Homing problems for diffusion processes with random resettings

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Abstract: The problem of minimizing the expected time spent by one-dimensional stochastic processes in a given interval is considered in the case of diffusion processes with random resettings. At random times that occur according to a Poisson process, the controlled diffusion process jumps from its current position to a fixed value. The differential equation satisfied by the value function is given and particular problems are solved explicitly.

Key-Words: Wiener process, stochastic control, dynamic programming, jump-diffusion process, special functions, integro-differential equation.

Received: March 30, 2026. Revised: April 13, 2026. Accepted: June 7, 2026. Published: July 28, 2026.

1 Introduction

We consider the one-dimensional continuous-time stochastic process $\{X(t), t \geq 0\}$ defined by

$$\begin{aligned} X(t) &= X(0) + \int_0^t f[X(s)]\mathrm{d}s \qquad (1) \\ &\quad + \int_0^t \{v[X(s)]\}^{1/2}\mathrm{d}B(s) + \sum_{i=1}^{N(t)} Y_i, \end{aligned}$$

where $f(\cdot)$ and $v(\cdot) > 0$ are real functions, $\{B(t), t \geq 0\}$ is a standard Brownian motion, $\{N(t), t \geq 0\}$ is a Poisson process with rate λ, and $Y_1, Y_2, \ldots$ are independent random variables that are identically distributed as the random variable Y. Moreover, we assume that $\{B(t), t \geq 0\}$ and $\{N(t), t \geq 0\}$ are independent and that $f(\cdot)$ and $v(\cdot)$ are such that if $\lambda = 0$, then is $\{X(t), t \geq 0\}$ a (pure) diffusion process. Hence, $\{X(t), t \geq 0\}$ is a jump-diffusion process if $\lambda > 0$.

We suppose that $X(0) = x \in (a, b)$ and we define the first-passage time

$$\tau(x) = \inf\{t > 0 : X(t) \notin (a, b)\}. \qquad (2)$$

In this paper, we assume that the probability density function of the random variable Y is given by

$$f_Y(y) = \delta[y - (x_t - x_r)], \qquad (3)$$

where $\delta(\cdot)$ is the Dirac delta function, x_t is the current value of $X(t)$, and $x_r \in (a, b)$ is a fixed constant. That is, at random times that occur according to a Poisson process, the process will jump from its current position to x_r. This type of stochastic process is called a *diffusion process with random resettings* and has attracted considerable attention in recent years; see, [1], [2], [3] and, [4].

Next, we consider the controlled version of the process $\{X(t), t \geq 0\}$ defined as follows:

$$\begin{aligned} X(t) &= X(0) + \int_0^t b[X(s)]\,u[X(s)]\mathrm{d}s \qquad (4) \\ &\quad + \int_0^t f[X(s)]\mathrm{d}s + \int_0^t \{v[X(s)]\}^{1/2}\mathrm{d}B(s) \\ &\quad + \sum_{i=1}^{N(t)} Y_i, \end{aligned}$$

where $b(\cdot)$ is a real function and $u(\cdot)$ is the control variable, which is assumed to be a continuous function. We look for the value $u^*(\cdot)$ of the control variable that minimizes the expected value of the cost functional

$$J(x) := \int_0^{\tau(x)} \left\{\frac{1}{2} q[X(t)]\,u^2[X(t)] + \theta\right\} \mathrm{d}t, \qquad (5)$$

where $q(\cdot)$ is a positive function and θ is a positive constant. Therefore, the aim is to minimize the expected time spent by the controlled process in the continuation region (a, b), while taking the quadratic control costs into account. This type of problem is known as a *homing* problem in stochastic control; see, [5], [6] and, [7].

In the next section, we will give the differential equation satisfied by the value function

$$F(x) := \inf_{\substack{u[X(t)] \\ t \in [0, \tau(x))}} E[J(x)] \qquad (6)$$

in this problem. Then, we will show how it is possible to obtain approximations to the function $F(x)$ that are quite accurate near the point x_r.

In Section 3, we will solve two particular homing problems explicitly and exactly. Finally, we will conclude this paper with a few remarks in Section 4.

2 Dynamic programming

In the general case of jump-diffusion processes, making use of dynamic programming, we can prove the following proposition (see, [8]).

