A New Perspective on Compactness via D-Sets in Bitopological Spaces

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Abstract:- This paper presents fundamental concepts and general examples of $\mathrm{D}-$sets, along with numerous facts and properties about them. It also discusses the concept of $\mathrm{D}-$compactness in topological spaces, as well as compactness in bitopological spaces, and several theories related to these concepts. Here, we examine the concept of $\mathrm{D}-$compactness in bitopological spaces, providing various definitions and examples to illustrate the main ideas, facts, and theories. Our discussion deals with special types of covering properties, namely the concept of compactness in bitopological spaces, including. $P^*-\mathrm{D}-$compactness and $S^*-\mathrm{D}-$ compactness. We examine the main properties of these concepts and how these properties relate to each other through established relationships with illustrative examples supporting these ideas.

Key-Words: - D-sets, $\tau_1^*\tau_2^*-$ $\mathrm{D}-$cover, P^*- $\mathrm{D}-$cover, $P^*-\mathrm{D}-$compact, $S^*-\mathrm{D}-$compact, P^*-continuous function, P^*-open function, P^*-closed function, P^*-homeomorphism.

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1 Introduction

[1] laid the foundation for the study of bitopological spaces, introducing the concept of a set equipped with two topologies, expressed as $(X^*, \tau_1^*, \tau_2^*)$ and provided fundamental definitions, such as pairwise regular and pairwise normal spaces. Kelly's work further included the generalization of several standard results from classical topology, including Urysohn's Lemma and the Tietze Extension Theorem, to bitopological spaces. From that time, the concept of bitopological spaces has attracted the attention of many topologists. Researchers have explored various aspects of bitopological spaces, focusing on problems such as defining compactness in this framework. Significant contributions to these studies can be found in the works of [2], [3], [4], [5], and [6]. The concept of compactness has attracted particular attention from researchers, who have made substantial efforts to study and generalize it to bitopological spaces. [3] introduced the concepts of $\tau_1^*\tau_2^*$-open sets and P^*-open covers in bitopological spaces, along with the notion of P^*-compact spaces. In the same year, [4] developed B^*-compact spaces. Later, [7] proposed the concept of S^* -compact spaces. The study of these compactness concepts was further advanced by [8], who analysed these definitions and explored the relationships between them. [9] introduced a new concept called the D-set, which is generated by open sets. He used this to present new separation axioms, denoted by D_0, D_1, and D_2 spaces. Later, [10] expanded this work by introducing S^*-D_i (i = 0, 1, 2) spaces by using semi-open sets. These advancements opened new directions of research in topology and its applications. [11] introduced the notions of D-sets, DS-sets, D-continuity and DS-continuity. [12] introduced the concepts of (i,j) semi-compact and (i,j) semi-closed sets in the context of bitopological spaces. During the same year, [13] presented the notion of b-open sets and used this to define b-continuous functions and pairwise b-continuous functions.

Later, [14] continued the study of pairwise almost Lindelöf subspaces and subsets. They investigated some of their properties and derived new results related to these types of spaces. [15] Introduced the concept of N-continuous functions in bitopological spaces, defining these functions via the notion of N-open sets. They also explored the properties of N-compact spaces, comparing these properties to those of S^*-compactness and pairwise compactness. [16] introduced the concept of D −sets and D −covers and used them to define D −compact spaces. They explored the properties of D −compact spaces and their relationships with other types of compactness in topological spaces. [17] worked on the definitions of metacompact spaces and D −metacompact spaces in bitopological spaces. Metacompact spaces are those in which every open cover has a point of accumulation, and D −metacompact spaces are a generalization with specific conditions.

[18] introduced several new concepts related to Lindelöf spaces in bitopological settings. These included pairwise strongly Lindelöf spaces, pairwise nearly Lindelöf spaces, pairwise almost Lindelöf spaces, and pairwise weakly strongly Lindelöf spaces. These concepts depend on the new idea of pairwise preopen countable covers, where the closures of these covers cover the bitopological spaces. [19] developed the concept of ij-pre-generalized closed sets in bitopological spaces. They also introduced the idea of pairwise operation pc-open sets and pairwise operation pc-separation axioms. Their work involved discussing the characteristics of these spaces and providing a detailed analysis of these new concepts. [20] introduced strongly star g-compactness, a novel topological covering property, and explored its structural relationships with analogous topological frameworks. They investigated and defined characteristics of strongly star -compact subsets and strongly star-compact subspaces, offering a comprehensive analysis of their behaviour and interactions within the broader realm of topology. Additionally, they presented finite intersection-like properties that provide conditions akin to strongly star g-compactness.

[21] introduced the concept of Delta-Compact spaces and examined their fundamental properties. They focused on the behaviour of Delta-Compact spaces under continuous mappings and investigated Locally Delta-Compactness. [22] contributed further to the study of operations on bitopological spaces. They introduced and examined the concept of operations on generalized semi-open sets in bitopological spaces. Additionally, two closure operators associated with this concept were defined, and their properties, along with the relationships between them, were discussed in detail. [23] developed new low separation axioms in bitopological spaces, introducing the concepts of (τ_1^*, τ_2^*)-Di spaces (for i = 0, 1, 2). They also provided characterizations of these spaces and explored related concepts like (τ_1^*, τ_2^*)-T_{14} spaces, (τ_1^*, τ_2^*)-T_{38} spaces, and weak (τ_1^*, τ_2^*)-Ro spaces. These new separation axioms offer additional ways to analyse the separation properties in bitopological spaces. In [24], the authors introduced several separation axioms using the D-set in a bitopological space and relationships linking D_1, D_2 classes of bitopological spaces.

