The Exact Solutions to the Nonlinear Caputo Multi-Fractional Differential Equations with Variable Coefficients Using the General Adomian Decomposition Method

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Abstract: This investigation presents the exact solution to a nonlinear Caputo multi-fractional differential equation with variable coefficients via the General Adomian decomposition method. The proposed approach integrates the Adomian decomposition technique with a general integral transformation, enabling its application to nonlinear problems characterized by a wide range of nonlinear terms. Notably, the method yields exact solutions within a few iterations, highlighting both the precision and computational efficiency of the approach. These findings underscore the potential of the process for addressing complex nonlinear multi-fractional differential equations.

Key-Words: General Transform; Nonlinear Equation; Caputo Fractional Derivative; Exact Solution; Adomian Polynomial

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1 Introduction

The fractional derivative has been recognized as one of the most developed mathematical disciplines, according to several research studies published in the last few decades. Some researchers have been interested in this field of study because of its potential to describe natural phenomena in depth and contribute to mathematical models that can predict events more accurately. The use of fractional derivatives enabled a new and powerful method of modeling complex physical phenomena from the past, such as non locality or memory effects. Some outstanding applications of fractional derivatives in science and engineering have been shown, emphasizing the interest in studying this field of study, [1].

The fractional differential equation is an emerging technique that evolved from the fractional derivative. This non-integer ordered differential equation arises frequently in a variety of mathematical and scientific models such as viscoelastic materials, control theory, signal processing, damping methods, and so on. Even though the equation is crucial in understanding real-world phenomena, solving it is a challenging endeavor. Scholars have tried to propose methods to overcome this equation, both analytically and numerically, including homotopy analysis transform, [2], variational iteration method, [3], integral transform method, and fractional residual power series method, [4].

Integral transform is a productive instrument for dealing with issues in science and engineering. Due to its direct application to problems and ability to convert differential equations into algebraic equations, this instrument has gained much attention in the academic community. Numerous integral transforms are proposed, and some of the dominant transforms are the Laplace transform, the Fourier transform, the Elzaki transform, the Mellin transform, and the Sumudu transform.

Recently, the general integral transform, or Jafari transform, was initially introduced in 2021 by [5], for resolving integral equations. By providing a parameter, one can simplify this transform generalization to others, such as the Laplace transform, the Elzaki transform, the Sumudu transform, and the Mellin transform. Mathematicians attempt to utilize the transformation to figure out the scientific problem. In 2021, [6], investigated some valuable properties and theories related to the transform and applied them to solve the model describing the cell population dynamics in the colonic crypt and colorectal cancer. The studies, [7], [8], applied the general integral transform of error function for computing improper integrals and answering Abel's integral equation. The application of this kind of transformation to models in health sciences is presented, [9]. Moreover, the fascinating application of the general integral transform can be found in [10], [11]. Beside the integer-order differential equation, this kind of transform is applied to fractional calculus. The study, [12], employed the general transform to solve the Caputo fractional differential equations. The authors in [13], examined the applicability of this transform on

Atangana-Baleau derivatives. The authors in [14], used the Jafari transform and ADM to nonlinear fractional-order partial differential equations arising in plasma.

One of the most substantial types of fractional differential equations is the multi-order one, which finds application in numerous fields. Many scientists have started researching their characteristics and solutions, both exact and numerical. The following are some notable advancements. An eigenfunction method combined with Legendre wavelets was used to solve boundary value problems for multi-order fractional differential equations, [15]. A numerical method based on the conjunction of Paraskevopoulos's algorithm and operational matrices was improved to solve multi-order linear and nonlinear fractional differential equations, [16]. Nonlinear systems of multi-order fractional differential equations were comprehensively studied from both theoretical and numerical approximation perspectives, [17].

The aforementioned research has inspired the search for an exact solution to a multi-order nonlinear differential equation with variable coefficients. The significant contribution of this task is implementing a straightforward procedure for handling the nonlinear multi-order fractional differential equation with variable coefficients. A variety of nonlinear problems are impressively overcome by utilizing the shifting property. The study has encompassed an approach that incorporates techniques integrating various integral transforms and the Adomian decomposition method. This approach not only provides the closed-form solution to the problem but also reduces the cost of computation. Linearization or perturbation is unnecessary.

The main objective of the current investigation is to implement the General Adomian decomposition strategy to tackle the initial value problem for a nonlinear multi-fractional order differential equation of the form

$$D^{\alpha}y(\xi)+\sum_{i=1}^{k}a_i(\xi)D^{\alpha_i}y(\xi)+b(\xi)y(\xi)+c(\xi)N(y(\xi))=g(\xi) \tag{1}$$

$$y^{(j)}(0)=\chi_j,\ j=0,1,\ldots,n-1 \tag{2}$$

$n-1<\alpha\leqslant n$, $0<\alpha_i<\alpha$. Here D^{α} is the Caputo fractional derivative of order α with respect to ξ, $a_i(\xi),b(\xi),c(\xi)$ are continuous functions, $N(y)$ is the general nonlinear term, and $g(\xi)$ is the source term. This system is highly significant in the fields of science and technology.

The rest of the article is organized as follows. In Section 2, the basic notions of fractional calculus and the general integral transform were reviewed. This section also presented the important properties of the integral transform and Caputo fractional derivative. How to carry out the suggested strategy for problems was addressed in Section 3. Several nonlinear fractional differential equations with variable coefficients are considered. The last section contained the study's discussion and conclusion.

2 Preliminary

This section introduces the fundamental concepts and essential properties of fractional calculus. Furthermore, the general integral transform and its related results are provided.

2.1 Fractional Calculus

Definition 1 *The fractional integral operator of order $\gamma(\gamma\geqslant 0)$ of $\phi(t)$ of Riemann-Liouville type is defined as*

$$I^{\gamma}\phi(t)=\begin{cases}\frac{1}{\Gamma(\gamma)}\displaystyle\int_0^t\frac{\phi(\tau)}{(t-\tau)^{1-\gamma}}d\tau,\ \gamma>0,\ t>0,\\ \phi(t),\ \gamma=0.\end{cases}$$

Definition 2 *The Caputo fractional derivative operator D^{γ} of order γ, $(n-1<\gamma\leqslant n, n\in\mathbb{N})$ is defined in the following form,*

$$D^{\gamma}\phi(t)=\frac{1}{\Gamma(n-\gamma)}\int_0^t(t-\tau)^{-\gamma+n-1}\phi^{(n)}(\tau)d\tau, \tag{3}$$

$\gamma>0$, $t>0$, *where the function $\phi(t)$ has absolutely continuous derivatives up to order $n-1$. In particular, if $0<\gamma<1$, we have*

$$D^{\gamma}\phi(t)=\frac{1}{\Gamma(1-\gamma)}\int_0^t(t-\tau)^{-\gamma}\phi'(\tau)d\tau.$$

Furthermore, the following basic features can be demonstrated.

