(t)),$$

Further, from (11) and (13) we obtain:

$$\mu_{k,n}^2(u''_{k,n}(t)+\mu_{k,n}^2u_{k,n}(t))=F_{k,n}(t;u,a,b,c)-u''_{k,n}(t)=$$
$$=\frac{\mu_{k,n}^4}{1+\mu_{k,n}^2}u_{k,n}(t)+\left(1-\frac{1}{1+\mu_{k,n}^2}\right)\times$$
$$\times F_{k,n}(t;u,a,b,c)=\frac{\mu_{k,n}^4}{1+\mu_{k,n}^2}u_{k,n}(t)+$$
$$+\frac{\mu_{k,n}^2}{1+\mu_{k,n}^2}F_{k,n}(t;u,a,b,c)=$$
$$=\frac{\mu_{k,n}^4}{1+\mu_{k,n}^2}\left[\varphi_{k,n}\cos\beta_{k,n}+\frac{1}{\beta_{k,n}}\psi_{k,n}sin\beta_{k,n}t+\right.$$
$$+\frac{1}{\beta_{k,n}(1+\mu_{k,n}^2)}\int_0^t F_{k,n}(\tau;u,a,b,c)\times$$
$$\left.\times\sin\beta_{k,n}(t-\tau)dt\right]+$$
$$+\frac{\mu_{k,n}^2}{1+\mu_{k,n}^2}F_{k,n}(t;u,a,b,c).\tag{19}$$

Now from (19) and (16), (17) and (18) respectively, we find:

$$a(t)=[h(t)]^{-1}\ \{q_1(t)+$$
$$+\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}q_{1k,n}(t)\left[\frac{\mu_{k,n}^4}{1+\mu_{k,n}^2}\left[\varphi_{k,n}\cos\beta_{k,n}+\right.\right.$$
$$+\frac{1}{\beta_{k,n}}\psi_{k,n}sin\beta_{k,n}t+\frac{1}{\beta_{k,n}(1+\mu_{k,n}^2)}\times$$
$$\left.\int_0^t F_{k,n}(\tau;u,a,b,c)\sin\beta_{k,n}(t-\tau)d\tau\right]+$$
$$\left.+\frac{\mu_{k,n}^2}{1+\mu_{k,n}^2}F_{k,n}(t;u,a,b,c)\right]\tag{20}$$

$$b(t)=[h(t)]^{-1}\ \{q_2(t)+$$
$$+\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}q_{2k,n}(t)\left[\frac{\mu_{k,n}^4}{1+\mu_{k,n}^2}\left[\varphi_{k,n}\cos\beta_{k,n}\right.\right.$$
$$+\frac{1}{\beta_{k,n}}\psi_{k,n}sin\beta_{k,n}t+$$
$$+\frac{1}{\beta_{k,n}(1+\mu_{k,n}^2)}\int_0^t F_{k,n}(\tau;u,a,b,c)\times$$
$$\left.\times\sin\beta_{k,n}(t-\tau)d\tau\right]+$$
$$\left.\left.+\frac{\mu_{k,n}^2}{1+\mu_{k,n}^2}F_{k,n}(t;u,a,b,c)\right]\right\},\tag{21}$$

$$c(t)=[h(t)]^{-1}\ \{q_3(t)+$$
$$+\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}q_{3k,n}(t)\left[\frac{\mu_{k,n}^4}{1+\mu_{k,n}^2}\left[\varphi_{k,n}\cos\beta_{k,n}\right.\right.$$
$$+\frac{1}{\beta_{k,n}}\psi_{k,n}sin\beta_{k,n}t+$$
$$+\frac{1}{\beta_{k,n}(1+\mu_{k,n}^2)}\int_0^t F_{k,n}(\tau;u,a,b,c)\times$$
$$\left.\times\sin\beta_{k,n}(t-\tau)d\tau\right]+$$
$$\left.\left.+\frac{\mu_{k,n}^2}{1+\mu_{k,n}^2}F_{k,n}(t;u,a,b,c)\right]\right\}.\tag{22}$$

This means that to study problem (1)-(4),(6) we need to study systems (15), (20), (21).

We will prove the following lemma about the uniqueness of the solution to problem (1)-(4), (6).

Lemma. *If* $\{u(x,t),a(t),b(t),c(t)\}$ *is arbitrary classical solution of problem (1)-(4), (6), then the function*

$$u_{k,n}(t)=4\int_0^1\int_0^1 u(x,y,t)\sin\lambda_k x\sin\gamma_n x dxdy$$
$$(k=1,2,\ldots,n=1,2,\ldots)$$

satisfies system (14) in [0, T].

Proof. Let $\{u(x,t),a(t),b(t),c(t)\}$ be any solution to problem (1)-(4), (6). Then multiplying both sides of equation (1) by the function

$4\sin\lambda_k x \sin\gamma_n y \ (k = 1,2,\ldots, n = 1,2,\ldots)$,

integrating the obtained equality over x and y from 0 to 1 we obtain that equation (12) is satisfied.
Similarly, the fulfilment of (13) is obtained from (2).

Thus, $u_{k,n}(t)$ $(k = 1,2,...; n = 1,2,...)$ is a solution to problems (12) and (13). Solving problem (12),(13) we obtain (14). Lemma is proved.

This lemma implies the validity of the following

Consequence. *Problem (1)-(4), (6) cannot have more than one solution if system (15), (20), (21), (22) has a unique solution.*

Now, to study problems (1)-(4), (6), we consider the following spaces.