Here, we introduce an overview notation of D-sets in topological space and some basic concepts of bitopological spaces, including compactness, P^* −compactness and S^* −compactness. Then, we introduce a new concept of D-sets in bitopological space and develop various compactness properties by using D −sets, such as D −compactness, $P^* - \mathrm{D}$ −compactness and $S^* - \mathrm{D}$ −compactness. The relationships between these concepts are investigated, supported by illustrative examples, proven results, and counterexamples that highlight the connections and distinctions between them. Several theories have been developed and established for these new concepts, and the impact of various types of functions on them has been examined.

In this paper, we use the following notations: ($\mathbb{R}, \mathbb{Q}, \mathbb{Z}, I_{rr}, \mathbb{N}, \mathbb{N}^e, \mathbb{N}^o$) to denote (real, rational, integer, irrational, natural, even, odd) numbers, respectively. Denote ($\tau_{l.r}^*$, τ_{rr}^*, τ_u^*, τ_{cof}^*, τ_{coc}^*, τ_{dis}^*, τ_s^*, τ_{ind}^*, τ_{sup}^*) be the (left-ray, right-ray, usual, co-finite, co-countable, discrete, Sorgenfrey line, indiscrete, supremum) topologies , respectively. And let (p^*, S^*, $p^* - D$ −compact, $S^* - D$ −compact) be (pair, semi, pair wise D −compact, semi D −compact) ,respectively.

2 Basic Concepts of D −set in Topological Space and Compactness in Bitopological Spaces

Throughout this section, we initiate with a detailed review of the concept of D −sets and their properties, including their behaviour under set-theoretic operations such as intersection, union, and difference in topological spaces. The provided examples illustrate the form of D −sets in various topological spaces and we also introduce the concept of bitopological spaces, including various compactness properties, such as compactness, P^* −compactness and S^* −compactness.

Definition 2.1. [9] Consider (X^*, τ^*) a topological space with $W \subseteq X^*$. If W can be expressed as W= γ_1 - γ_2, where both γ_1 and γ_2 are open in τ^* and $\gamma_1 \neq X^*$, then W is a D −set formed by γ_1 and γ_2.
Result: It follows directly that any open set $\gamma_1 \neq X^*$ forms a D-set when we set W= γ_1 and $\gamma_2 = \phi$.

Examples 2.1.

1. Let $X^* = \{\omega_1, \omega_2, \omega_3\}$ and $\tau^* = \{\phi, X^*, \{\omega_1\}, \{\omega_1, \omega_2\}\}$, then the D −sets in X^* are: ϕ, $\{\omega_1\}$, $\{\omega_1, \omega_2\}$ and $\{\omega_2\}$.
2. Consider (X^*, τ^*_{dis}), then any proper subset of X^* is a D −set, so all members of $P(X^*)$ are D −sets in X^*.
3. Consider $(\mathbb{R}, \tau^*_{rr})$, then the forms of D −sets are:
$\{\emptyset, (\mu, \infty)\}$ and $(\mu, \vartheta]$. Since
$(\mu, \infty) - (\vartheta, \infty) = \begin{cases} (\mu, \vartheta] & if \quad \mu < \vartheta \\ \emptyset & if \quad \mu \geq \vartheta \end{cases}$
4. Consider $(\mathbb{R}, \tau^*_{l.r})$, then the forms of D −sets are:
$\{\emptyset, (-\infty, \mu)\}$ and $[\mu, \beta)$. Since
$(-\infty, \mu) - (-\infty, \vartheta) =$
$\begin{cases} [\mu, \vartheta) & if \quad \mu > \vartheta \\ \emptyset & if \quad \mu \leq \vartheta \end{cases}.$
5. Consider $(\mathbb{R}, \tau^*_{cof})$, Then the forms of D −sets are: ϕ, {ℝ −finite set} and any finite set. For instance, $(\mathbb{R} - \{1, 2, 3\}) - (\mathbb{R} - \{1, 5, 6) = \{5, 6\}$ is a finite set, which is a D −set .

6. Consider $(\mathbb{R}, \tau^*_{coc})$, Then the forms of D −sets are: ϕ, { ℝ −countable set} and any countable set.
7. Consider$(\mathbb{R}, \tau^*_u)$, then we have various forms of D −sets, including:
τ^*_u- {ℝ}, $(-\infty, \mu]$, $[\mu, \infty)$, $(\mu, \vartheta]$, $[\mu, \vartheta]$, $[\mu, \vartheta)$ $\{[\mu, \vartheta) \cup (\rho, \sigma] : \vartheta < \rho\}$, $\{\mu\}$, $\{(-\infty, \mu] \cup [\vartheta, \rho) : \mu < \vartheta\}$, $\{(\mu, \vartheta] \cup [\rho, \infty) : \vartheta < \rho\}$,…etc.