1.) $D^{\gamma}C=0$, where C is a constant.

2.) For $p\in\mathbb{N}_0=\{0,1,2,\ldots\}$ and the ceiling function $\lceil\gamma\rceil$ refers to the smallest integer greater than or equal to γ

$$D^{\gamma}x^p=\begin{cases}0, & \text{for}\, p<\lceil\gamma\rceil\\ \dfrac{\Gamma(p+1)}{\Gamma(p+1-\gamma)}x^{p-\gamma}, & \text{for } p\geqslant\lceil\gamma\rceil.\end{cases}$$

3.) $D^{\gamma}I^{\gamma}\phi(t)=\phi(t)$.

4.) $I^{\gamma}D^{\gamma}\phi(t)=\phi(t)-\displaystyle\sum_{k=1}^{n-1}\frac{\phi^{(k)}(0)}{k!}t^k$.

2.2 General Integral transform

Definition 3 *[5] Let $f(t)$ be a integrable function defined for $t \geq 0$, $p(s) \neq 0$ and $q(s)$ are positive real functions. The general integral transform $\mathcal{F}(s)$ of $f(t)$ is defined by*

$$\mathbb{J}\{f(t)\} = \mathcal{F}(s) = p(s)\int_0^\infty f(t)e^{-q(s)t}dt$$

provide the improper integral exists for some $q(s)$.

The general integral transform of some basic functions are presented in the Table 1.

Table 1: Show the general integral transform of some basic functions.

Functions $f(t)$	General integral transform $\mathcal{F}(s) = \mathbb{J}\{f(t)\}$
1	$\dfrac{p(s)}{q(s)}$
t	$\dfrac{p(s)}{q^2(s)}$
$t^\alpha,\ \alpha > 0$	$\dfrac{\Gamma(\alpha+1)p(s)}{q^{\alpha+1}(s)}$
e^{at}	$\dfrac{p(s)}{q(s)-a}$ if $q(s) > a$
$\sin at$	$\dfrac{ap(s)}{a^2+q^2(s)}$ if $q(s) > \lvert\Im a\rvert$
$\cos at$	$\dfrac{q(s)p(s)}{a^2+q^2(s)}$

Source: [5].

Theorem 4 *[5] Let $f(t)$ is differentiable and $p(s)$ and $q(s)$ are positive real functions, then*

$$\mathbb{J}\{f^{(n)}(t)\} = q^n(s)\mathbb{J}\{f(t)\} - p(s)\sum_{k=0}^{n-1} q^{n-1-k}f^{(k)}(0).$$

Theorem 5 *[12] If $n \in \mathbb{Z}^+$ where $n-1 < \alpha \leqslant n$ and $\mathcal{F}(s)$ be the general integral transform of the function $f(t)$, then the general integral transform of the Caputo fractional derivative of order $\alpha > 0$, is*

$$\mathbb{J}\{D^\alpha f(t)\} = q^\alpha(s)\mathcal{F}(s) - p(s)\sum_{k=0}^{n-1} q^{\alpha-1-k}(s)f^{(k)}(0).$$

Theorem 6 (Shifting Property) *[7] If $\mathbb{J}\{f(t)\} = \mathcal{F}(p(s), q(s))$, then*

$$\mathbb{J}\{e^{at}f(t)\} = \mathcal{F}(p(s), q(s)-a) = \mathbb{J}\{f(t)\}_{q(s)\to q(s)-a}.$$

Theorem 7 *[7] The general transform of the error function,*

$$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt,$$

is given by $\mathbb{J}\{erf(\sqrt{t})\} = \dfrac{p(s)}{q(s)\sqrt{q(s)+1}}$.

3 Main Results

This section suggests employing the General Adomian decomposition method for nonlinear multi-fractional differential equations. Furthermore, several nonlinear problems are examined to support the proposed concept.

3.1 Application of Proposed Method to Multi-Fractional Differential Equations

Consider the nonlinear multi-fractional differential equation (1)

$$D^\alpha y(\xi) + \sum_{i=1}^{k} a_i(\xi)D^{\alpha_i}y(\xi) + b(\xi)y(\xi) + c(\xi)N(y(\xi)) = g(\xi)$$

with initial conditions $y^{(j)}(0) = \chi_j\ j = 0, \ldots, n-1$. Applying the General integral transform and utilizing the transformation of fractional derivative yields

$$\begin{aligned}\mathbb{J}\{D^\alpha y(\xi)\} &+ \sum_{i=1}^{k}\mathbb{J}\{a_i(\xi)D^{\alpha_i}y(\xi)\} + \mathbb{J}\{b(\xi)y(\xi)\} \\ &+ \mathbb{J}\{c(\xi)N(y(\xi))\} = \mathbb{J}\{g(\xi)\} \\ q^\alpha(s)\mathcal{Y}(s) &- p(s)\sum_{j=0}^{n-1} q^{\alpha-1-j}(s)y^{(j)}(0) \\ &+ \sum_{i=1}^{k}\mathbb{J}\{a_i(\xi)D^{\alpha_i}y(\xi)\} + \mathbb{J}\{b(\xi)y(\xi)\} \\ &+ \mathbb{J}\{c(\xi)N(y(\xi))\} = \mathcal{G}(s),\end{aligned}$$

where $\mathcal{Y}(s) = \mathbb{J}\{y(\xi)\}, \mathcal{G}(s) = \mathbb{J}\{g(\xi)\}$. Imposing the initial condition and rearranging the eqution, one obtains

$$\begin{aligned}q^\alpha\mathcal{Y}(s) &= p\sum_{j=0}^{n-1} q^{\alpha-1-j}\chi_j - \sum_{i=1}^{k}\mathbb{J}\{a_i(\xi)D^{\alpha_i}y(\xi)\} \\ &- \mathbb{J}\{b(\xi)y(\xi)\} - \mathbb{J}\{c(\xi)N(y(\xi))\} + \mathcal{G}(s), \\ \mathcal{Y}(s) &= \frac{\mathcal{G}(s)}{q^\alpha} + p(s)\sum_{j=0}^{n-1}\frac{\chi_j}{q^{j+1}} \\ &- \frac{1}{q^\alpha}\Big\{\sum_{i=1}^{k}\mathbb{J}\{a_i(\xi)D^{\alpha_i}y(\xi)\} + \mathbb{J}\{b(\xi)y(\xi)\} \\ &+ \mathbb{J}\{c(\xi)N(y(\xi))\}\Big\}.\end{aligned}$$