1. To study problems (1)-(4), (6), we introduce the spaces. Denote by $B_{2,T}^{5,3}$ the set of all functions $u(x,y,t)$ of the form

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} u_{k,n}(t)\sin\lambda_k x \sin\gamma_n x$$

$$\left(\lambda_k = \frac{\pi}{2}(2k-1), \gamma_n = \frac{\pi}{2}(2n-1)\right)$$

defined on D_T, where each of the functions $u_{k,n}(t) \in C^1\ [0,T] (k = 1,2,\ldots; n = 1,2,\ldots)$ and

$$J_T(u) \equiv \left(\sum_{k=1}^{\infty}\left(\mu_{k,n}^5 \|u_{k,n}(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} + \\ +\left(\sum_{k=1}^{\infty}(\mu_{k,n}^3 \|u'_{k,n}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} < +\infty.$$

The norm in this space is defined as:

$$\|u(x,y,t)\|_{B_{2,T}^{5,3}} = J(u).$$

2. By $E_T^{5,3}$ we denotes the space of the vector functions $\{u(x,t), a(t), b(t), c(t)\}$ such that$u(x,y,t) \in B_{2,T}^{5,3}$, $a(t) \in C[0,T]$, $b(t) \in C[0,T]$, $c(t) \in C[0,T]$ and equip this space according to the norm

$$\|z\|_{E_T^{5,3}} = \|u(x,y,t)\|_{B_{2,T}^{5,3}} + \|a(t)\|_{C[0,T]} + \\ +\|b(t)\|_{C[0,T]} + \|c(t)\|_{C[0,T]}.$$

Clearly, $B_{2,T}^{5,3}$and $E_T^{5,3}$are Banach spaces.

Now we consider in $E_T^{5,3}$ the operator

$$\Phi(u,a,b,c) = \\ = \{\Phi_1(u,a,b,c), \Phi_2(u,a,b,c), \Phi_3(u,a,b,c)\},$$

where

$$\Phi_1(u,a,b,c) = \tilde{u}(x,t) \equiv \sum_{k=1}^{\infty}\tilde{u}_k(t)\sin\lambda_k x,$$

$$\Phi_2(u,a,b,c) = \tilde{a}(t), \Phi_3(u,a,b,c) = \tilde{b}(t),$$

$\tilde{u}_k(t)$ $(k = 1,2,\ldots)$, $\tilde{a}(t)$ and $\tilde{b}(t)$ are the right-hand sides of (14) and (20), (21), (22) correspondingly.

Obviously

$$|U_i y_k(x)| \leq \left(1 + \|\omega_i(x,y)\|_{[(Q_{xy})}\right) \leq \\ \leq \max\left(1 + \|\omega_i(x,y)\|_{[(Q_{xy})}\right) \equiv p, \\ (i = 1,2,3)$$

$$\|q_{1k}(t)\|_{C[0,T]} \leq \\ \leq p\|h'_2(t)\, U_3(g)(t) - h'_3(t)\, U_2(g)(t)\|_{C[0,T]} + \\ + p\|h'_1(t)\ U_3(g)(t) - h'_3(t)\ U_1(g)(t)\|_{C[0,T]} \\ + p\|h'_1(t)\ \ U_2(g)(t) - \ h'_2(t)\ \ U_1(g)(t))\|_{C[0,T]} \equiv\equiv \\ p_1(T),$$

$$\|q_{2k}(t)\|_{C[0,T]} \leq p\|h_2(t)\ \ U_3(g)(t) \\ - h_3(t)\ \ U_2(g)(t)\|_{C[0,T]} + \\ + p\|h_1(t)\ U_3(g)(t) - h_3(t)\ U_1(g)(t)\|_{C[0,T]} + \\ + p\ \|h_1(t)\ \ U_2(g)(t) - \ h_2(t)\ \ U_1(g)(t)\|_{C[0,T]} \equiv \\ \equiv p_2(T),$$

$$\|q_{3k}(t)\|_{C[0,T]} \leq \\ \leq p\|h_2(t)\, h'_3, t) - h_3(t)h'_2(t)\|_{C[0,T]} + \\ + p\|h_1(t)\ \ h'_3(t) - \ h_3(t)\ \ h'_1(t)\|_{C[0,T]} + \\ p\|h_1(t)\ \ h'_2(t) - \ h_2(t)\ \ h'_1(t)\,\|_{C[0,T]} \equiv p_3(T),$$

$$\mu_{k,n}^3 \leq \left(\lambda_k^2 + \gamma_n^2\right)\ (\lambda_k + \gamma_n) = \\ = \lambda_k^3 + \lambda_k^2\gamma_n + \gamma_n^2\lambda_k + \gamma_n^3,$$

$$\mu_{k,n}^5 \leq\ 2\left(\lambda_k^4 + \gamma_k^4\right)(\lambda_k + \gamma_n) = \\ = 2(\lambda_k^5 + \lambda_k^4\,\gamma_n + \gamma_n^4\lambda_k + \gamma_n^5).$$