Remark 2.1. All open sets are D −sets, but not every D-set is open, as examples will show.
Let $X^* = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ and $\tau^* = \{\phi, X^*, \{\omega_1, \omega_2, \omega_3\}, \{\omega_1, \omega_3\}\}$, then $\{\omega_1, \omega_2, \omega_3\} - \{\omega_1, \omega_3\} = \{\omega_2\}$ classifies as a D−set, but it fails to be an open set.
Let $(\mathbb{R}, \tau^*_{cof})$, then the collection $\{\{\mu\}: \mu \in \mathbb{R}\}$ consists of D −sets, but it is not a collection of open sets.
Let $(\mathbb{R}, \tau^*_{l.r})$, then the collection $\{[\mu, \vartheta): \mu, \vartheta \in \mathbb{R}\}$ consists of D −sets, but it is not a collection of open sets.

Definition 2.2. In the indiscrete topological space (X^*, τ^*_{ind}), X^* is considered a weakly D −set, since no other non-empty open sets exist in τ^*_{ind}.

Theorem 2.1. [16] Let D_1 and D_2 be two D −sets, then their intersection $D_1 \cap D_2$ is also a D −set.

Proof: Let $D_1 = \beta_1 - \beta_2$ and $D_2 = \gamma_1 - \gamma_2$ be any two D −sets, where β_1, β_2, γ_1 and γ_2 are open sets, then:
$$\begin{aligned} D_1 \cap D_2 &= (\beta_1 - A_2) \cap (\gamma_1 - \gamma_2) \\ &= (\beta_1 \cap \beta_2{}^c) \cap (\gamma_1 \cap \gamma_2{}^c) \\ &= (\beta_1 \cap \gamma_1) \cap (\beta_2{}^c \cap \gamma_2{}^c) \\ &= (\beta_1 \cap \gamma_1) \cap (\beta_2 \cup \gamma_2)^c \\ &= (\beta_1 \cap \gamma_1) - (\beta_2 \cup \gamma_2) \end{aligned}$$
which is a D −set, being the difference of two open sets.

Corollary 2.1. The finite intersection of D −sets in any topological space is a D −set. This can be proven by mathematical induction.

Remark 2.2.

1. A finite union of D-sets may fail to be a D-set.
   - For example, consider the topological space. $(\mathbb{R}, \tau^*_{r.r})$. The sets $(2, 5]$ and$(10, \infty)$ are D −sets, but their union $(2, 5] \cup (10, \infty)$ is not a D −set.
   - Another illustrative example is the topological space $X^* = \{\omega_1, \omega_2, \omega_3\}$ with $\tau^* = \{\phi, X^*, \{\omega_1\}, \{\omega_3\}, \{\omega_1, \omega_3\}, \{\omega_2, \omega_3\}\}$. The sets $D_1 = \{\omega_1\} = \{\omega_1, \omega_3\} - \{\omega_2, \omega_3\}$ and $D_2 = \{\omega_2\} = \{\omega_2, \omega_3\} - \{\omega_3\}$ are two D −sets. However, their union $D_1 \cup D_2 = \{\omega_1, \omega_2\}$ is not a D −set.
2. The difference of any two D −sets need not be a D −set.
   Consider the topological space $(\mathbb{R}, \tau^*_{l.r})$, where the D −sets have the forms : ϕ, $(-\infty, \varepsilon)$, $[\mu, \vartheta)$ where $\mu, \vartheta, \varepsilon \in \mathbb{R}$. The sets $[1, 8)$ and $[3, 5)$ are D −sets in $(\mathbb{R}, \tau^*_{l.r})$. However, $[1, 8) - [3, 5) = [1, 3) \cup [5, 8)$ is not a D −set

Definition 2.3. [16] A family $\widetilde{C} = \{D_\alpha : \alpha \in \Omega\}$ of a topological space (X^*, τ^*) is classified as a D −cover provided that D_α is a D −set for every $\alpha \in \Omega$.

Note 2.1 Open covers are D-covers because D-sets include open sets, but the converse fails. For instance, the family $\{\{\mu\}: \mu \in \mathbb{R}\}$ is a D-cover of $(\mathbb{R}, \tau^*_u)$ which fails to be an open cover.

Definition 2.4. [16] D-compactness of (X^*, τ^*) means every D-cover of X^* has a finite subcover.

Theorem 2.2. [16] If (X^*, τ^*) It is a D −compact space, it is compact.

Remark 2.2. The converse of the preceding theorem does not hold. For instance, $(\mathbb{R}, \tau^*_{cof})$ *It* is compact, but fails to be D −compact. This is because $\widetilde{C} = \{\{\mu\}: \mu \in \mathbb{R}\}$ is a D −cover with no finite subcover.

Definition 2.5. [16] (X^*, τ^*) is locally indiscrete if all open sets are clopen.

Remark 2.3. Every discrete space is locally indiscrete, however the converse is not always true.

Example 2.2. Let $X^* = \mathbb{R}$ and $\tau^* = \{\mathbb{R}, \emptyset, \mathbb{Q}, I_{rr}\}$, then (X^*, τ^*) It is a locally indiscrete space, but it is not a discrete space.

Definition 2.6. [1]Given a non-empty set X^* and two topologies τ_1^*, τ_2^* on X^*, the structure $(X^*, \tau_1^*, \tau_2^*)$ It is called a bitopological space.