Hence, taking the inverse transform leads to

$$y(\xi) = \mathbb{J}^{-1}\Big\{\frac{\mathcal{G}(s)}{q^{\alpha}} + p\sum_{j=0}^{n-1}\frac{\chi_j}{q^{j+1}}\Big\} - \mathbb{J}^{-1}\Big\{\frac{1}{q^{\alpha}}\Big\{\sum_{i=1}^{k}\mathbb{J}\{a_i(\xi)D^{\alpha_i}y(\xi)\}\Big\}\Big\} - \mathbb{J}^{-1}\Big\{\frac{1}{q^{\alpha}}\Big\{\mathbb{J}\{b(\xi)y(\xi)\} + \mathbb{J}\{c(\xi)N(y)\}\Big\}\Big\}. \quad (4)$$

Suppose the solution is represented in the form

$$y(\xi) = \sum_{k=0}^{\infty} y_k(\xi), \quad (5)$$

and the nonlinear term $N(y)$ is expressed in the infinite series

$$N(y) = \sum_{k=0}^{\infty} A_k, \quad (6)$$

where A_k refers to Adomian polynomials. This sort of polynomial can be found by

$$A_k = \frac{1}{k!}\Big[\frac{d^k}{d\lambda^k}N(\sum_{n=0}^{\infty}\lambda^n y_n)\Big]_{\lambda=0}.$$

Substituting equations (5) and (6) into equation (4) leads to

$$\sum_{k=0}^{\infty} y_k(\xi) = \mathbb{J}^{-1}\Big\{\frac{\mathcal{G}(s)}{q^{\alpha}} + p(s)\sum_{j=0}^{n-1}\frac{\chi_j}{q^{j+1}}\Big\} - \mathbb{J}^{-1}\Big\{\frac{1}{q^{\alpha}}\Big\{\sum_{i=1}^{k}\mathbb{J}\{a_i(\xi)D^{\alpha_i}(\sum_{k=0}^{\infty}y_k(\xi))\}\Big\}\Big\} - \mathbb{J}^{-1}\Big\{\frac{1}{q^{\alpha}}\Big\{\mathbb{J}\{b(\xi)(\sum_{k=0}^{\infty}y_k(\xi))\} + \mathbb{J}\{c(\xi)\sum_{k=0}^{\infty}A_k\}\Big\}\Big\}.$$

The classic iterative relation is defined by

$$y_0(\xi) = \mathbb{J}^{-1}\Big\{\frac{\mathcal{G}(s)}{q^{\alpha}} + p\sum_{j=0}^{n-1}\frac{\chi_j}{q^{j+1}}\Big\} = \mathcal{H}(\xi),$$

$$y_{k+1}(\xi) = -\mathbb{J}^{-1}\Big\{\frac{1}{q^{\alpha}}\Big\{\sum_{i=1}^{k}\mathbb{J}\{a_i(\xi)D^{\alpha_i}(y_k(\xi))\}\Big\}\Big\} - \mathbb{J}^{-1}\Big\{\frac{1}{q^{\alpha}}\Big\{\mathbb{J}\{b(\xi)(y_k(\xi))\}\Big\}\Big\} - \mathbb{J}^{-1}\Big\{\frac{1}{q^{\alpha}}\Big\{\mathbb{J}\{c(\xi)(A_k(\xi))\}\Big\}\Big\}.$$

In order to expedite the calculation, we break down the function $\mathcal{H}(\xi) = \mathcal{H}_0(\xi) + \mathcal{H}_1(\xi)$. The mentioned procedure is therefore enhanced

$$\left.\begin{aligned} y_0(\xi) &= \mathcal{H}_0(\xi), \\ y_1(\xi) &= \mathcal{H}_1(\xi) - \mathbb{J}^{-1}\Big\{\frac{1}{q^{\alpha}}\sum_{i=1}^{k}\mathbb{J}\{a_i(\xi)D^{\alpha_i}(y_0(\xi))\}\Big\} - \mathbb{J}^{-1}\Big\{\frac{1}{q^{\alpha}}\Big\{\mathbb{J}\{b(\xi)(y_0(\xi))\}\Big\}\Big\} - \mathbb{J}^{-1}\Big\{\frac{1}{q^{\alpha}}\Big\{\mathbb{J}\{c(\xi)(A_0(\xi))\}\Big\}\Big\}, \\ y_{k+1} &= -\mathbb{J}^{-1}\Big\{\frac{1}{q^{\alpha}}\sum_{i=1}^{k}\mathbb{J}\{a_i(\xi)D^{\alpha_i}(y_k)\}\Big\} - \mathbb{J}^{-1}\Big\{\frac{1}{q^{\alpha}}\Big\{\mathbb{J}\{b(\xi)(y_k(\xi))\}\Big\}\Big\} - \mathbb{J}^{-1}\Big\{\frac{1}{q^{\alpha}}\Big\{\mathbb{J}\{c(\xi)(A_k(\xi))\}\Big\}\Big\}, \ k \geqslant 1. \end{aligned}\right\} \quad (7)$$

One of the keys to reducing computation costs is iteration (7). The convergence of the solution may be impacted by the initial function selection. In this case, it is advised to use the initial function, $\mathcal{H}_0$, which fulfilled the initial condition.

Criterior convergence for general Adomian decomposition method

The convergence criterion for the general Adomian decomposition approach is described in [18]. The results are reviewed below.

Theorem 8 *[18] Let the Banach space $B \equiv C[a,b]$ is defined on $[a,b]$. Then Eq. (5) as $y_n(\xi) = \sum_{k=0}^{n} y_k(\xi)$ is convergent series, if $y_0 \in B$ is bounded and $\forall y_k \in B, \|y_{k+1}\| \leqslant \sigma\|y_k\|$, where $0 < \sigma < 1$ and $\|y\| = \sup_{\xi\in[a,b]}|y(\xi)|$.*

Theorem 9 *[18] The maximum absolute error for (5) is determined by*

$$\Big\|y(\xi) - \sum_{k=0}^{n} y_k(\xi)\Big\| \leqslant \frac{\sigma^{n+1}}{1-\sigma}\big\|y_0(\xi)\big\|.$$

3.2 Applications

In this part, various nonlinear multi-fractional differential equations have been solved by the suggested strategies. Additionally, the shifting property is required to overcome some challenging issues.