Then we have:

$$\left\{\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\mu_{k,n}^5\|\tilde{u}_{k,n}(t)\|_{C[0,T]}\right)^2\right\}^{\frac{1}{2}} \leq \\ \leq 2\sqrt{10}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k^5\ \ |\varphi_{k,n}|\right)^2\right)^{\frac{1}{2}} +$$

$$+2\sqrt{10}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k^4\,\gamma_n\,\left|\varphi_{k,n}\right|\right)^2\right)^{\frac{1}{2}}+$$
$$2\sqrt{10}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k\gamma_n^4\left|\varphi_{k,n}\right|\right)^2\right)^{\frac{1}{2}}+$$
$$+2\sqrt{10}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\gamma_n^5\left|\varphi_{k,n}\right|\right)^2\right)^{\frac{1}{2}}+$$
$$+4\sqrt{10}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k^4\,\left|\psi_{k,n}\right|\right)^2\right)^{\frac{1}{2}}+$$
$$4\sqrt{10}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\gamma_n^4\,\left|\psi_{k,n}\right|\right)^2\right)^{\frac{1}{2}}+$$
$$+\sqrt{10T}\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_k^2\left|f_{k,n}(\tau)\right|)^2d\tau\right)^{\frac{1}{2}}+$$
$$+\sqrt{10T}\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\gamma_n^2\left|f_{k,n}(\tau)\right|)^2d\tau\right)^{\frac{1}{2}}+$$
$$+\sqrt{10}\;T\|a(t)\|_{C[0,T]}\times$$
$$\times\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\mu_{k,n}^5\|u_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}$$
$$+\sqrt{10}\;T\|b(t)\|_{C[0,T]}\times$$
$$\times\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\mu_{k,n}^3\|u'_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$
$$+\sqrt{10T}\|c(t)\|_{C[0,T]}\times$$
$$\times\left(\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_k^2\left|g_{k,n}(\tau)\right|)^2d\tau\right)^{\frac{1}{2}}+\right.$$
$$\left.+\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\gamma_n^2\left|g_{k,n}(\tau)\right|)^2d\tau\right)^{\frac{1}{2}}\right),\quad(23)$$

$$\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\mu_{k,n}^3\left\|\tilde{u}'_{k,n}(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}\leq$$
$$\leq 4\sqrt{3}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k^4\,\left|\varphi_{k,n}\right|\right)^2\right)^{\frac{1}{2}}+$$
$$+4\sqrt{3}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\gamma_n^4\,\left|\varphi_{k,n}\right|\right)^2\right)^{\frac{1}{2}}+$$
$$+2\sqrt{3}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k^3\,\left|\psi_{k,n}\right|\right)^2\right)^{\frac{1}{2}}+$$
$$+2\sqrt{3}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k^2\gamma_n\left|\psi_{k,n}\right|\right)^2\right)^{\frac{1}{2}}+$$
$$+2\sqrt{3T}\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_k\left|f_{k,n}(\tau)\right|)^2d\tau\right)^{\frac{1}{2}}+$$
$$+2\sqrt{3T}\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\gamma_n\left|f_k(\tau)\right|)^2d\tau\right)^{\frac{1}{2}}+$$
$$+2\sqrt{3}\;T\|a(t)\|_{C[0,T]}\times$$
$$\times\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\mu_{k,n}^5\left\|u_{k,n}(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}+$$
$$+2\sqrt{3}\,T\|b(t)\|_{C[0,T]}\times$$
$$\times\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\mu_{k,n}^3\left\|u'_{k,n}(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}+$$
$$+2\sqrt{3T}\|c(t)\|_{C[0,T]}\times$$
$$\times\left(\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_k\left|g_{k,n}(\tau)\right|)^2d\tau\right)^{\frac{1}{2}}+\right.$$
$$\left.+\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\gamma_n\left|g_k(\tau)\right|)^2d\tau\right)^{\frac{1}{2}}\right),\quad(24)$$
$$\|\tilde{a}(t)\|_{C[0,T]}\leq\;\|[h(t)]^{-1}\|_{C[0,T]}\{\|q_1(t)\,\|_{C[0,T]}+$$

$$+p_1(T)\left(\sum_{n=1}^{\infty}\sum_{K=1}^{\infty}\mu_{k.n}^{-2}\right)^{\frac{1}{2}}\left[\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k^3\ \left|\varphi_{k,n}\right|\right)^2\right)^{\frac{1}{2}}\right.$$

$$+\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k^2\gamma_n\left|\varphi_{k,n}\right|\right)^2\right)^{\frac{1}{2}}+$$

$$+\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k\gamma_n^2\ \left|\varphi_{k,n}\right|\right)^2\right)^{\frac{1}{2}}+$$

$$+\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\gamma_n^3\left|\varphi_{k,n}\right|\right)^2\right)^{\frac{1}{2}}+$$

$$+\sqrt{T}\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_k^2\left|f_{k,n}(\tau)\right|)^2d\tau\right)^{\frac{1}{2}}+$$

$$+\sqrt{T}\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\gamma_n^2\left|f_{k,n}(\tau)\right|)^2d\tau\right)^{\frac{1}{2}}+$$

$$+T\|a(t)\|_{C[0,T]}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\mu_{k,n}^5\|u_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}$$

$$+\ T\|b(t)\|_{C[0,T]}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\mu_{k,n}^3\|u'_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$

$$+\sqrt{T}\|c(t)\|_{C[0,T]}\left(\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_k^2\left|g_{k,n}(\tau)\right|)^2d\tau\right)^{\frac{1}{2}}\right.$$

$$\left.+\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\gamma_n^2\left|g_{k,n}(\tau)\right|)^2d\tau\right)^{\frac{1}{2}}\right)+$$

$$+\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k^2\left\|f_{k,n}(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}+$$

$$+\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\gamma_n^2\left\|f_{k,n}(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}+$$

$$+\|a(t)\|_{C[0,T]}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\mu_{k,n}^5\|u_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}$$

$$+\ \|b(t)\|_{C[0,T]}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\mu_{k,n}^3\|u'_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}$$

$$+\|c(t)\|_{C[0,T]}\times$$

$$\times\left(\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_k^2\left\|g_{k,n}(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}+\right.$$