Definition 2.7. A set ω is open in $(X^*, \tau_1^*, \tau_2^*)$ If ω belongs to τ_1^* or τ_2^*. We call its complement a closed set.

Definition 2.8. [3] Assume $(X^*, \tau_1^*, \tau_2^*)$ is a bitopological space then:

1. A cover $\widetilde{C}$ is classified as a $\tau_1^*\tau_2^*$ −open cover if it is a subset of τ_1^* or τ_2^*.
2. A cover $\widetilde{C}$ is classified as a P^* −open cover if $\widetilde{C}$ includes at least one non-empty set of each τ_1^* and τ_2^*.

Definition 2.9. A space $(X^*, \tau_1^*, \tau_2^*)$ is classified as compact if (X^*, τ_1^*) and (X^*, τ_2^*) are compact spaces.

Definition 2.10. [3] If all P^* −open covers of $(X^*, \tau_1^*, \tau_2^*)$ have finite subcovers, then it is P^* −compact

Definition 2.11. [7] If all $\tau_1^*\tau_2^*$ −open covers of $(X^*, \tau_1^*, \tau_2^*)$ have finite subcovers, then it is S^* −compact.

Theorem 2.3. Every S^* −compact space is a P^* −compact space.

Note 2.2. The converse of the state theorem may not hold.

Example 2.3 $(\mathbb{N}, \tau^*_{dis}, \tau^*_{ind})$ is P^* −compact. Since, any P^*−The open cover must contain $\{\mathbb{N}\}$ as a finite subcover. But it fails to be S^* −compact, because $\{\{\mu\} : \mu \in \mathbb{N}\}$ is $\tau^*_{dis}\tau^*_{ind}$−open cover without a finite subcover.

Theorem 2.4. [8] If $(X^*, \tau_1^*, \tau_2^*)$ is an S^* −compact space, then both (X^*, τ_1^*) and (X^*, τ_2^*) are compact.

Note 2.3. The converse of the preceding theorem does not hold.

Example 2.4. Let $X^* = [0, \infty)$ and let $\tau_1^* = \{\emptyset,\ X^*,\ \{0\}\}$ and
$\tau_2^* = \{\emptyset, X^*, (0, \infty)\} \cup \left\{\left(\frac{1}{m}, \infty\right): m \in \mathbb{N}\right\}$. Then (X^*, τ_1^*) and (X^*, τ_2^*) are compact spaces, but $(X^*, \tau_1^*, \tau_2^*)$ is not S^* −compact, since the S^* −open cover $\{\{0\}\} \cup \left\{\left(\frac{1}{m}, \infty\right): m \in \mathbb{N}\right\}$ of X^*has no finite subcover.

Theorem 2.5. [8] Given compact (X^*, τ_1^*) and (X^*, τ_2^*), then S^* −compactness and P^* −compactness of $(X^*, \tau_1^*, \tau_2^*)$ are equivalent.

3 D −Compactness in Bitopological Space

We study the concept of D-compactness in bitopological spaces, providing definitions and examples to clarify key ideas. Our discussion covers various types of compactness, including P*-D-compactness and S*-D-compactness, and investigates the relationships between these properties**.**

Definition 3.1. Assume $(X^*, \tau_1^*, \tau_2^*)$ is a bitopological space, then we define the following concepts:

1. τ_1^* − D −sets: Are all D −sets formed by open sets in (X^*, τ_1^*) and denoted by $D_{\tau_1^*}$ −sets.
2. τ_2^* − D −sets: Are all D −sets formed by open sets in (X^*, τ_2^*) and denoted by $D_{\tau_2^*}$ −sets.
3. $\tau_1^*\tau_2^*$ − D −cover: A cover $\widetilde{C} = \{D_\alpha : \alpha \in \Omega\}$ of the bitopological space $(X^*, \tau_1^*, \tau_2^*)$ which is a subset of $D_{\tau_1^*}$ or $D_{\tau_2^*}$.
4. P^* − D −cover: A cover $\widetilde{C} = \{D_\alpha : \alpha \in \Omega\}$ including at least one non-empty element from each of the $D_{\tau_1^*}$ −sets and $D_{\tau_2^*}$ −sets.

Example 3.1. Consider $(\mathbb{R}, \tau^*_{dis}, \tau^*_{r.r})$ then $\{\{\mu\}: \mu \in \mathbb{R}\}$ is one of the forms of $D_{\tau^*_{dis}}$ −sets and any of these forms $\{\emptyset, (a, \infty), (b, \beta]\}$ are $D_{\tau^*_{r.r}}$ −sets.

Remark 3.1. Each P^* − D −cover of $(X^*, \tau_1^*, \tau_2^*)$ is also a $\tau_1^*\tau_2^*$ − D −cover.

Note 3.1. The converse of the preceding remark does not hold.