Example 10 *Consider the nonlinear multi-fractional differential equation*

$$D^{\alpha+1}y(\xi) + D^{\alpha}y(\xi) + (1+\xi^2)y^2(\xi) = \frac{\xi^{1-\alpha}}{\Gamma(2-\alpha)} + \xi^4 + 2\xi^3 + 2\xi^2 + 2\xi + 1$$

$0 < \alpha \leqslant 1,$ *with initial conditions* $y(0) = y'(0) = 1.$

$$y(0) = 1,\ y'(0) = 1.$$

The exact solution to the problem is $y(\xi) = 1 + \xi$.

Note that $a_1(\xi) = 1, b(\xi) = 0, c(\xi) = 1 + \xi^2$, $N(y) = y^2, \chi_0 = 1, \chi_1 = 1$ and $g(\xi) = \frac{\xi^{1-\alpha}}{\Gamma(2-\alpha)} + \xi^4 + 2\xi^3 + 2\xi^2 + 2\xi + 1$. Hence, function $\mathcal{H}(\xi) = 1+\xi + \mathbb{J}^{-1}\Big[\frac{1}{q^{\alpha+1}}\left(\frac{p}{q^{2-\alpha}} + \frac{4!p}{q^5} + \frac{12p}{q^4} + \frac{4p}{q^3} + \frac{2p}{q^2} + \frac{p}{q}\right)\Big]$. By choosing the initial approximation $\mathcal{H}_0(\xi) = 1+\xi$, the recurrence relation for this problem is

$$\begin{aligned}
y_0 &= 1 + \xi,\\
y_1 &= \mathbb{J}^{-1}\Big[\frac{1}{q^{\alpha+1}}\left(\frac{p}{q^{2-\alpha}} + \frac{4!p}{q^5} + \frac{12p}{q^4} + \frac{4p}{q^3}\right)\Big]\\
&\quad + \mathbb{J}^{-1}\Big[\frac{1}{q^{\alpha+1}}\left(\frac{2p}{q^2} + \frac{p}{q}\right)\Big]\\
&\quad - \mathbb{J}^{-1}\Big[\frac{1}{q^{\alpha+1}}\mathbb{J}\Big\{D^{\alpha}y_0\Big\} + \frac{1}{q^{\alpha+1}}\mathbb{J}\Big\{(1+\xi^2)A_0\Big\}\Big],\\
y_{i+1} &= -\mathbb{J}^{-1}\Big[\frac{1}{q^{\alpha+1}}\mathbb{J}\Big\{D^{\alpha}y_i\Big\}\Big]\\
&\quad - \mathbb{J}^{-1}\Big[\frac{1}{q^{\alpha+1}}\mathbb{J}\Big\{(1+\xi^2)A_i\Big\}\Big],\ i \geqslant 1.
\end{aligned}$$

Here, the Adomian polynomials for the nonlinear term $N(y) = y^2$ are defined as follows: $A_0 = y_0^2$, $A_1 = 2y_0y_1, A_2 = 2y_0y_2 + y_1^2, \ldots$. As it can be observed

$$\begin{aligned}
y_1(\xi) &= \mathbb{J}^{-1}\Big[\frac{1}{q^{\alpha+1}}\left(\frac{p}{q^{2-\alpha}} + \frac{4!p}{q^5} + \frac{12p}{q^4} + \frac{4p}{q^3}\right)\Big]\\
&\quad + \mathbb{J}^{-1}\Big[\frac{1}{q^{\alpha+1}}\left(\frac{2p}{q^2} + \frac{p}{q}\right)\Big] - \mathbb{J}^{-1}\Big[\frac{1}{q^{\alpha+1}}\mathbb{J}\Big\{D^{\alpha}(1+\xi)\Big\}\Big]\\
&\quad - \mathbb{J}^{-1}\Big[\frac{1}{q^{\alpha+1}}\mathbb{J}\Big\{(1+\xi^2)(1+\xi)^2\Big\}\Big],\\
&= \mathbb{J}^{-1}\Big[\frac{1}{q^{\alpha+1}}\left(\frac{p}{q^{2-\alpha}} + \frac{4!p}{q^5} + \frac{12p}{q^4} + \frac{4p}{q^3} + \frac{2p}{q^2} + \frac{p}{q}\right)\Big]\\
&\quad - \mathbb{J}^{-1}\Big[\frac{1}{q^{\alpha+1}}\mathbb{J}\Big\{\frac{\xi^{1-\alpha}}{\Gamma(2-\alpha)} + 1 + 2(\xi+\xi^2+\xi^3) + \xi^4\Big\}\Big],\\
&= \mathbb{J}^{-1}\Big[\frac{1}{q^{\alpha+1}}\left(\frac{p}{q^{2-\alpha}} + \frac{4!p}{q^5} + \frac{12p}{q^4} + \frac{4p}{q^3} + \frac{2p}{q^2} + \frac{p}{q}\right)\Big]\\
&\quad - \mathbb{J}^{-1}\Big[\frac{1}{q^{\alpha+1}}\left(\frac{p}{q^{\alpha-2}} + \frac{p}{q} + \frac{2p}{q^2} + \frac{4p}{q^3} + \frac{12p}{q^4} + \frac{4!p}{q^5}\right)\Big]\\
&= 0.
\end{aligned}$$

It can be seen after a few iterations that $y_i = 0,\ i \geqslant 1$. Therefore, the solution of the problem is $y(\xi) = y_0 + y_1 + y_2 + \ldots = 1 + \xi$.

Example 11 *Let solve the nonlinear multi-fractional differential equation*

$$D^{\frac{5}{3}}y(\xi) + \xi D^{\frac{3}{2}}y(\xi) + e^{2\xi}y^2(\xi) = \frac{4\xi^{\frac{3}{2}}}{\sqrt{\pi}} + \frac{6\xi^{\frac{1}{3}}}{\Gamma\left(\frac{1}{3}\right)} + \xi^4 e^{2\xi}$$

subject to the initial conditions $y(0) = 0,\ y'(0) = 0$. *The exact solution to this problem is* $y(\xi) = \xi^2$.