Proposition 2.1. *Assuming that the value function $F(x)$ defined in Eq. (6) is twice differentiable, it satisfies the second-order, non-linear integro-differential equation*

$$\begin{aligned} 0 &= \theta - \frac{1}{2}\frac{b^2(x)}{q(x)}[F'(x)]^2 + f(x)F'(x) \\ &+ \frac{1}{2}v(x)F''(x) \\ &+ \lambda\left\{\int_{-\infty}^{\infty} F(x+y)f_Y(y)\mathrm{d}y - F(x)\right\}. \end{aligned} \tag{7}$$

This equation is subject to the boundary condition

$$F(x) = 0 \quad \textit{for } x \notin (a,b). \tag{8}$$

Moreover, the optimal control is given by

$$u^*(x) = -\frac{b(x)}{q(x)}F'(x). \tag{9}$$

In the case of the diffusion processes with random resettings defined in the previous section, the integro-differential equation reduces to an ordinary differential equation (ODE):

$$\begin{aligned} 0 &= \theta - \frac{1}{2}\frac{b^2(x)}{q(x)}[F'(x)]^2 + f(x)F'(x) \\ &+ \frac{1}{2}v(x)F''(x) + \lambda[F(x_r) - F(x)]. \end{aligned} \tag{10}$$

The boundary condition in Eq. (8) is still valid.

Now, finding explicit and exact solutions to the above second-order, non-linear ODE is a difficult task. In the next section, we will solve two particular problems exactly by considering the inverse problem: we will fix the form of the function $F(x)$ and try to find functions $f(x)$ and $v(x)$ for which this function $F(x)$ is indeed the solution to a specific homing problem.

In what follows, we will show how we can at least obtain approximate solutions near the point x_r. Indeed, for x close to x_r, making use of Taylor's theorem, we can write that

$$\begin{aligned} F(x_r) &= F\big(x + (x_r - x)\big) \approx F(x) \\ &\quad + (x_r - x)F'(x) + \frac{(x_r - x)^2}{2}F''(x), \end{aligned} \tag{11}$$

so that Eq. (10) becomes

$$\begin{aligned} 0 &\approx \theta - \frac{1}{2}\frac{b^2(x)}{q(x)}[F'(x)]^2 \\ &\quad + [f(x) + \lambda(x_r - x)]F'(x) \\ &\quad + \frac{1}{2}[v(x) + \lambda(x_r - x)^2]F''(x). \end{aligned} \tag{12}$$

Example 1. Suppose that $f(x) = \lambda x$ and $v(x) \equiv 1$. Moreover, we set $[a,b] = [-1,1]$ and $x_r = 0$. With $b(x) \equiv 1$, $q(x) \equiv 1$ and $\theta = \lambda = 1$, we must then solve the ODE

$$0 \approx 1 - \frac{1}{2}[F'(x)]^2 + \frac{1+x^2}{2}F''(x), \tag{13}$$

subject to the boundary conditions $F(-1) = F(1) = 0$. Furthermore, by symmetry, we can state that $F'(0) = 0$.

We find that the function $F(x)$ can be expressed as follows:

$$F(x) = \sqrt{2}\int_1^x \frac{1 - e^{2\sqrt{2}\arctan(z)}}{1 + e^{2\sqrt{2}\arctan(z)}}\,\mathrm{d}z. \tag{14}$$

Next, we substitute this function into the exact ODE

$$0 = 1 - \frac{1}{2}[F'(x)]^2 + xF'(x) + \frac{1}{2}F''(x) + F(0) - F(x). \tag{15}$$

Let $\mathrm{Error}(x)$ denote the right-hand side of the above equation. This function is displayed in Figure 1. We observe that the approximate solution is quite precise near the origin, as expected. Moreover, we also present in Figure 1 the right-hand side of Eq. (15) when $\lambda = 1/2$ and when $\lambda = 1/10$. As was also expected, the smaller λ is, the more precise the approximation. For instance, for $x = 1/2$, the error decreases from 0.1338 ($\lambda = 1$) to -0.0228 ($\lambda = 1/2$) and to -0.0043 ($\lambda = 1/10$).