Example 3.2. Given $(\mathbb{R}, \tau^*_{cof}, \tau^*_{r.r})$ Then the covers $\widetilde{C}_1 = \{(\mu, \infty): \mu \in \mathbb{R}\}$, $\widetilde{C}_2 = \{\{\mu\}: \mu \in \mathbb{R}\}$, $\widetilde{C}_3 = \{\mathbb{R} - finite\ set\}$ and $\widetilde{C}_4 = \{(\mu, \vartheta]: \mu, \vartheta \in \mathbb{R}\}$ are examples of $\tau_1^*\tau_2^*$ − D −covers. But they are not P^* − D −covers. In contrast $\widetilde{C}_5 = \{\{\{\mu\}: \mu < 1\} \cup (0, \infty)\}$ is a P^* − D −cover.

Example 3.3. $(\mathbb{R}, \tau^*_{coc}, \tau^*_{dis})$ is a bitopological space on $\mathbb{R}$,
$\tilde{C}_1 = \{\mathbb{R} - countable\ set\}$, $\widetilde{C}_2 = \{\{\mu\}: \mu \in \mathbb{R}\}$ form $\tau_1^*\tau_2^*$ − D −covers. Furthermore, $\widetilde{C}_2$ forms a P^* − D −cover.

Remark 3.2. Clearly, every $\tau_1^*\tau_2^*$ −open cover is a $\tau_1^*\tau_2^* -$ D −cover, while the converse fails generally. For instance, see this case in Example 3.2. the family $\{(b\,,\ \beta]: \alpha\,,\beta\ \in \mathbb{R}\}$ forms a $\tau_1^*\tau_2^* -$ D −cover that it does not represent a $\tau_1^*\tau_2^*$ −open cover.

Also, all P^* −open covers are $P^* - D$ −covers, while the converse fails generally. For instance, in the Example 3.3 the family $\{\{\mu\}: \mu\ \in \mathbb{R}\}$ forms a $P^* - D$ −cover that it does not represent a P^* −open cover.

Definition 3.2. A space $(X^*, \tau_1^*\,, \tau_2^*)$ is D −compact when both (X^*, τ_1^*) and (X^*, τ_2^*) are D-compact spaces.

Definition 3.3. A space $(X^*, \tau_1^*, \tau_2^*)$ is P^*− D −compact if it's P^*−D−covers of $(X^*, \tau_1^*, \tau_2^*)$ have finite subcovers.

Definition 3.4. A space $(X^*, \tau_1^*, \tau_2^*)$ is S^*− D −compact if it's $\tau_1^*\tau_2^* -$ D −covers of $(X^*, \tau_1^*, \tau_2^*)$ have finite subcovers.

Example 3.4. The space $(\mathbb{R}\,, \tau_{\text{cof}}^*, \tau_{\text{dis}}^*)$ fails to be P^*− D −compact and S^*− D −compact. Since the $P^* -$ D −cover $\{\{\mu\}: \mu\ \in\ \mathbb{R}\}$ has no a finite subcover.

Theorem 3.1. Every $S^* - D$ −compact space is a $P^* - D$ −compact space.

Proof: The result comes from Remark 3.1.

Note 3.2. The converse of the preceding theorem does not hold.

Example 3.5. Let $\left(\mathbb{R}\,, \tau_{cof}^*, \tau_{ind}^*\right)$, then it is $P^* - D$ −compact. But it is not $S^* - D$ −compact. Since the $\tau_{\text{cof}}^*\tau_{\text{ind}}^* - D$ −cover $\{\{\mu\}: \mu \in \mathbb{R}\,\}$ has no a finite subcover.

Theorem 3.2. The $S^* - D$ −compactness of $(X^*, \tau_1^*\,, \tau_2^*)$ ensure that both (X^*, τ_1^*) and (X^*, τ_2^*) are D −compact.

Proof: To verify the D-compactness of (X^*, τ_1^*), we need to prove that every $\tau_1^* - D -$ cover of X^* has a finite subcover. Let $\widetilde{C}$ be any $\tau_1^* -\ D$ −cover of X^*. It follows that $\widetilde{C}$ is a $\tau_1^*\,\tau_2^*\ D$ −cover of X^*. As $(X^*, \tau_1^*\,, \tau_2^*)$ is a $S^* - D$ −compact space implies there is a finite subcover of X^*, hence (X^*, τ_1^*) is D −compact. The same applies to (X^*, τ_2^*).

Corollary 3.1. $S^* - D$ −compactness of $(X^*, \tau_1^*\,, \tau_2^*)$ ensures (X^*, τ_1^*) and (X^*, τ_2^*) are compact.

Note 3.3. The converse of the stated theorem does not hold.

Example 3.6. Suppose $X^*= [0,1]$
And $\tau_1^* = \{\emptyset\,,\ X^*,\ \{0\}\}$ and $\tau_2^* = \{\ \emptyset\,, X^*, (0\,, 1]\} \cup \left\{\left(\frac{1}{m}, 1\right]: m \in \mathbb{N}\right\}$. Both (X^*, τ_1^*) and (X^*, τ_2^*) are D −compact spaces, but $(X^*, \tau_1^*\,, \tau_2^*)$ is not $S^* - D$ −compact as $\{\{0\}\} \cup \left\{\left(\frac{1}{m}, 1\right]: m \in \mathbb{N}\right\}$ is an S^* −D−cover with no finite subcover.

We add a certain condition to make the converse true.

Theorem 3.3. $(X^*, \tau_1^*\,, \tau_2^*)$ is $S^* -\ D$ −compact iff it is $P^* - D$ −compact, provided that (X^*, τ_1^*) and (X^*, τ_2^*) are D −compact .

