# Cascade (1+1) and (2+1) Dagum Reliabilities for a Strength between Two Level of Stresses

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*Abstract:* - The current research examines the reliability of the P(T<X<Z) system for two stress -strength models which have Dagum distribution, a (1+1) cascade model, an active component and a backup component, and a (2+1) cascade model, consisting of three elements, two active components and a backup component. The mathematical formula for the reliability was reached of the cascade system (1+1) and (2+1), is expressed by a component of the strength X that be between of two stresses T and Z. It is shown that the reliability of the system increases as the reliability of the components connected increased, and sometimes reliability reached to certain limits. Through the use of the probability P(T<X<Z), stress and strength variables follow the Dagum distribution with unknown shape parameters and known scale parameters. In the current study, the maximum likelihood estimation (MLE) and the Method of Moments were used to estimate the unknown shape parameters of the system reliabilities $R_{11}$ and $R_{21}$ for the stress-strength model. The maximum likelihood and moment methods were evaluated through a comparative study of experimental simulations using the mean square error criterion. Besides, this investigation showed through simulation results that the maximum likelihood estimation is the best in the studied experiments at different sample sizes n except some very simple cases. The reliability of the systems that were derived and formulated in the research was compared.

*Key-Words:* - Dagum Distribution, Reliability, Estimation, Probability P(T<X<Z), Stress-Strength, Cascade.

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## 1 Introduction

There are many practical applications in the field of mechanical engineering, including factories and devices used in machinery and equipment that require the replacement of damaged units and the use of high-level backup systems to compensate when failures occur. These modern systems use additional technologies to further improve devices performance, making them more complex, [1]. In many studies, the reliability of a life model of a component with a strength X subjected to a certain stress Y has been studied. It is shown that the reliability of the system increases as the reliability of the components connected to each other increases, and sometimes reliability achieved certain limits. The reliability of the P(T<X<Z) system was studied for two stress-strength models having Dagum distribution, a (1+1) cascade model, an active component and a backup component, and a (2+1) cascade model, consisting of three components, two active components and a backup component, many studies and researches have been conducted on the cascade model and it has been studied in the literature such as [2], studied of the reliability of the system P(X<Y<Z) for the stress-strength cascade model in the cases (1+1) and (2+1) when the random variables stress-strength are based on the Exponential distribution and Weibull distribution. Furthermore, [3], derived mathematical formulas for the reliability of the cascade model (2+1), and reached formulas for the reliability of the model when generalizing the force and stress variables to the inverse random Rayleigh distribution. [4], used the (2+1) cascade model to estimate of the reliability of a system following the inverse Weibull distribution. Additionally, [5], derived of the mathematical formula for the (3+1) cascade model and obtain

the reliability of the stress strength as random variables following an Exponential distribution. [6], used statistical inference for a (1+1) cascade system, and the extent to which the joint attenuation factors between of stress-strength for the Exponential distribution. The Dagum distribution was introduced first by [7], and it was vastly used in different of life domains in income and wealth data, climate data, survival and reliability analysis. Moreover, the Dagum distribution is known as the inverse Bohr XII, and it is one of the important features of the distribution. Its special hazard function such that the function is monotonically decreasing, or an inverted bathtub. Depending on the probability of the strength component yielding between two random stresses $R = P(T < X < Z)$ within a given time interval [0, t] can be interpreted as the stress T. Z stress is independent random variables and the strength X being a random variable independent of T and Z according to [8]. The expression for a cumulative density function (cdf) of the Dagum distribution with an independent random strength variable X is [9]:

$$F(x) = (1 + \lambda x^{-\gamma})^{-\beta} \quad \lambda, \gamma, \beta > 0; x > 0 \tag{1}$$

According to the probability density function (pdf) for the Dagum distribution by:

$$f(x) = \beta\lambda\gamma x^{-\gamma-1}\left(1 + \frac{\lambda}{x^{\gamma}}\right)^{-\beta-1} \quad \lambda, \gamma, \beta > 0; x > 0 \tag{2}$$

We can rewrite equation (2) as:

Then $$\int_0^{\infty} \lambda\gamma x^{-\gamma-1}\left(1 + \frac{\lambda}{x^{\gamma}}\right)^{-\beta-1} = \frac{1}{\beta} \tag{3}$$

where the shape parameter is $\beta$ and the common known scale parameters are $\lambda, \gamma$.