If we set $\lambda = 0$, so that there are no resettings to 0, the function $F(x)$ satisfies the ODE

$$0 = 1 - \frac{1}{2}[F'(x)]^2 + \frac{1}{2}F''(x). \tag{16}$$

The solution that satisfies the boundary conditions $F(\pm 1) = 0$ is

$$F(x) = -1 + x^2 + \ln\left(\frac{1 + e\sqrt{\pi}\,\mathrm{erf}(1)}{1 + e^{x^2}\sqrt{\pi}x\,\mathrm{erf}(x)}\right), \tag{17}$$

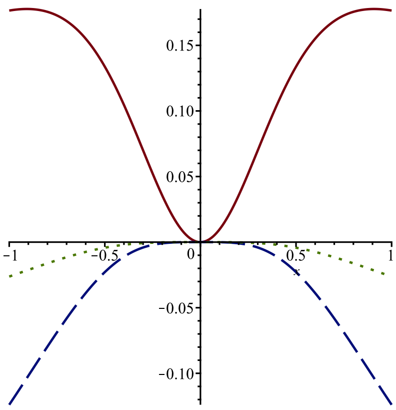

Figure 1: Error(x) for x in the interval $[-1, 1]$ obtained with $\lambda = 1$ (solid line), $\lambda = 1/2$ (dashed line) and $\lambda = 1/10$ (dotted line). Source: created by the author.

where erf is the *error function*:

$$\mathrm{erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \mathrm{d}t. \tag{18}$$

In Figure 2, we see the effect of the resettings on the value function.

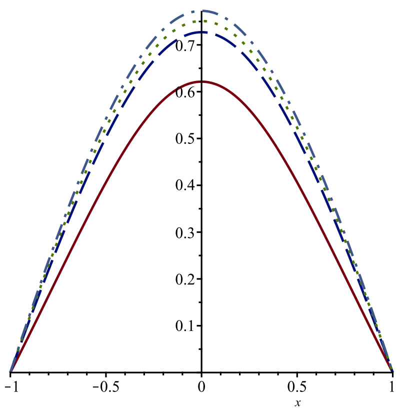

Figure 2: Value function $F(x)$ in Example 1 when $\lambda = 0$ (solid line), and approximate value function when $\lambda = 1$ (dashed line), $\lambda = 1/2$ (dotted line) and $\lambda = 1/10$ (dash-dotted line), for x in the interval $[-1, 1]$. Source: created by the author.

Finally, the optimal control when $\lambda = 0$ and the (approximate) optimal control when $\lambda = 1$, $\lambda = 1/2$ and $\lambda = 1/10$ are presented in Figure 3. The four functions are almost equal around the origin.

Example 2. Suppose now that $f(x) = x$ and $v(x) = x^2$, so that the continuous part of the jump-diffusion process $\{X(t), t \geq 0\}$ is a geometric Brownian motion. Let us take $[a, b] = [1/2, 3/2]$ and $x_r = 1$.

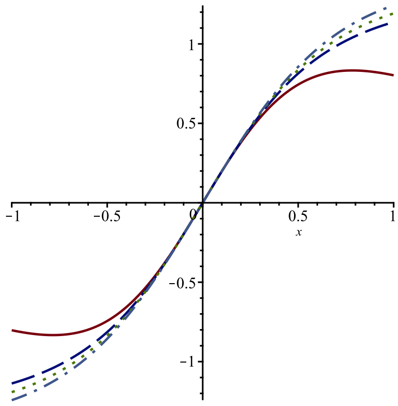

Figure 3: Optimal control $u^*(x)$ in Example 1 when $\lambda = 0$ (solid line), and approximate optimal control when $\lambda = 1$ (dashed line), $\lambda = 1/2$ (dotted line) and $\lambda = 1/10$ (dash-dotted line), for x in the interval $[-1, 1]$. Source: created by the author.

Then, if $\theta = \lambda = 1$, and if we assume that $q(x) \equiv 1$ and $b(x) = \sqrt{2x^2 - 2x + 1}$, we find that the function $G(x) := e^{-F(x)}$ satisfies the linear ODE

$$(-2x^2 + 2x - 1)G''(x) - G'(x) + 2G(x) = 0. \tag{19}$$

The boundary conditions are

$$G(1/2) = G(3/2) = 1. \tag{20}$$

The solution to this boundary value problem can be expressed in terms of the generalized hypergeometric function. It is shown in Figure 4. From the function $G(x)$, we can compute approximations to the value function $F(x)$ and the optimal control $u^*(x)$. These approximations will be valid near $x = 1$.