Proof: Given that $\widetilde{C}$ is an $\tau_1^*\tau_2^* -$ D −cover of X^*, Then we have three cases:

1. $\widetilde{C}$ is a τ_1^* − D −cover and it has a finite subcover of X^*, because (X^*, τ_1^*) is D −compact.
2. $\widetilde{C}$ is a τ_2^* − D −cover and it has a finite subcover of X^*, because (X^*, τ_2^*) is D −compact.
3. $\widetilde{C}$ is a P^* − D −cover and it has a finite subcover of X^*, because $(X^*, \tau_1^*\,, \tau_2^*)$ is $P^* - D$ −compact.

Therefore, $(X^*, \tau_1^*\,, \tau_2^*)$ is an $S^* - D$ −compact.

Theorem 3.4. P^*− D −compactness of $(X^*, \tau_1^*, \tau_2^*)$ implies P^* −compact space.

Proof: Given that $\widetilde{C} = \{D_\alpha : \alpha \in \Omega\}$ is a P^*−open the cover of $(X^*, \tau_1^*, \tau_2^*)$, It follows that $\widetilde{C}$ is a P^*− D −cover and it has a finite subcover being X^* is a P^*− D −compactness. Making $(X^*, \tau_1^*, \tau_2^*)$ is P^*−compact.

Example 3.7. Consider $(\mathbb{R}\,, \tau_u^*, \tau_{ind}^*)$, then it is $P^* -$ D −compact. So it is P^* −compact. But it is not $S^* -$ D −compact. Since the family $\{(-\mu\,, \mu): \mu \in \mathbb{N}\}$ is a $\tau_{\text{u}}^*\tau_{\text{ind}}^* -$ D −cover with no finite subcover.

The converse of the stated theorem may not hold, as illustrated by the following example.

Example 3.8. Let $\left(\mathbb{R}\,, \tau_{r.r}^*\,, \tau_{cof}^*\right)$ be a bitopological space, then it is P^* −compact that fails to be P^*− D −compact. As the P^*− D −cover $\{(-\beta\,, 0]: \beta \in \mathbb{N}\} \cup \{\{\mu\}: \mu > 0\}$ with no a finite subcover.

Example 3.9. Consider $\left(\mathbb{R}\,, \tau_u^*, \tau_{cof}^*\right)$ be a bitopological space, then it is P^* −compact that fails to be P^*− D −compact. As the P^*− D −cover $\{\{\mu\}: \mu < 1\} \cup \{(0\,, \infty)\,\}$ has no a finite subcover.

Definition 3.5. $(X^*, \tau_1^*, \tau_2^*)$ is locally indiscrete provided that all τ_{i}^*−open sets are also τ_{i}^*−closed (i = 1 , 2).

Every τ_{i}^*− D −set is clopen in a locally indiscrete space $(X^*, \tau_1^*, \tau_2^*)$.

An additional condition makes the converse of the stated theorem true, as shown in the next theorem.

Theorem 3.5. For a space $(X^*, \tau_1^*, \tau_2^*)$, if it is locally indiscrete and P^*−compact, then it is P^*− D −compact.

Proof: Given that $(X^*, \tau_1^*, \tau_2^*)$ is a locally indiscrete P^*−compact space and $\widetilde{C}$ be a P^*− D −cover of

$(X^*, \tau_1^*, \tau_2^*)$. Then $\widetilde{C} = \{V_\alpha : \alpha \in \Omega\} \cup \{W_\beta : \beta \in \Delta\}$, where $V_\alpha \in D_{\tau_1^*}$ for each $\alpha \in \Omega$ and $W_\beta \in D_{\tau_2^*}$ for each $\beta \in \Delta$. Since $(X^*, \tau_1^*, \tau_2^*)$ is locally indiscrete, V_α is clopen set for each $\alpha \in \Delta$ and W_β Is clopen set for each $\beta \in \Delta$. Therefore, $\widetilde{C} = \{V_\alpha : \alpha \in \Omega\} \cup \{W_\beta : \beta \in \Delta\}$ is a P^*−open cover. By compactness of X*, $\widetilde{C}$ has a finite subcover. Hence the result.

Example 3.10. Let $(\mathbb{R}, \tau_{\text{ind}}^*, \tau_{\text{dis}}^*)$ be a locally indiscrete bitopological space, then it is P^*−D−compact, because it is P^*−compact.

Definition 3.6. In $(X^*, \tau_1^*, \tau_2^*)$, τ_{sup}^* is the smallest topology that includes the union of τ_1^* and τ_2^*.

Remake 3.3. In $(X^*, \tau_1^*, \tau_2^*)$, all open sets of (X^*, τ_1^*) and (X^*, τ_2^*) are open in $(X^*, \tau_{\text{sup}}^*)$.

Theorem 3.6. The D −compactness of $(X^*, \tau_{\text{sup}}^*)$ implies (X^*, τ_1^*) and (X^*, τ_2^*) are D −compact spaces.