This research aims to extract a mathematical form of the reliability for successive stress and strength models (1+1) and (2+1) according to the Dagum distribution and to study the results of the estimators between two methods (maximum likelihood and moment). The different sample sizes n = 15, 30, 90 and different parameter values as shown in Table 1 (Appendix), a comparison was made between the estimators using the MSE that will be obtained from the simulation.

# 2 Reliability Formulation

To formula reliability system of the (1+1) cascade model with two components (one activated and other reserve) be working under the impact of strength $x_i \sim DaD\,(\beta_i, \lambda, \gamma), i = 1,2;$ where are independently and identically distributed. Let $T \sim DaD\,(\alpha, \lambda, \gamma)$ and $Z \sim DaD\,(\theta, \lambda, \gamma)$ where $\alpha, \theta > 0$; are an unknown shape parameters of the lower and upper stress, respectively and the expression of the cumulative density function (cdf), as:

$G_T(t) = \left(1 + \frac{\lambda}{t^{\gamma}}\right)^{-\alpha}$ and $H_Z(z) = \left(1 + \frac{\lambda}{z^{\gamma}}\right)^{-\theta}$

$$R_{11} = R_1 + R_2 \tag{4}$$

$$R_1 = P(T < x_1 < Z)$$

$$R_1 = \int_{x_1} G_T(x_1)(1 - H_z(x_1)) f(x_1) dx_1 = \int_{x_1} G_T(x_1) H_z(x_1) f(x_1) dx_1$$

$$= \int_0^{\infty}\left(1 + \frac{\lambda}{x_1^{\gamma}}\right)^{-\alpha}\left(1 - \left(1 + \frac{\lambda}{x_1^{\gamma}}\right)^{-\theta}\right)\beta_1\lambda\gamma x_1^{-\gamma-1}\left(1 + \frac{\lambda}{x_1^{\gamma}}\right)^{-\beta_1-1} dx_1$$

$$= \beta_1\lambda\gamma\int_0^{\infty} x_1^{-\gamma-1}\left(1 + \frac{\lambda}{x_1^{\gamma}}\right)^{-\alpha-\beta_1-1} dx_1 - \beta_1\lambda\gamma\int_0^{\infty} x_1^{-\gamma-1}\left(1 + \frac{\lambda}{x_1^{\gamma}}\right)^{-\alpha-\theta-\beta_1-1} dx_1$$

Then using equation (3) and substitution it becomes as follows:

$$R_1 = \beta_1\frac{1}{\alpha+\beta_1} - \beta_1\frac{1}{\alpha+\theta+\beta_1} \tag{5}$$

Therefore, after each failure in the cascade system, the strength is updated by a factor k [11], so that $X_i = k^{i-1}x_1$; for the modified a factor k=1 then $x_2 = kx_1\ and\ x_3 = kx_2 = k(kx_1) = k^2x_1$. By this, once distribution $x_1$ is selected, distribution $x_2$ is automatically selected.

By this, once the distribution $x_1$ is determined, the distribution of $x_2, x_3, \ldots, x_n$ are automatically determined, as follows:

$$R_2 = (1 - R_1)P(T < kx_1 < Z) \Rightarrow R_2 = (1 - R_1)\left(\frac{\beta_2}{\alpha+\beta_2} - \frac{\beta_2}{\alpha+\theta+\beta_2}\right) \tag{6}$$

where $x_2 = kx_1$ since $k = 1$ then $x_2 = x_1$

Now, it is given that the reliability expression for the (2 + 1) cascade model with three components (two active and one reserve) such that the strength $x_i$ ; $i = 1,2,3$ are independently and identically distributed, so the reliability formula can be obtained by [10].

$$R_{21} = R_1 + R_2 + R_3 \quad (7)$$

The reliability estimate $R_1$ will be given by the probability of the first component and the second component working, as follows:

$$R_1 = P(T < x_1 < Z)P(T < x_2 < Z) = c_1c_2$$

$$R_1 = \left(\frac{\beta_1}{\alpha+\beta_1} - \frac{\beta_1}{\alpha+\theta+\beta_1}\right)\left(\frac{\beta_2}{\alpha+\beta_2} - \frac{\beta_2}{\alpha+\theta+\beta_2}\right) \quad (8)$$