It can be observed that $a_1(\xi) = \xi, b(\xi) = 0, c(\xi) = e^{2\xi}, N(y) = y^2, \chi_0 = 0, \chi_1 = 0$ and $g(\xi) = \frac{4}{\sqrt{\pi}}\xi^{\frac{3}{2}} + \frac{6}{\Gamma\left(\frac{1}{3}\right)}\xi^{\frac{1}{3}} + \xi^4 e^{2\xi}$. By using shifting property, $\mathcal{H}(\xi) = \xi^2 + \mathbb{J}^{-1}\Big[\frac{3p}{q^{25/6}} + \frac{4!p}{q^{25/3}(q-2)^5}\Big]$. Let $\mathcal{H}_0(\xi) = \xi^2$, the iterative relation (7) is defined as follows

$$\begin{aligned}
y_0 &= \xi^2,\\
y_1 &= \mathbb{J}^{-1}\Big[\frac{3p}{q^{\frac{25}{6}}} + \frac{4!p}{q^{\frac{25}{3}}(q-2)^5}\Big]\\
&\quad - \mathbb{J}^{-1}\Big[\frac{1}{q^{\frac{5}{3}}}\mathbb{J}\Big\{\xi D^{\frac{3}{2}}y_0\Big\} + \frac{1}{q^{\frac{5}{3}}}\mathbb{J}\Big\{e^{2\xi}A_0\Big\}\Big],\\
y_{i+1} &= -\mathbb{J}^{-1}\Big[\frac{1}{q^{\frac{5}{3}}}\mathbb{J}\Big\{\xi D^{\frac{3}{2}}y_i\Big\} + \frac{1}{q^{\frac{5}{3}}}\mathbb{J}\Big\{e^{2\xi}A_i\Big\}\Big], i \geqslant 1.
\end{aligned}$$

Note that by applying the shifting and the Gamma properties,

$$\begin{aligned}
y_1 &= \mathbb{J}^{-1}\Big[\frac{3p}{q^{\frac{25}{6}}} + \frac{4!p}{q^{\frac{5}{3}}(q-2)^5}\Big]\\
&\quad - \mathbb{J}^{-1}\Big[\frac{1}{q^{\frac{5}{3}}}\mathbb{J}\Big\{\xi D^{\frac{3}{2}}\xi^2\Big\} + \frac{1}{q^{\frac{5}{3}}}\mathbb{J}\Big\{e^{2\xi}\xi^4\Big\}\Big],\\
&= \mathbb{J}^{-1}\Big[\frac{3p}{q^{\frac{25}{6}}} + \frac{4!p}{q^{\frac{5}{3}}(q-2)^5}\Big]\\
&\quad - \mathbb{J}^{-1}\Big[\frac{1}{q^{\frac{5}{3}}} \cdot \frac{4}{\sqrt{\pi}}\mathbb{J}\Big\{\xi^{\frac{3}{2}}\Big\} + \frac{1}{q^{\frac{5}{3}}}\mathbb{J}\Big\{\xi^4\Big\}_{q\to q-2}\Big],\\
&= \mathbb{J}^{-1}\Big[\frac{3p}{q^{\frac{25}{6}}} + \frac{4!p}{q^{\frac{5}{3}}(q-2)^5}\Big]\\
&\quad - \mathbb{J}^{-1}\Big[\frac{1}{q^{\frac{5}{3}}} \cdot \frac{4}{\sqrt{\pi}}\frac{\Gamma(\frac{5}{2})p}{q^{\frac{5}{2}}} + \frac{1}{q^{\frac{5}{3}}} \cdot \frac{4!p}{(q-2)^5}\Big],\\
&= \mathbb{J}^{-1}\Big[\frac{3p}{q^{\frac{25}{6}}} + \frac{4!p}{q^{\frac{5}{3}}(q-2)^5}\Big]\\
&\quad - \mathbb{J}^{-1}\Big[\frac{3p}{q^{\frac{25}{6}}} + \frac{4!p}{q^{\frac{25}{3}}(q-2)^5}\Big],\\
&= 0.
\end{aligned}$$

Following the previously mentioned repetition, the outcome results are $y_k(\xi) = 0, k \geqslant 2$. Therefore, the solution is $y(\xi) = y_0 + y_1 + y_2 + \ldots = \xi^2$.

Example 12 *Consider the nonlinear multi-fractional differential equation*

$$D^3y(\xi)+D^{\frac{5}{2}}y(\xi)-5^{\xi}y(\xi)+\ln y(\xi) = \xi+\big(1+erf(\sqrt{\xi})\big)e^{\xi}-e^{(\ln 5+1)\xi}$$

with the initial conditions

$$y(0)=1,\ y'(0)=1,\ y''(0)=1.$$

The exact solution to this problem is $y(\xi)=e^{\xi}$.

Here, $a_1(\xi)=1, b(\xi)=-5^{\xi}, c(\xi)=1, N(y)=\ln y$ $\chi_0=1,\chi_1=1,\chi_2=1$ and $g(\xi)=\xi+\big(1+\mathrm{erf}\sqrt{\xi}\big)e^{\xi}-e^{(\ln 5+1)\xi}$. Hence, $\mathcal{G}(s)=\frac{p}{q^2}+\frac{p}{q-1}$ $+\frac{p}{(q-1)\sqrt{q}}-\frac{p}{q-(\ln 5+1)}$ and

$$\begin{aligned}
\mathcal{H}(\xi) &= \mathbb{J}^{-1}\Big\{\frac{p}{q^5}+\frac{p}{q^3(q-1)}+\frac{p}{q^3(q-1)\sqrt{q}} \\
&\quad -\frac{p}{q^3(q-(\ln 5+1))}+p(\frac{1}{q}+\frac{1}{q^2}+\frac{1}{q^3})\Big\} \\
&= \mathbb{J}^{-1}\Big\{\frac{p}{q^5}+\frac{p}{q^3(q-1)}+\frac{p}{q^3(q-1)\sqrt{q}} \\
&\quad -\frac{p}{q^3(q-(\ln 5+1))}+\frac{p}{q^3}(q^2+q+1)\Big\} \\
&= \mathbb{J}^{-1}\Big\{\frac{p}{q^5}+\frac{p}{q^3(q-1)}+\frac{p}{q^3(q-1)\sqrt{q}} \\
&\quad -\frac{p}{q^3(q-(\ln 5+1))}+\frac{p(q^3-1)}{q^3(q-1)}\Big\} \\
&= \mathbb{J}^{-1}\Big\{\frac{p}{q^5}+\frac{p}{(q-1)}+\frac{p}{q^3(q-1)\sqrt{q}} \\
&\quad -\frac{p}{q^3(q-(\ln 5+1))}\Big\} \\
&= \frac{\xi^4}{4!}+e^{\xi}+\mathbb{J}^{-1}\Big[\frac{p}{q^3(q-1)\sqrt{q}}-\frac{p}{q^3(q-(\ln 5+1))}\Big].
\end{aligned}$$