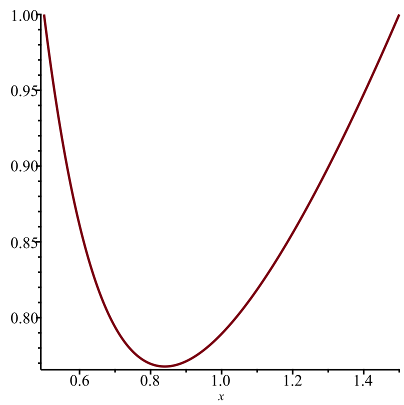

Figure 4: Solution of (19), (20) for x in the interval $[1/2, 3/2]$. Source: created by the author.

We can also obtain an approximate solution to

Eq. (10) by writing

$$F(x) \approx F(x_r)+(x-x_r)F'(x_r)+\frac{(x-x_r)^2}{2}F''(x_r). \quad (21)$$

Then, the equation becomes

$$\begin{aligned} 0 \ &\approx \ \theta - \frac{1}{2}\frac{b^2(x)}{q(x)}[F'(x)]^2 + f(x)F'(x) \quad (22) \\ &+ \frac{1}{2}v(x)F''(x) - \lambda\Big[(x-x_r)F'(x_r) \\ &- \frac{(x-x_r)^2}{2}F''(x_r)\Big], \end{aligned}$$

which can be rewritten as follows:

$$\begin{aligned} 0 \ &\approx \ -\frac{1}{2}\frac{b^2(x)}{q(x)}[F'(x)]^2 + f(x)F'(x) \quad (23) \\ &+ \frac{1}{2}v(x)F''(x) + c_0 + c_1 x + c_2 x^2, \end{aligned}$$

where c_0, c_1 and c_2 are constants (that depend on $F(x)$).

3 Exact solutions

In this section, we will consider the inverse homing problem: we will assume that the value function $F(x)$ is of a certain form, and we will try to find homing problems for which the function $F(x)$ is indeed the value function we are looking for.

Problem 1. Suppose that $\theta = \lambda = 1$, $b(x) \equiv 1$ and $q(x) \equiv 1$. Moreover, $[a, b] = [0, \infty)$. We deduce from (10) that we must solve the non-linear ODE

$$\begin{aligned} 0 \ &= \ 1 - \frac{1}{2}[F'(x)]^2 + f(x)F'(x) \quad (24) \\ &+ \frac{1}{2}v(x)F''(x) + F(x_r) - F(x). \end{aligned}$$

The equation is subject to the boundary condition $F(0) = 0$. Furthermore, the function $F(x)$ must be strictly increasing in x.

Let us assume that $F(x)$ is of the form $F(x) = F_0 x$, where F_0 is a positive constant. Substituting this function into the above ODE, we find that we must have

$$0 = 1 - \frac{1}{2}F_0^2 + F_0 f(x) + F_0 x_r - F_0 x. \quad (25)$$

Hence, if we choose $f(x) = x$, the function $F(x) = F_0 x$ is a solution to our homing problem if and only if $F_0 = x_r + \sqrt{x_r^2 + 2}$, where $x_r > 0$. It follows that the optimal control would be $u^*(x) \equiv -x_r - \sqrt{x_r^2 + 2}$.

Remark. Notice that the value function $F(x)$ and the optimal control $u^*(x)$ do not depend on the function $v(x)$ in the above problem.