$$\left.\left.\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\gamma_n^2\left\|g_{k,n}(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right)\right]\Bigg\}, \quad (25)$$

$$\left\|\tilde{b}(t)\right\|_{C[0,T]}\leq\ \|[h(t)]^{-1}\|_{C[0,T]}\{\|q_1(t)\ \|_{C[0,T]}+$$

$$+p_2(T)\left(\sum_{n=1}^{\infty}\sum_{K=1}^{\infty}\mu_{k.n}^{-2}\right)^{\frac{1}{2}}\left[\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k^3\ \left|\varphi_{k,n}\right|\right)^2\right)^{\frac{1}{2}}\right.$$

$$+\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k^2\gamma_n\left|\varphi_{k,n}\right|\right)^2\right)^{\frac{1}{2}}+$$

$$+\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k\gamma_n^2\ \left|\varphi_{k,n}\right|\right)^2\right)^{\frac{1}{2}}+$$

$$+\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\gamma_n^3\left|\varphi_{k,n}\right|\right)^2\right)^{\frac{1}{2}}+$$

$$+2\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k^2\ \left|\psi_{k,n}\right|\right)^2\right)^{\frac{1}{2}}+$$

$$+2\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\gamma_n^2\ |\psi_{k,n}|\right)^2\right)^{\frac{1}{2}}+$$

$$+\sqrt{T}\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_k^2\,|f_{k,n}(\tau)|)^2 d\tau\right)^{\frac{1}{2}}+$$

$$+\sqrt{T}\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\gamma_n^2\,|f_{k,n}(\tau)|)^2 d\tau\right)^{\frac{1}{2}}+$$

$$+T\|a(t)\|_{C[0,T]}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\mu_{k,n}^5\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$

$$+\ T\|b(t)\|_{C[0,T]}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\mu_{k,n}^3\|u'_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$

$$+\sqrt{T}\|c(t)\|_{C[0,T]}\left(\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_k^2\,|g_{k,n}(\tau)|)^2 d\tau\right)^{\frac{1}{2}}\right.$$

$$\left.+\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\gamma_n^2\,|g_{k,n}(\tau)|)^2 d\tau\right)^{\frac{1}{2}}\right)+$$

$$+\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k^2\|f_{k,n}(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}+$$

$$+\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\gamma_n^2\|f_{k,n}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}+$$

$$+\|a(t)\|_{C[0,T]}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\mu_{k,n}^5\|u_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}$$

$$+\|b(t)\|_{C[0,T]}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\mu_{k,n}^3\|u'_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}+$$

$$\|c(t)\|_{C[0,T]}\times \qquad \times$$

$$\left(\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k^2\|g_{k,n}(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}+\right.$$

$$\left.\left.\left.+\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\gamma_n^2\|g_{k,n}(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}\right)\right]\right\} \tag{26}$$

$$\|\tilde{c}(t)\|_{C[0,T]}\leq\ \|[h(t)]^{-1}\|_{C[0,T]}\{\|q_3(t)\ \|_{C[0,T]}+$$

$$+p_3(T)\left(\sum_{n=1}^{\infty}\sum_{K=1}^{\infty}\mu_{k.n}^{-2}\right)^{\frac{1}{2}}\left[\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k^3\ |\varphi_{k,n}|\right)^2\right)^{\frac{1}{2}}\right.$$

$$+\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k^2\gamma_n\,|\varphi_{k,n}|\right)^2\right)^{\frac{1}{2}}+$$

$$+\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k\gamma_n^2\ |\varphi_{k,n}|\right)^2\right)^{\frac{1}{2}}+$$

$$+\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\gamma_n^3\,|\varphi_{k,n}|\right)^2\right)^{\frac{1}{2}}+$$

$$+2\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k^2\ |\psi_{k,n}|\right)^2\right)^{\frac{1}{2}}+$$

$$+2\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\gamma_n^2\ |\psi_{k,n}|\right)^2\right)^{\frac{1}{2}}+$$

$$+\sqrt{T}\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_k^2\,|f_{k,n}(\tau)|)^2 d\tau\right)^{\frac{1}{2}}+$$

$$+\sqrt{T}\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\gamma_n^2\,|f_{k,n}(\tau)|)^2 d\tau\right)^{\frac{1}{2}}+$$

$$+T\|a(t)\|_{C[0,T]}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\mu_{k,n}^{5}\|u_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}$$
$$+\ T\|b(t)\|_{C[0,T]}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\mu_{k,n}^{3}\|u'_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$
$$+\sqrt{T}\|c(t)\|_{C[0,T]}\left(\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_k^2\left|g_{k,n}(\tau)\right|)^2d\tau\right)^{\frac{1}{2}}\right.$$
$$\left.+\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\gamma_n^2\left|g_{k,n}(\tau)\right|)^2d\tau\right)^{\frac{1}{2}}\right)+$$
$$+\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k^2\left\|f_{k,n}(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}+$$
$$+\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\gamma_n^2\left\|f_{k,n}(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}+$$
$$+\|a(t)\|_{C[0,T]}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\mu_{k,n}^{5}\|u_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}$$
$$+\ \|b(t)\|_{C[0,T]}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\mu_{k,n}^{3}\|u'_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}$$
$$+\|c(t)\|_{C[0,T]}\times$$
$$\times$$
$$\left(\left(\sum\nolimits_{n=1}^{\infty}\sum\nolimits_{k=1}^{\infty}\left(\lambda_k^2\left\|g_{k,n}(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}+\right.$$
$$\left.\left(\sum\nolimits_{n=1}^{\infty}\sum\nolimits_{k=1}^{\infty}\left(\gamma_n^2\left\|g_{k,n}(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}\right)\Bigg]\Bigg\} \quad (27)$$

where

$$\widetilde{u'}_k(t)\ =\ -\beta_{k,n}\varphi_{k,n}\sin\beta_{k,n}t\ +\ \psi_{k,n}\sin\beta_{k,n}t\ +$$
$$+\frac{1}{1+\mu_{k,n}^2}\int_0^t F_{k,n}(\tau;u,a,b,c)\times$$
$$\times\cos\beta_{k,n}(t\ -\ \tau)d\tau\ \ (k\ =\ 1,2,\ldots;n=1,2,\ldots).$$