Choose function $\mathcal{H}_0(\xi)=e^{\xi}$ and $\mathcal{H}_1(\xi)=\frac{\xi^4}{4!}+\mathbb{J}^{-1}\Big[\frac{p}{q^3(q-1)\sqrt{q}}-\frac{p}{q^3(q-(\ln 5+1))}\Big]$, the iteration for this problem is

$$\begin{aligned}
y_0 &= e^{\xi}, \\
y_1 &= \frac{\xi^4}{4!}+\mathbb{J}^{-1}\Big[\frac{p}{q^3(q-1)\sqrt{q}}-\frac{p}{q^3(q-(\ln 5+1))}\Big] \\
&\quad -\mathbb{J}^{-1}\Big[\frac{1}{q^3}\mathbb{J}\Big\{D^{\frac{5}{2}}y_0\Big\}-\frac{1}{q^3}\mathbb{J}\Big\{5^{\xi}y_0\Big\}+\frac{1}{q^3}\mathbb{J}\Big\{A_0\Big\}\Big], \\
y_{i+1} &= -\mathbb{J}^{-1}\Big[\frac{1}{q^3}\mathbb{J}\Big\{D^{\frac{5}{2}}y_i\Big\}-\frac{1}{q^3}\mathbb{J}\Big\{5^{\xi}y_i\Big\}+\frac{1}{q^3}\mathbb{J}\Big\{A_i\Big\}\Big],
\end{aligned}$$

for $i\geqslant 1$.

The direct calculation using Theorem 5 yields

$$\begin{aligned}
y_1 &= \frac{\xi^4}{4!}+\mathbb{J}^{-1}\Big[\frac{p}{q^3(q-1)\sqrt{q}}\Big] \\
&\quad -\mathbb{J}^{-1}\Big[\frac{p}{q^3(q-(\ln 5+1))}\Big]-\mathbb{J}^{-1}\Big[\frac{1}{q^3}\mathbb{J}\Big\{D^{\frac{5}{2}}e^{\xi}\Big\}\Big] \\
&\quad +\mathbb{J}^{-1}\Big[\frac{1}{q^3}\mathbb{J}\Big\{5^{\xi}e^{\xi}\Big\}\Big]-\mathbb{J}^{-1}\Big[\frac{1}{q^3}\mathbb{J}\Big\{\ln e^{\xi}\Big\}\Big], \\
&= \frac{\xi^4}{4!}+\mathbb{J}^{-1}\Big[\frac{p}{q^3(q-1)\sqrt{q}}-\frac{p}{q^3(q-(\ln 5+1))}\Big] \\
&\quad -\mathbb{J}^{-1}\Big[\frac{1}{q^3}\Big[q^{\frac{5}{2}}\mathbb{J}\Big\{e^{\xi}\Big\}-p\sum_{m=0}^{2}q^{\frac{3}{2}-m}\Big]\Big] \\
&\quad +\mathbb{J}^{-1}\Big[\frac{1}{q^3}\mathbb{J}\Big\{e^{(\ln 5+1)\xi}\Big\}\Big]-\frac{\xi^4}{4!}, \\
&= \mathbb{J}^{-1}\Big[\frac{p}{q^3(q-1)\sqrt{q}}\Big]-\mathbb{J}^{-1}\Big[\frac{p}{q^3(q-(\ln 5+1))}\Big] \\
&\quad -\mathbb{J}^{-1}\Big[\frac{1}{q^3}\Big[\frac{q^{\frac{5}{2}}p}{q-1}-p(q^{\frac{3}{2}}+q^{\frac{1}{2}}+q^{-\frac{1}{2}})\Big]\Big] \\
&\quad +\mathbb{J}^{-1}\Big[\frac{p}{q^3(q-(\ln 5+1))}\Big], \\
&= \mathbb{J}^{-1}\Big[\frac{p}{q^3(q-1)\sqrt{q}}\Big] \\
&\quad -\mathbb{J}^{-1}\Big[\frac{1}{q^3}\Big[\frac{q^{\frac{5}{2}}p}{q-1}-p(q^{\frac{3}{2}}+q^{\frac{1}{2}}+q^{-\frac{1}{2}})\Big]\Big], \\
&= \mathbb{J}^{-1}\Big[\frac{p}{q^3(q-1)\sqrt{q}}\Big] \\
&\quad -\mathbb{J}^{-1}\Big[\frac{1}{q^3}\Big[\frac{q^{\frac{5}{2}}p}{q-1}-\frac{p(q^3-1)}{q^{\frac{1}{2}}(q-1)}\Big]\Big], \\
&= \mathbb{J}^{-1}\Big[\frac{p}{q^3(q-1)\sqrt{q}}\Big]-\mathbb{J}^{-1}\Big[\frac{p}{q^3(q-1)\sqrt{q}}\Big], \\
&= 0.
\end{aligned}$$

Then by repeating the iterations give $y_k=0, k\geqslant 2$. Therefore the solution for this initial problem is $y(\xi)=\sum_{k=0}^{\infty}y_k(\xi)=e^{\xi}$.

Example 13 *Let solve the singular multi-fractional differential equation*

$$D^{\frac{3}{2}}y(\xi)+6\xi D^{\frac{1}{4}}y(\xi)+\frac{y(\xi)}{\xi}-\sin y(\xi) = 1+\frac{8}{\Gamma\left(\frac{3}{4}\right)}\xi^{\frac{7}{4}}-\sin\xi$$

with the initial conditions

$$y(0)=0,\ y'(0)=1.$$

The exact solution to this problem is $y(\xi)=\xi$.