Proof: To verify that (X^*, τ_1^*) is D −compact space, we need to prove every $\tau_1^* -$ D −cover of X^*, has a finite subcover. Let $\widetilde{C}$ be any $\tau_1^* -$ D −cover of X^*, implies $\widetilde{C}$ is an D −cover of X^*, since (X^*, τ_{sup}^*) is a D −compact space, there is a finite subcover of X^*, so (X^*, τ_1^*) is D −compact. And in the same way, we can prove (X^*, τ_2^*) is D −compact.

Note 3.4. The converse of the state theorem does not hold.

Example 3.11. Consider $(\mathbb{N}, \tau_1^*, \tau_2^*)$, $\tau_1^* = \{\mathbb{N}\} \cup p(\mathbb{N}^{od})$ and $\tau_2^* = \{\mathbb{N}\} \cup p(\mathbb{N}^{ev})$. Then $\tau_1^* \cup \tau_2^* = (\mathbb{N}, \tau_{sup}^*) = (\mathbb{N}, \tau_{dis}^*)$, where $p(\mathbb{N}^{od})$ and $p(\mathbb{N}^{ev})$ are the power sets of $\mathbb{N}^{od}$ and $\mathbb{N}^{ev}$ respectively. We note that both. $(\mathbb{N}, \tau_1^*)$ and $(\mathbb{N}, \tau_2^*)$ are D −compact spaces, but $(\mathbb{N}, \tau_{sup}^*)$ fails to be D −compact. Because the τ_{sup}^* open cover $\{\{\mu\} : \mu \in \mathbb{N}\}$ of $\mathbb{N}$ has no finite subcover.

Theorem 3.7. If (X^*, τ_{sup}^*) is a D −compact space, then $(X^*, \tau_1^*, \tau_2^*)$ is $S^* - D$ −compact.

Proof: To confirm that $(X^*, \tau_1^*, \tau_2^*)$ is an $S^* - D$ −compact space, need to prove every $\tau_1^*\tau_2^* - D$ −cover of X^*, has a finite subcover. Let $\widetilde{C}$ be any $\tau_1^*\tau_2^* - D$ −cover of X^*. Then $\widetilde{C}$ is a $\tau_{sup}^* - D$ −cover of X^*, as (X^*, τ_{sup}^*) is a D −compact space, there is a finite subcover of X^*, making $(X^*, \tau_1^*, \tau_2^*)$ $S^* - D$ −compact.

Theorem 3.8. If a space is D-compact, then its continuous image is also D-compact.

Proof: Assume f: $(X^*, \tau_1^*, \tau_2^*) \to (Y^*, T_1^*, T_2^*)$ is a continuous onto function and X*It is a D−compact. let $\widetilde{C} = \{V_\alpha : \alpha \in \Omega\}$ and $\widetilde{\beta} = \{U_\gamma : \gamma \in \Delta\}$ be two D −covers of (Y^*, T_1^*) and (Y^*, T_2^*) respectively. Then for each $\alpha \epsilon \Omega$, $f^{-1}(V_\alpha)$ and $\gamma \in \Delta$, $f^{-1}(U_\alpha)$ are D−sets in X*, because f is continuous. Now, $\widetilde{V} = \{\{f^{-1}(V_\alpha) : \alpha \in \Omega\}\}$ and $\widetilde{U} = \{f^{-1}(U_\alpha) : \alpha \in \Omega\}$ are D−covers of (X^*, τ_1^*) and (X^*, τ_2^*) respectively. Since (X^*, τ_1^*) and (X^*, τ_2^*) are D−compact, $\widetilde{V}$ and $\widetilde{U}$ have a finite subcover,

$\widetilde{V}^* = \{f^{-1}(V_{\alpha_1}), f^{-1}(V_{\alpha_2}), \ldots, f^{-1}(V_{\alpha_n})\}$ and

$\widetilde{U}^* = f^{-1}(U_{\gamma_1}), f^{-1}(U_{\gamma_2}), \ldots, f^{-1}(U_{\gamma_n})$.Therefore, $\widetilde{C}^* = \{V_{\alpha_1}, V_{\alpha_2}, \ldots, V_{\alpha_n}\}$ and $\widetilde{\beta}^* = \{U_{\gamma_1}, U_{\gamma_2}, \ldots, U_{\gamma_n}\}$ are finite subcovers of $\widetilde{C}$ and $\widetilde{\beta}$ for(Y^*, T_1^*) and (X^*, τ_2^*) respectively. Hence, (Y^*, T_1^*, T_2^*) is D−compact.

Definition 3.7. Assume $(X^*, \tau_1^*, \tau_2^*)$ and (Y^*, T_1^*, T_2^*) are two bitopological spaces, a function f: $(X^*, \tau_1^*, \tau_2^*) \to (Y^*, T_1^*, T_2^*)$ is P^* −continuous function (P^*−open, P^*−closed, P^*−homeomorphism, respectively) provided that f: $(X^*, \tau_1^*) \to (Y^*, T_1^*)$ and f: $(X^*, \tau_2^*) \to (Y^*, T_2^*)$ are D −continuous (open, closed, homeomorphism, respectively).

Theorem 3.9. Suppose f: $(X^*, \tau_1^*, \tau_2^*) \to (Y^*, T_1^*, T_2^*)$ is a P^*−continuous, P^*−closed, onto function and Y^* It is a locally indiscrete space. If X^* is P^*−D−compact, then Y^* is P^*−D−compact.