Assume that the data of problem (1)-(4), (6) satisfy the following conditions:

1. $\varphi(x,y)\in C^{4,4}(Q_{xy}),\ \varphi_{xxxxy}(x,y),$ $\varphi_{xyyyy}(x,y)\varphi_{xxxxx}(x,y),$ $\varphi_{yyyyy}(x,y)\in L_2(Q_{xy}),\ \varphi(0,y)=\varphi_x(1,y)$ $=\varphi_{xx}(0,y)=\varphi_{xxx}(1,y)$ $=\varphi_{xxxx}(0,y)=0$ $(0\le y\le 1),$ $\varphi(x,0)=\varphi_y(x,1)=\varphi_{yy}(x,0)=\varphi_{yyy}(x,1)=$ $\varphi_{yyyy}(x,0)=0\ \ (0\le x\le 1)$,
2. $\psi(x,y)\in C^{3,3}(Q_{xy}),\psi_{xxxx}(x,y),\psi_{yyyy}(x,y)\in$ $L_2(Q_{xy}),\ \psi(0,y)=\psi_x(1,y)=\psi_{xx}(0,y)=$ $\psi_{xxx}(1,y)=0\ \ (0\le y\le 1),$ $\psi(x,0)=\psi_y(x,1)=\psi_{yy}(x,0)=\psi_{yyy}(x,1)=0$ $(0\le x\le 1)$,
3. $f(x,y,t),f_x(x,y,t),f_y(x,y,t)\ \in$ $C(D_T),\ f_{xx}(x,y,t)\,,f_{yy}(x,y,t)\ \in\ L_2(D_T),$ $f(0,y,t)\ =f_x(1,y,t)=0\ \ (0\le y\le 1,0\le t\le$ $T),f(x,0,t)\ =f_y(x,1,t)=0\ \ (0\le x\le 1,0\le t\le$ $T),$
4. $g(x,y,t),g_x(x,y,t),g_y(x,y,t)\ \in\ C(D_T),$ $g_{xx}(x,y,t)\,,g_{yy}(x,y,t)\ \in\ L_2(D_T),$ $g(0,y,t)\ =g_x(1,y,t)=0$ $(0\le y\le 1,0\le t\le T),$ $g(x,0,t)\ =g_y(x,1,t)=0$ $(0\le x\le 1,0\le t\le T),$
5. $h_i(t)\in C^2[0,T]\ \ \ (i=1,2,3),$
$$h(t)\equiv\begin{vmatrix}h_1(t) & h'_1(t) & U_1(g)(t)\\ h_2(t) & h'_2(t) & U_2(g)(t)\\ h_3(t) & h'_3(t) & U_3(g)(t)\end{vmatrix}\neq 0$$
$(0\le t\le T).$

Then following [13] we have

$$\|\tilde{u}(x,t)\|_{B_{2,T}^{5,3}}\le A_1(T)+$$
$$+B_1(T)\left(\|a(t)\|_{C[0,T]}+\|b(t)\|_{C[0,T]}\right)\ \|u(x,t)\|_{B_{2,T}^{5,3}}$$
$$+D_{\ 1}(T)\ \|c(t\,)\|_{C[0,T]}, \qquad (23)$$

$$\|\tilde{a}(t)\|_{C[0,T]}\le A_2(T)+$$
$$+B_2(T)\left(\|a(t)\|_{C[0,T]}+\|b(t)\|_{C[0,T]}\right)\times$$
$$\times\|u(x,t)\|_{B_{2,T}^{5,3}}\ +D_{\ 2}(T)\ \|c(t\,)\|_{C[0,T]}, \qquad (24)$$

$$\left\|\tilde{b}(t)\right\|_{C[0,T]}\le A_3(T)+$$
$$+B_2(T)\left(\|a(t)\|_{C[0,T]}+\|b(t)\|_{C[0,T]}\right)\times$$

$$\times \|u(x,t)\|_{B_{2,T}^{5,3}} + D_3(T)\, \|c(t)\|_{C[0,T]}, \tag{25}$$

$$\|\tilde{c}(t)\|_{C[0,T]} \le A_4(T) + +B_4(T)\left(\|a(t)\|_{C[0,T]} + \|b(t)\|_{C[0,T]}\right)\times \times \|u(x,t)\|_{B_{2,T}^{5,3}} + D_4(T)\, \|c(t)\|_{C[0,T]}, \tag{26}$$

where

$$A_1(T) = A_{11}(T) + A_{12}(T), A_{11}(T) = = 2\sqrt{10}\|\varphi_{xxxxx}(x,y)\|_{L_2(Q_{xy})} + +2\sqrt{10}\|\varphi_{xxxxy}(x,y)\|_{L_2(Q_{xy})} + +2\sqrt{10}\|\varphi_{xyyyy}(x,y)\|_{L_2(Q_{xy})} + +2\sqrt{10}\|\varphi_{xxxxx}(x,y)\|_{L_2(Q_{xy})} + +4\sqrt{10}\|\psi_{xxxx}(x,y)\|_{L_2(Q_{xy})} + +4\sqrt{10}\|\psi_{yyyy}(x,y)\|_{L_2(Q_{xy})} + +\sqrt{10T}\|f_{xx}(x,y,t)\|_{L_2(D_T)} + +\sqrt{10T}\|f_{yy}(x,y,t)\|_{L_2(D_T)},$$