As for the estimate of $R_2$, it is the probability the first component will fail and be replaced by the third component while the second component continues to operate, since $x_3 = kx_1$ and $k = 1$; as follows:

$$R_2 = \left(P(T < x_1 < Z)\right)^c P(T < x_3 < Z)P(T < x_2 < Z)$$

$$R_2 = (1 - c_1)P(T < x_1 < Z)c_2 = (1 - c_1)c_1c_2 = (1 - c_1)R_1 \quad (9)$$

Finally, we estimate the reliability $R_3$ when the second component fails there is a possibility of replacing with the third component while the first component continues to work, where $x_3 = kx_2$ and $k = 1$ ;as follows:

$$R_3 = P(T < x_1 < Z)\left(P(T < x_2 < Z)\right)^c P(T < x_3 < Z)$$

$$R_3 = c_1(1 - c_2)P(T < x_2 < Z) = c_1(1 - c_2)c_2 = (1 - c_2)R_1 \quad (10)$$

# 3 Estimation of Reliability for Cascade (1+1) and (2+1) Dagum

In this section, it used maximum likelihood estimation (MLE) and the Method of Moment to estimate the unknown shape parameters of the system reliabilities $R_{11}$and $R_{21}$ for stress-strength model.

## 3.1 The Maximum Likelihood Estimation (MLE)

Let the random sample $x_1, x_2, \dots, x_n$ is representing the strength from size n of the pdf in eq.(2) the $Dag(\beta, \lambda, \gamma)$ distribution, a parameter $\beta$ is an unknown and the parameters $\lambda, \gamma$ are known, and the general function L is as shown, [12]:

$$L(x_1, x_2, \dots, x_n; \beta, \lambda, \gamma) = (\beta\lambda\gamma)^n \prod_{i=1}^{n} x_i^{-\gamma-1}\left(1 + \frac{\lambda}{x_i^{\gamma}}\right)^{-\beta-1} \quad (11)$$

Use the natural logarithm function for equation (11) as;

$$lnL = nln(\beta\lambda\gamma) - (\gamma + 1)\sum_{i=1}^{n} ln(x_i) - (\beta + 1)\sum_{i=1}^{n} ln\left(1 + \lambda x_i^{-\gamma}\right)$$

$$= nln\beta + nln\lambda + nln\gamma - \gamma\sum_{i=1}^{n} ln(x_i) - \sum_{i=1}^{n} \ln(x_i) - \beta\sum_{i=1}^{n} ln\left(1 + \lambda x_i^{-\gamma}\right) - \sum_{i=1}^{n} ln\left(1 + \lambda x_i^{-\gamma}\right) \quad (12)$$

By taking the partial derivative with respect to an unknown parameter $\beta$ in equation (12) and then setting the result equal to zero, we get;

$$\frac{\partial ln}{\partial \beta} = \frac{n}{\beta} - \sum_{i=1}^{n} ln\left(1 + \lambda x_i^{-\gamma}\right)$$

So, $\frac{n}{\beta} = \sum_{i=1}^{n} ln\left(1 + \lambda x_i^{-\gamma}\right)$

Then $\beta = \frac{n}{\sum_{i=1}^{n} ln\left(1+\lambda x_i^{-\gamma}\right)}$

The ML estimator for the first system (1+1) cascade which $x_{\omega_i}$; $i = 1,2, \dots, n$ be strength random samples from $\beta_{\omega_i}$ with the sample size $n$ ; $\omega = 1,2$ the ML estimator for $\beta_{\omega_i}$, as follows:

$$\hat{\beta}_{\omega i_{MLE}} = \frac{n}{\sum_{\omega i=1}^{n} ln\left(1+\lambda x_{\omega i}^{-\gamma}\right)} \quad (13)$$

let $t_1, t_2, \dots, t_m$ and $z_1, z_2, \dots, z_m$ be random samples of stress components from size m for $T \sim Dag(\alpha, \lambda, \gamma)$ and $Z \sim Dag(\theta, \lambda, \gamma)$, respectively. As for ML estimators with unknown parameters $\alpha, \theta$. They will be;

$$\hat{\alpha}_{MLE} = \frac{m}{\sum_{j=1}^{m} ln\left(1+\lambda t_j^{-\gamma}\right)} \quad (14)$$

and

$$\hat{\theta}_{MLE} = \frac{m}{\sum_{j=1}^{m} ln\left(1+\lambda z_j^{-\gamma}\right)} \quad (15)$$