Note that $a_1(\xi) = 6\xi, b(\xi) = \frac{1}{\xi}, c(\xi) = -1$, $N(y(\xi)) = \sin y(\xi), \chi_0 = 0, \chi_1 = 1$ and $g(\xi) = 1 + \frac{8}{\Gamma(\frac{3}{4})}\xi^{\frac{7}{4}} - \sin\xi$. The initial conditions and source term yield
$\mathcal{H}(\xi) = \xi + \frac{\xi^{\frac{3}{2}}}{\Gamma(\frac{5}{2})} + \frac{21}{2}\cdot\frac{\xi^{\frac{13}{4}}}{\Gamma(\frac{17}{4})} - \mathbb{J}^{-1}\Big[\frac{p(s)}{q^{\frac{3}{2}}(s)(q^2(s)+1)}\Big]$.
By letting $\mathcal{H}_0(\xi) = \xi$, we define the iteration

$$\begin{aligned}
y_0 &= \xi,\\
y_1 &= \frac{\xi^{\frac{3}{2}}}{\Gamma(\frac{5}{2})} + \frac{21}{2}\cdot\frac{\xi^{\frac{13}{4}}}{\Gamma(\frac{17}{4})} - \mathbb{J}^{-1}\Big[\frac{p}{q^{\frac{3}{2}}(q^2+1)}\Big]\\
&\quad - \mathbb{J}^{-1}\Big[\frac{1}{q^{\frac{3}{2}}}\mathbb{J}\Big\{6\xi D^{\frac{1}{4}}y_0\Big\}\Big]\\
&\quad - \mathbb{J}^{-1}\Big[-\frac{1}{q^{\frac{3}{2}}}\mathbb{J}\Big\{\frac{y_0}{\xi}\Big\} + \frac{1}{q^{\frac{3}{2}}}\mathbb{J}\Big\{A_0\Big\}\Big],\\
y_{i+1} &= -\mathbb{J}^{-1}\Big[\frac{1}{q^{\frac{3}{2}}}\mathbb{J}\Big\{6\xi D^{\frac{1}{4}}y_i\Big\} + \frac{1}{q^{\frac{3}{2}}}\mathbb{J}\Big\{\frac{y_i}{\xi}\Big\}\Big]\\
&\quad + \mathbb{J}^{-1}\Big[\frac{1}{q^{\frac{3}{2}}}\mathbb{J}\Big\{A_i\Big\}\Big], \quad i = 1,\ 2, \ldots
\end{aligned}$$

where the Adomian polynomials for nonlinear term $N(y) = \sin y$ are $A_0 = \sin y_0,\ A_1 = y_1\cos y_0, \ldots$ From the proposed relation, one finds that

$$\begin{aligned}
y_1 &= \frac{\xi^{\frac{3}{2}}}{\Gamma(\frac{5}{2})} + \frac{21}{2}\cdot\frac{\xi^{\frac{13}{4}}}{\Gamma(\frac{17}{4})} - \mathbb{J}^{-1}\Big[\frac{p}{q^{\frac{3}{2}}(q^2+1)}\Big]\\
&\quad - \mathbb{J}^{-1}\Big[\frac{1}{q^{\frac{3}{2}}}\mathbb{J}\Big\{6\xi D^{\frac{1}{4}}\xi\Big\} + \frac{1}{q^{\frac{3}{2}}}\mathbb{J}\Big\{\frac{\xi}{\xi}\Big\}\Big]\\
&\quad - \mathbb{J}^{-1}\Big[\frac{1}{q^{\frac{3}{2}}}\mathbb{J}\Big\{\sin\xi\Big\}\Big]\\
&= \frac{\xi^{\frac{3}{2}}}{\Gamma(\frac{5}{2})} + \frac{21}{2}\cdot\frac{\xi^{\frac{13}{4}}}{\Gamma(\frac{17}{4})} - \mathbb{J}^{-1}\Big[\frac{p}{q^{\frac{3}{2}}(q^2+1)}\Big]\\
&\quad - \mathbb{J}^{-1}\Big[\frac{1}{q^{\frac{3}{2}}\Gamma(\frac{7}{4})}\mathbb{J}\Big\{6\xi^{\frac{7}{4}}\Big\}\Big] - \frac{\xi^{\frac{3}{2}}}{\Gamma(\frac{5}{2})}\\
&\quad + \mathbb{J}^{-1}\Big[\frac{1}{q^{\frac{3}{2}}}\frac{p}{(q^2+1)}\Big]\\
&= \frac{21}{2}\cdot\frac{\xi^{\frac{13}{4}}}{\Gamma(\frac{17}{4})} - \mathbb{J}^{-1}\Big[\frac{1}{q^{\frac{3}{2}}\Gamma(\frac{7}{4})}\mathbb{J}\Big\{6\xi^{\frac{7}{4}}\Big\}\Big]\\
&= \frac{21}{2}\cdot\frac{\xi^{\frac{13}{4}}}{\Gamma(\frac{17}{4})} - \frac{21}{2}\cdot\frac{\xi^{\frac{13}{4}}}{\Gamma(\frac{17}{4})}\\
&= 0.
\end{aligned}$$

The later iterations are delivered the solutions $y_2 = 0, y_3 = 0, \ldots$. Therefore the solution for this problem is $y(\xi) = \sum_{n=0}^{\infty} y_n = \xi$.

Example 14 *Consider the nonlinear multi-fractional differential equation*

$$\begin{aligned}
&D^{\frac{3}{2}}y(\xi) + e^{\xi}Dy(\xi) + D^{\frac{1}{2}}y(\xi) + \xi^2 y(\xi) + e^{y(\xi)}\\
&\qquad = \frac{4\sqrt{\xi}}{\sqrt{\pi}} + \frac{8}{3\sqrt{\pi}}\xi^{\frac{3}{2}} + \xi^4 + 2\xi e^{\xi} + e^{\xi^2}
\end{aligned}$$

with the initial conditions

$$y(0) = 0,\ y'(0) = 0.$$

The exact solution to this problem is $y(\xi) = \xi^2$.