$$A_{12}(T) = 4\sqrt{3}\|\varphi_{xxxx}(x,y)\|_{L_2(Q_{xy})} + +4\sqrt{3}\|\varphi_{yyyy}(x,y)\|_{L_2(Q_{xy})} + +2\sqrt{3}\|\psi_{xxx}(x,y)\|_{L_2(Q_{xy})} + +2\sqrt{3}\|\psi_{xxy}(x,y)\|_{L_2(Q_{xy})} + +2\sqrt{3}\|\psi_{xyy}(x,y)\|_{L_2(Q_{xy})} + +2\sqrt{3}\|\varphi_{yyy}(x,y)\|_{L_2(Q_{xy})} + +2\sqrt{3T}\|f_x(x,y,t)\|_{L_2(D_T)} + 2\sqrt{3T}\|f_y(x,y,t)\|_{L_2(D_T)},$$

$$B_1(T) = (\sqrt{10} + 2\sqrt{3})T,$$

$$D_1(T) = \sqrt{10T}\|g_{xx}(x,y,t)\|_{L_2(D_T)} + +\sqrt{10T}\|g_{yy}(x,y,t)\|_{L_2(D_T)} + +2\sqrt{3T}\|g_x(x,y,t)\|_{L_2(D_T)} + +2\sqrt{3T}\|g_y(x,y,t)\|_{L_2(D_T)},$$

$$A_2(T) = \|[h(t)]^{-1}\|_{C[0,T]}\Big\{\|q_1(t)\|_{C[0,T]} + +p_1(T)\left(\sum_{n=1}^{\infty}\sum_{K=1}^{\infty}\mu_{k.n}^{-2}\right)^{\frac{1}{2}}\Big[\|\varphi_{xxx}(x,y)\|_{L_2(Q_{xy})} + \|\varphi_{xyy}(x,y)\|_{L_2(Q_{xy})} + \|\varphi_{xyy}(x,y)\|_{L_2(Q_{xy})} + +\|\varphi_{yyy}(x,y)\|_{L_2(Q_{xy})} + \|\psi_{xx}(x,y)\|_{L_2(Q_{xy})} + +\|\psi_{yy}(x,y)\|_{L_2(Q_{xy})} + +(\sqrt{T}+1)\left(\|f_{xx}(x,y,t)\|_{L_2(D_T)} + \left\|\|f_{xx}(x,y,t)\|_{C[0,T]}\right\|_{L_2(0.1)}\right) + +(\sqrt{T}+1)\times \left(\|f_{yy}(x,y,t)\|_{L_2(D_T)} + \left\|\|f_{yy}(x,y,t)\|_{C[0,T]}\right\|_{L_2(0.1)}\right)\Big]\Big\},$$

$$B_2(T) = \|[h(t)]^{-1}\|_{C[0,T]}\times \left(\sum_{n=1}^{\infty}\sum_{K=1}^{\infty}\mu_{k.n}^{-2}\right)^{\frac{1}{2}} p_1(T)(T+1),\ D_2(T) = \|[h(t)]^{-1}\|_{C[0,T]}\left(\sum_{n=1}^{\infty}\sum_{K=1}^{\infty}\mu_{k.n}^{-2}\right)^{\frac{1}{2}} p_1(T)(\sqrt{T}+1)\left(\|g_{xx}(x,t)\|_{L_2(D_T)} + \left\|\|g_{xx}(x,t)\|_{C[0,T]}\right\|_{L_2(0.1)}\right),$$

$$A_3(T) = \|[h(t)]^{-1}\|_{C[0,T]}\Big\{\|q_2(t)\|_{C[0,T]} + +p_2(T)\left(\sum_{n=1}^{\infty}\sum_{K=1}^{\infty}\mu_{k.n}^{-2}\right)^{\frac{1}{2}}\times \Big[\|\varphi_{xxx}(x,y)\|_{L_2(Q_{xy})} + \|\varphi_{xyy}(x,y)\|_{L_2(Q_{xy})} + +\|\varphi_{xyy}(x,y)\|_{L_2(Q_{xy})} + \|\varphi_{yyy}(x,y)\|_{L_2(Q_{xy})} + +\|\psi_{xx}(x,y)\|_{L_2(Q_{xy})} + \|\psi_{yy}(x,y)\|_{L_2(Q_{xy})} + +(\sqrt{T}+1)\left(\|f_{xx}(x,y,t)\|_{L_2(D_T)} + \left\|\|f_{xx}(x,y,t)\|_{C[0,T]}\right\|_{L_2(0.1)}\right) + +(\sqrt{T}+1)\times \left(\|f_{yy}(x,y,t)\|_{L_2(D_T)} + \left\|\|f_{yy}(x,y,t)\|_{C[0,T]}\right\|_{L_2(0.1)}\right)\Big]\Big\},$$

$$B_3(T) = \|[h(t)]^{-1}\|_{C[0,T]}\times \times\left(\sum_{n=1}^{\infty}\sum_{K=1}^{\infty}\mu_{k.n}^{-2}\right)^{\frac{1}{2}} p_2(T)(T+1),$$