Problem 2. Suppose now that $[a, b] = [0, 2]$ in Problem 1. Moreover, we assume that $f(x) \equiv 0$ and $v(x) \equiv 2$. It follows that Eq. (24) reduces to

$$0 = 1 - \frac{1}{2}[F(x)]^2 + F''(x) + F(x_r) - F(x). \quad (26)$$

We look for a solution of the above ODE such that $F(0) = F(2) = 0$. If $x_r = 1$, one can check that the function

$$F(x) = x\left(1 - \frac{x}{2}\right) \quad \text{for } 0 \le x \le 2 \quad (27)$$

is a valid solution to our problem. The optimal control is

$$u^*(x) = x - 1 \quad \text{for } 0 < x < 2. \quad (28)$$

4 Concluding remarks

In this paper, a stochastic optimal control problem known as a homing problem has been studied for one-dimensional diffusion processes with random resettings. We saw in Section 2 that to obtain the value function $F(x)$, from which the optimal control follows at once, we need to solve a non-linear ODE of order 2. Moreover, this ODE involves the term $F(x_r)$, where x_r is the point to which the diffusion process is reset at random times.

In Section 2, we also explained how it is sometimes possible to obtain approximate solutions to the control problem that are accurate in the vicinity of the point x_r. Then, in Section 3, we presented two problems that we were able to solve explicitly and exactly by assuming that the value function was of a given (simple) form.

Even in the simplest cases, such as when all the functions that appear in Eq. (10) are constants, it is not easy to find an exact analytical solution to this non-linear ODE. We could however use numerical methods to solve particular problems.

As a continuation of this work, in addition to the random resettings, we could also have jumps according to a second (independent) Poisson process. Finally, we could consider the homing problem when the diffusion process is replaced by a discrete-time stochastic process, for instance a Markov chain; see, [9] and, [10].

Acknowledgments
The author would like to thank the anonymous reviewers for their constructive comments.

References:

[1] M. Abundo, The first-passage area of Wiener process with stochastic resetting, *Methodology and Computing in Applied Probability*, Vol. 25, No. 4, Paper No. 92, 25 p., 2023. https://doi.org/10.1007/s11009-023-10069-4

[2] M. R. Evans and S. N. Majumdar, Diffusion with stochastic resetting, *Physical Review Letters*, Vol. 106, No. 16, 160601, 4 p., 2011. https://doi.org/10.1103/PhysRevLett.106.160601

[3] M. R. Evans, S. N. Majumdar and G. Schehr, Stochastic resetting and applications, *Journal of Physics A: Mathematical and Theoretical*, Vol. 53, 193001, 67 p., 2020. https://doi.org/10.1088/1751-8121/ab7cfe

[4] A. Dubey and A. Pal, First-passage functionals for Ornstein-Uhlenbeck process with stochastic resetting, *Journal of Physics A: Mathematical and Theoretical*, Vol. 56, No. 43, Article ID 435002, 19 p., 2023. https://doi.org/10.1088/1751-8121/acf748

[5] J. Kuhn, The risk-sensitive homing problem, *Journal of Applied Probability*, Vol. 22, No. 4, pp. 796–803, 1985. https://doi.org/10.2307/3213947

[6] C. Makasu, Homing problems with control in the diffusion coefficient, *IEEE Transactions on Automatic Control*, Vol. 67, No. 7, pp. 3770–3772, 2022, https://doi.org/10.1109/TAC.2022.3157077

[7] M. Kounta and N. J. Dawson, Linear quadratic Gaussian homing for Markov processes with regime switching and applications to controlled population growth/decay, *Methodology and Computing in Applied Probability*, Vol. 23, No. 3, pp. 1155–1172, 2021. https://doi.org/10.1007/s11009-020-09800-2

[8] M. Lefebvre, First-passage times and optimal control of integrated jump-diffusion processes, *Fractal and Fractional*, Vol. 7, No. 2, Article 152, 2023. https://doi.org/10.3390/fractalfract7020152

[9] S. N. Majumdar, S. Sabhapandit and G. Schehr, Random walk with random resetting to the maximum position, *Physical Review E*, Vol. 92, No. 5, 052126, 2015. https://doi.org/10.1103/PhysRevE.92.052126

[10] M. Montero and J. Villarroel, Directed random walk with random restarts: The Sisyphus random walk, *Physical Review E*, Vol. 94, No. 3, 032132, 2016. https://doi.org/10.1103/PhysRevE.94.032132

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The author contributed in the present research, at all stages from the formulation of the problem to the final findings and solution.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

No funding was received for conducting this study.

Conflict of Interest

The author has no conflict of interest to declare that is relevant to the content of this article.