It is observed that $a_1(\xi) = e^{\xi}, a_2(\xi) = 1, b(\xi) = \xi^2$, $c(\xi) = 1, N(y) = e^y, \chi_0 = 0, \chi_1 = 0$ and the source term is $g(\xi) = \frac{4\sqrt{\xi}}{\sqrt{\pi}} + \frac{8}{3\sqrt{\pi}}\xi^{\frac{3}{2}} + \xi^4 + 2\xi e^{\xi} + e^{\xi^2}$. The transform of initial conditions and source term provided that
$\mathcal{H}(\xi) = \xi^2 + \frac{\xi^3}{3} + \mathbb{J}^{-1}\Big[\frac{4!p}{q^{\frac{13}{2}}} + \frac{2p}{q^{\frac{3}{2}}(q-1)^2} + \frac{1}{q^{\frac{3}{2}}}\mathbb{J}\{e^{\xi^2}\}\Big]$.
Choose $\mathcal{H}_0(\xi) = \xi^2$. The iterative relation for the problem is defined as

$$\begin{aligned}
y_0 &= \xi^2,\\
y_1 &= \frac{\xi^3}{3} + \mathbb{J}^{-1}\Big[\frac{4!p}{q^{\frac{13}{2}}} + \frac{2p}{q^{\frac{3}{2}}(q-1)^2} + \frac{1}{q^{\frac{3}{2}}}\mathbb{J}\{e^{\xi^2}\}\Big]\\
&\quad - \mathbb{J}^{-1}\Big[\frac{1}{q^{\frac{3}{2}}}\mathbb{J}\Big\{e^{\xi}Dy_0\Big\} + \frac{1}{q^{\frac{3}{2}}}\mathbb{J}\Big\{D^{\frac{1}{2}}y_0\Big\}\Big]\\
&\quad - \mathbb{J}^{-1}\Big[\frac{1}{q^{\frac{3}{2}}}\mathbb{J}\Big\{\xi^2 y_0\Big\} + \frac{1}{q^{\frac{3}{2}}}\mathbb{J}\Big\{A_0\Big\}\Big],\\
y_{i+1} &= -\mathbb{J}^{-1}\Big[\frac{1}{q^{\frac{3}{2}}}\mathbb{J}\Big\{e^{\xi}Dy_i\Big\} + \frac{1}{q^{\frac{3}{2}}}\mathbb{J}\Big\{D^{\frac{1}{2}}y_i\Big\}\Big]\\
&\quad - \mathbb{J}^{-1}\Big[\frac{1}{q^{\frac{3}{2}}}\mathbb{J}\Big\{\xi^2 y_i\Big\} + \frac{1}{q^{\frac{3}{2}}}\mathbb{J}\Big\{A_i\Big\}\Big], \quad i \geqslant 1.
\end{aligned}$$

One can find that

$$\begin{aligned}
y_1 &= \frac{\xi^3}{3} + \mathbb{J}^{-1}\Big[\frac{4!p}{q^{\frac{13}{2}}} + \frac{2p}{q^{\frac{3}{2}}(q-1)^2} + \frac{1}{q^{\frac{3}{2}}}\mathbb{J}\{e^{\xi^2}\}\Big]\\
&\quad - \mathbb{J}^{-1}\Big[\frac{1}{q^{\frac{3}{2}}}\mathbb{J}\Big\{e^{\xi}D\xi^2\Big\} + \frac{1}{q^{\frac{3}{2}}}\mathbb{J}\Big\{D^{\frac{1}{2}}\xi^2\Big\}\Big]\\
&\quad - \mathbb{J}^{-1}\Big[\frac{1}{q^{\frac{3}{2}}(s)}\mathbb{J}\Big\{\xi^4\Big\} + \frac{1}{q^{\frac{3}{2}}(s)}\mathbb{J}\Big\{e^{\xi^2}\Big\}\Big],\\
&= \frac{\xi^3}{3} + \mathbb{J}^{-1}\Big[\frac{4!p}{q^{\frac{13}{2}}} + \frac{2p}{q^{\frac{3}{2}}(q-1)^2}\Big] + \mathbb{J}^{-1}\Big[\frac{1}{q^{\frac{3}{2}}}\mathbb{J}\{e^{\xi^2}\}\Big]\\
&\quad - \mathbb{J}^{-1}\Big[\frac{1}{q^{\frac{3}{2}}}\mathbb{J}\Big\{2\xi e^{\xi}\Big\} + \frac{1}{q^{\frac{3}{2}}}\mathbb{J}\Big\{\frac{2\xi^{\frac{3}{2}}}{\Gamma(\frac{5}{2})}\Big\}\Big]\\
&\quad - \mathbb{J}^{-1}\Big[\frac{4!p}{q^{\frac{13}{2}}} + \frac{1}{q^{\frac{3}{2}}}\mathbb{J}\Big\{e^{\xi^2}\Big\}\Big],
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\xi^3}{3} + \mathbb{J}^{-1}\Big[\frac{2p}{q^{\frac{3}{2}}(q-1)^2}\Big] - \mathbb{J}^{-1}\Big[\frac{1}{q^{\frac{3}{2}}}\mathbb{J}\Big\{2\xi e^{\xi}\Big\}\Big] - \frac{\xi^3}{3}\\
&= \mathbb{J}^{-1}\Big\{\frac{2p}{q^{\frac{3}{2}}(q-1)^2}\Big\} - \mathbb{J}^{-1}\Big[\frac{1}{q^{\frac{3}{2}}}\mathbb{J}\Big\{2\xi e^{\xi}\Big\}\Big]\\
&= \mathbb{J}^{-1}\Big\{\frac{2p}{q^{\frac{3}{2}}(q-1)^2}\Big\} - \mathbb{J}^{-1}\Big[\frac{2}{q^{\frac{3}{2}}}\mathbb{J}\Big\{\xi\Big\}_{q\to q-1}\Big]\\
&= \mathbb{J}^{-1}\Big\{\frac{2p}{q^{\frac{3}{2}}(q-1)^2}\Big\} - \mathbb{J}^{-1}\Big[\frac{2p}{q^{\frac{3}{2}}(q-1)^2}\Big]\\
&= 0.
\end{aligned}
$$

Proceed the rest iteration, the results are $y_n = 0, n \geqslant 2$. Hence, the solution for this problem is obtained $y(\xi) = \sum_{n=0}^{\infty} y_n = \xi^2$.

4 Conclusion

In this challenge, we are able to handle nonlinear multi-fractional differential equations with variable coefficients by employing the general Adomain decomposition approach. In a few repetitions, the suggested method yielded an exact solution to the nonlinear problem with a variety of nonlinear terms. The key to success is to define iterative relations. Choosing an initial approximation is also important; a simple one with a satisfied initial condition is recommended. The Adomian polynomial may appear costly to compute, but it is not required in subsequent iterations if a suitable initial approximation is selected. Additionally, as seen in Example 13, the singular fractional differential equation is tested, and the result is established.

The general Adomian decomposition method is a beneficial approach for resolving multi-fractional differential equations with variable coefficients, as demonstrated by the discussion previously. This straightforward and fewer costly algorithm dominates the solution of several nonlinear problems and actively proposes further fractional calculus problems.

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Conflicts of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.