$$D_3(T) = \|[h(t)]^{-1}\|_{C[0,T]}\times \times\left(\sum_{n=1}^{\infty}\sum_{K=1}^{\infty}\mu_{k.n}^{-2}\right)^{\frac{1}{2}} p_2(T)(\sqrt{T}+1)\times \times\left(\|g_{xx}(x,t)\|_{L_2(D_T)} + \left\|\|g_{xx}(x,t)\|_{C[0,T]}\right\|_{L_2(0.1)}\right),$$

$$A_4(T) = \|[h(t)]^{-1}\|_{C[0,T]}\Big\{\|q_3(t)\|_{C[0,T]} + +p_3(T)\left(\sum_{n=1}^{\infty}\sum_{K=1}^{\infty}\mu_{k.n}^{-2}\right)^{\frac{1}{2}}\Big[\|\varphi_{xxx}(x,y)\|_{L_2(Q_{xy})} +$$

$$+\left\|\varphi_{xyy}(x,y)\right\|_{L_2(Q_{xy})}+\left\|\varphi_{xyy}(x,y)\right\|_{L_2(Q_{xy})}+$$
$$+\left\|\varphi_{yyy}(x,y)\right\|_{L_2(Q_{xy})}+\|\psi_{xx}(x,y)\|_{L_2(Q_{xy})}+$$
$$+\left\|\psi_{yy}(x,y)\right\|_{L_2(Q_{xy})}+$$
$$+(\sqrt{T}+1)\left(\|f_{xx}(x,y,t)\|_{L_2(D_T)}\right.$$
$$\left.+\left\|\|f_{xx}(x,y,t)\|_{C[0,T]}\right\|_{L_2(0.1)}\right)+$$
$$+(\sqrt{T}+1)\times$$
$$\times\left(\left\|f_{yy}(x,y,t)\right\|_{L_2(D_T)}+\left\|\left\|f_{yy}(x,y,t)\right\|_{C[0,T]}\right\|_{L_2(0.1)}\right)\Big]\Big\}$$
$$B_4(T)=\|[h(t)]^{-1}\|_{C[0,T]}\times$$
$$\times\left(\sum_{n=1}^{\infty}\sum_{K=1}^{\infty}\mu_{k.n}^{-2}\right)^{\frac{1}{2}}p_3(T)(T+1)\ ,$$
$$D_4(T)=\ \|[h(t)]^{-1}\|_{C[0,T]}\times$$
$$\times\left(\sum_{n=1}^{\infty}\sum_{K=1}^{\infty}\mu_{k.n}^{-2}\right)^{\frac{1}{2}}p_3(T)(\sqrt{T}+1)\times$$
$$\times\left(\|g_{xx}(x,t)\|_{L_2(D_T)}+\left\|\|g_{xx}(x,t)\|_{C[0,T]}\right\|_{L_2(0.1)}\right),$$

From inequalities (23)-(26) we conclude

$$\|\tilde{u}(x,y,t)\|_{B_{2,T}^{5,3}}+\|\tilde{a}(t)\|_{C[0,T]}+\left\|\tilde{b}(t)\right\|_{C[0,T]}\le$$
$$\le A(T)+B(T)\big(\|a(t)\|_{C[0,T]}$$
$$+\|b(t)\|_{C[0,T]}\big)\|u(x,y,t)\|_{B_{2,T}^{5,3}}+$$
$$+D(T)\ \|c(t)\|_{C[0,T]}, \tag{27}$$

where

$$A(T)=A_1(T)+A_2(T)+A_3(T),$$
$$B(T)=B_1(T)+B_2(T)+B_3(T),$$
$$D(T)=D_1(T)+D_2(T)+D_3(T).$$

So, we can prove the following theorem:

Theorem 2. *Let conditions 1-5 be satisfied and*

$$(B(T)(A(T)+2)+D(T))(A(T)+2)<1. \tag{28}$$

The problem (1)-(4),(6) has a unique solution in the ball $K=K_R(||z||_{E_T^{5,3}}\le R=A(T)+2)$ *of the space* $E_T^{5,3}$.

Proof. In the space $E_T^{5,3}$ consider the equation

$$z=\Phi z \tag{29}$$

where $z=\{u,a,b,c\}$, the components $\Phi(u,a,b,c)$ $(i=\overline{1,4})$ of the operator $\Phi(u,a,b,c)$ are defined by the right-hand sides of equations (15), (20), (21), (22)

Consider the operator $\Phi(u,a,b,c)$ in the ball $K=K_R$ from $E_T^{5,3}$. Similarly to (27) we obtain that the estimations

$$\|\Phi z\|_{E_T^{5,3}}\le A(T)+B(T)\big(\|a(t)\|_{C[0,T]}+$$
$$+\|b(t)\|_{C[0,T]}\big)\|u(x,y,t)\|_{B_{2,T}^{5,3}}+$$
$$+D(T)\|c(t)\|_{C[0,T]}\le A(T)+$$
$$+(B(T)(A(T)+2)+$$
$$+D(T))(A(T)+2), \tag{30}$$

$$\|\Phi z_1-\Phi z_2\|_{E_T^{5,3}}\le\ +D(T)\|c_1(t)-c_2(t)\|_{C[0,T]}+$$
$$+B(T)R\left(\|a_1(t)-a_2(t)\|_{C[0,T]}+\|b_1(t)-\right.$$
$$\left.-b_2(t)\|_{C[0,T]}+\|u_1(x,y,t)-u_2(x,y,t)\|_{B_{2,T}^{5,3}}\right) \tag{31}$$

for the arbitrary $z,z_1,z_2\in K_R$. Then, from estimates (30), (31), taking into account (28), it follows that the operator Φ acts in the ball and is contractive. Therefore, in the ball $K=K_R$ the operator Φ has a single fixed point $\{u,a,b,c\}$ which is a unique solution to equation (29) in the ball $K=K_R$, i.e. $\{u,a,b,c\}$ is a unique solution to system *(1)-(4),(6)* in the ball $K=K_R$.