Proof: Consider an arbitrary $\widetilde{C} = \{V_\alpha : \alpha \in \Omega\} \cup \{U_\gamma : \gamma \in \Delta\}$ P^*−D−cover of Y^*, comprising $\{V_\alpha : \alpha \in \Omega\}$ are $T_1^* - D$ −members of $\widetilde{C}$ and $\{U_\gamma : \gamma \in \Delta\}$ are $T_2^* - D$ −members of $\widetilde{C}$. Since f is P^*−continuous, onto function, the set $\widetilde{\beta} = \{\{f^{-1}(V_\alpha) : \alpha \in \Omega\} \cup \{f^{-1}(U_\gamma) : \gamma \in \Delta\}\}$ is a P^*−open the cover of X^*. As X^* is P^*−D−compact space, there exists a finite subset $\Delta^* \subseteq \Delta$, $\Omega^* \subseteq \Omega$, such that $\widetilde{\beta}^* = \{\{f^{-1}(V^*{}_\alpha) : \alpha \in \Omega^*\} \cup \{f^{-1}(U^*{}_\gamma) : \gamma \in \Delta^*\}\}$ is a finite subcover of $\widetilde{\beta}$ .Therefore, $\widetilde{C}^* = \{V^*{}_\alpha : \alpha \in \Omega^*\} \cup \{U^*{}_\gamma : \gamma \in \Delta^*\}$ is a finite subcover of Y^*. Hence, Y* is P^* −D−compact.

Corollary 3.2. Given a P^*−continuous, P^*−closed, onto function f: $(X^*, \tau_1^*, \tau_2^*) \to (Y^*, T_1^*, T_2^*)$, if X^* is P^*−compact, then Y^* is P^*−D−compact.

Corollary 3.3. Suppose f: $(X^*, \tau_1^*, \tau_2^*) \to (Y^*, T_1^*, T_2^*)$ is a P^*−continuous, P^*−closed, onto function. Then P^*−compactness of X^* implies P^*−compactness of Y^*.

Theorem 3.10. The union of two $S^* -$ D −compact subsets G and H in $(X^*, \tau_1^*, \tau_2^*)$ is $S^* -$ D −compact.

Proof: To show that $H \cup G$ is $S^* -$ D −compact subset of X^*, We need to prove every $\tau_1^*\tau_2^* -$ D −cover of $H \cup G$ has a finite subcover. Let $\{V_\alpha : \alpha \in \Omega\}$ be any $\tau_1^*\tau_2^* -$ D −cover of $H \cup G$ then $H \cup G \subseteq \{\bigcup V_\alpha : \alpha \in \Omega\}$ and therefore, $H \subseteq$

$\cup V_\alpha$ and $G \subseteq \cup V_\alpha$, then $\{V_\alpha : \alpha \in \Omega\}$ is an $\tau_1^* \tau_2^* -$ D $-$cover of H and G.
But H and G are $S^* -$ D $-$compact subsets, therefore, there exists $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n \in \Omega$ and $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_m \in \Omega$ such that $\{\rho_{\alpha_1}, \rho_{\alpha_2}, \rho_{\alpha_3}, \dots, \rho_{\alpha_n}\}$and $\{\rho_{\alpha_1}, \rho_{\alpha_2}, \rho_{\alpha_3}, \dots, \rho_{\alpha_m}\}$ is a finite subcover of H and G, respectively, then $\{\rho_{\alpha_1}, \rho_{\alpha_2}, \rho_{\alpha_3}, \dots, \rho_{\alpha_n}\} \cup \{\rho_{\alpha_1}, \rho_{\alpha_2}, \rho_{\alpha_3}, \dots, \rho_{\alpha_m}\}$is a finite subcover of $H \cup G$.Thus, $H \cup G$ is a $S^* -$ D $-$compact subset of X^*.

Remark 3.4. $S^* -$ D $-$compactness is not preserved under intersection in a bitopological space.
For example: Let $X^* = \mathbb{N} \cup \{0, -1\}$ and let
$\tau_1^* = p(\mathbb{N}) \cup \{A \subseteq X^* : -1, 0 \in A \text{ where } (X^* - A) \text{ is finite}\}$ and
$\tau_2^* = \tau_1^* \cup \{A \subseteq X^* : -1 \in A \text{ or } 0 \in A \text{ where } (X^* - A) \text{ is finite}\}$.
Now, let $G = \mathbb{N} \cup \{0\}$ and $H = \mathbb{N} \cup \{-1\}$, then both G and H are $S^* -$ D $-$compact subsets of $(X^*, \tau_1^*, \tau_2^*)$, but $G \cap H = \mathbb{N}$ is not $S^* -$ D $-$compact set.

4 Conclusion

This study builds upon previous research to explore bitopological spaces by introducing D−sets, a concept previously defined in bitopological spaces. By extending D−sets to bitopological spaces and exploring their implications. The introduction of D−sets leads to new compactness concepts: D−compact, P^*−D−compact, and S^*−D−compact spaces. We have studied the properties of these spaces and in addition, we defined D−continuous functions and obtained new results and theories that support researchers in their studies related to the field of D−bitopological spaces. We also linked these results and theories to previously known results. This research opens up new paths for exploring the properties of bitopological spaces.

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