The function $u(x,t)$ as an element of the space $B_{2,T}^{5,3}$ has continuous derivatives $u(x,t),\ u_x(x,t),\ u_{xx}(x,t),u_{xxx}(x,t),u_{xxxx}(x,t)$, $u_t(x,t),\ u_{tx}(x,t),u_{txx}(x,t)$ in D_T.

As one can easily see from:

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\|u''_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}\le$$
$$\le\sqrt{2}\left(\sum_{k=1}^{\infty}(\lambda_k^5\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}+$$
$$+\sqrt{2}\left\|\|f_x(x,t)\ +\ a(t)u_x(x,t)+\ b(t)u_{tx}(x,t)\right.$$
$$\left.+\ c(t)g_x(x,t)\|_{C[0,T]}\right\|_{L_2(0,1)}.$$

It implies that $u_{tt}(x,t)$, $u_{ttx}(x,t)$, $u_{ttxx}(x,t)$ are continuous in D_T.

It is easy to check that equation (1) and conditions (2)-(4) and (6) are satisfied in the usual sense. Therefore, $\{u(x,t)\,,a(t),b(t),c(t)\}$is a solution to problem (1)-(4), (6), and, by virtue of the corollary of Lemma 1, it is unique in the ball $K=K_R$.

The theorem is proved.

Using Theorem 1, we prove the following

Theorem 3. Let all conditions of Theorem 2 be satisfied and

$$U_i(\phi) = h_i(0), U_i(\psi) = h'_i(0) \ (i = 1,2,3).$$

The problem (1)-(5) has unique classical solution in the ball $K = K_R(||z||_{E_T^{5,3}} \leq R = A(T) + 2)$ from $E_T^{5,3}$.

References:

[1] Tikhonov A.N. On the stability of inverse problems. Dokl. USSR Academy of Sciences, 1943, 39(5), p.195-198.

[2] Lavrent'ev M.M. On some incorrect problems of mathematical physics. Novosibirsk, Nauka, 1962.

[3] Ivanov V.K. O linear incorrect problems. DAN SSSR, 1962, 145(2), p.270-272.

[4] *(text in Russian)* Lavrent'ev M.M., Romanov V.G., Shishatsky S.T., Ill-Posed Problems of Mathematical Physics and Analysis, M. Nauka, 1980.

[5] Denisov A.M., Introduction to Theory of Inverse Problems, M: MSU, 1994

[6] Kamynin V.L. (2013). The inverse problem of determining the lower-order coefficient in parabolic equations with integral observation. *Mathematical Notes*, *94*(1), p.205-213.

[7] Kamynin, V.L. (). On inverse problems for strongly degenerate parabolic equations under the integral observation condition. *Computational Mathematics and Mathematical Physics*, *58*(12), 2018.

[8] He Yang. An inverse problem for the sixth-order linear boussinesq-type equation. U.P.B. Sci. Bull., Series A, Vol. 82, Iss. 2, 2020, pp.27-36.

[9] Mehraliyev Y.T., Huseynova A.F. On solvability of an inverse boundary value problem for pseudo hyperbolic equation of the fourth order, Journal of Mathematics Research, 7 (2015), no. 2, 101-109.Control, Automation, Communication and Systems (IJCACS), 2016, 1:21–9. https://doi.org/10.5121/IJCACS.2016.1203.

[10] Isgendarov N.Sh., Mehraliyev Y.T., Huseyinova A.F. On an Inverse Boundary Value Problem for the Boussinesq-Love Equation with an Integral Condition. Applied Mathematical Sciences, Vol. 10, 2016, no. 63, p.3119-3131.

[11] *(text in Russian)* Erofeev, V.I., Kazhaev, V.V., & Semerikova, N.P. Volny v sterzhniakh: Dispersiia. Dissipatsiia. Nelineinost [Waves in Rods: Dispersion. Dissipation. Nonlinearity]. Moscow: Fizmatlit,, 2002.

[12] Khudaverdiev K.I., Veliyev A.A. Study of a one-dimensional mixed problem for a class of third-order pseudohyperbolic equations with a nonlinear operator right-hand side. Baku, 2010, 168 p.

[13] Mehraliyev Yashar T., Alosaimi Moataz, Huseynova Afag. On an inverse boundary value problem for the sixth-order Boussinesq equation with integral conditions, Filomat, University of Nis, 2025, 39(27), pp. 9547–9560, doi: 10.2298/FIL2527547M.

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

Yashar T. Mehraliyev contributed to the conceptual development of the mathematical model, provided scientific supervision, and critically reviewed the theoretical framework and the final version of the manuscript.

Afag Huseynova made a major contribution to the research design, mathematical analysis, interpretation of the results, preparation of the manuscript, literature review, revision of the paper, and coordination of the publication process.

Anar Mammedov contributed substantially to the formulation of the mathematical problem, derivation and verification of the results, preparation of figures and equations, manuscript editing, and the implementation of the reviewers' revisions.

All authors discussed the results, contributed to the final version of the manuscript, approved the submitted version, and agree to be accountable for all aspects of the work.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

This research received no external funding. The study was conducted without financial support from any funding agency in the public, commercial, or not-for

Conflict of Interest

The authors declare that they have no conflict of interest regarding the publication of this paper.