by substitution (13) , (14) and (15) in the equations (5) and (6), can be the ML estimator for $R_{11}$ obtained as;

$$\hat{R}_{1MLE} = \frac{\hat{\beta}_{1MLE}}{\hat{\alpha}_{MLE}+\hat{\beta}_{1MLE}} - \frac{\hat{\beta}_{1MLE}}{\hat{\alpha}_{MLE}+\hat{\theta}_{MLE}+\hat{\beta}_{1MLE}}$$

and,

 

$$\hat{R}_{2MLE} = \left(1 - \hat{R}_{1MLE}\right)\left(\frac{\hat{\beta}_{2MLE}}{\hat{\alpha}_{MLE}+\hat{\beta}_{2MLE}} - \frac{\hat{\beta}_{2MLE}}{\hat{\alpha}_{MLE}+\hat{\theta}_{MLE}+\hat{\beta}_{2MLE}}\right)$$

We can get the reliability expression for cascade (1 + 1) system obtained as;

$$\hat{R}_{11_{MLE}} = \hat{R}_{1_{MLE}} + \hat{R}_{2_{MLE}}$$

In the second system (2+1) cascade can be obtained the ML estimator for $R_{21}$ by substitution (13) , (14) and (15) in the equations (8), (9) and (10), let $x_{\omega_i};\ i = 1,2,\dots,n$ be strength random samples taken from $Dag(\beta,\lambda,\gamma)$ distribution of the sample size $n\,;\omega = 1,2,3$ obtained as;

$$\hat{R}_{1MLE} = \left(\frac{\hat{\beta}_{1MLE}}{\hat{\alpha}_{MLE}+\hat{\beta}_{1MLE}} - \frac{\hat{\beta}_{1MLE}}{\hat{\alpha}_{MLE}+\hat{\theta}_{MLE}+\hat{\beta}_{1MLE}}\right)\left(\frac{\hat{\beta}_{2MLE}}{\hat{\alpha}_{MLE}+\hat{\beta}_{2MLE}} - \frac{\hat{\beta}_{2MLE}}{\hat{\alpha}_{MLE}+\hat{\theta}_{MLE}+\hat{\beta}_{2MLE}}\right)$$

$$\hat{R}_{2MLE} = (1-c_1)\left(\frac{\hat{\beta}_{1MLE}}{\hat{\alpha}_{MLE}+\hat{\beta}_{1MLE}} - \frac{\hat{\beta}_{1MLE}}{\hat{\alpha}_{MLE}+\hat{\theta}_{MLE}+\hat{\beta}_{1MLE}}\right)\left(\frac{\hat{\beta}_{2MLE}}{\hat{\alpha}_{MLE}+\hat{\beta}_{2MLE}} - \frac{\hat{\beta}_{2MLE}}{\hat{\alpha}_{MLE}+\hat{\theta}_{MLE}+\hat{\beta}_{2MLE}}\right)$$

$$\hat{R}_{3MLE} = (1-c_2)\left(\frac{\hat{\beta}_{1MLE}}{\hat{\alpha}_{MLE}+\hat{\beta}_{1MLE}} - \frac{\hat{\beta}_{1MLE}}{\hat{\alpha}_{MLE}+\hat{\theta}_{MLE}+\hat{\beta}_{1MLE}}\right)\left(\frac{\hat{\beta}_{2MLE}}{\hat{\alpha}_{MLE}+\hat{\beta}_{2MLE}} - \frac{\hat{\beta}_{2MLE}}{\hat{\alpha}_{MLE}+\hat{\theta}_{MLE}+\hat{\beta}_{2MLE}}\right)$$

Finally, the reliability expression for cascade (2 + 1) system obtained as;

$$\hat{R}_{21MLE} = \hat{R}_{1MLE} + \hat{R}_{2MLE} + \hat{R}_{3MLE}$$

## 3.2 The Method of Moment (MOM)

Suppose that the random sample $x_1, x_2, \dots, x_n$ is the strength from size n taken $Dag(\beta,\lambda,\gamma)$ and the random samples $t_1, t_2, \dots, t_m$ and $z_1, z_2, \dots, z_m$ are the stress from size m taken to each $Dag(\alpha,\lambda,\gamma)$ and $Dag(\theta,\lambda,\gamma)$, respectively. Thus, the mean population number corresponding to the sample when $\gamma > 1$ is given by [9], [10];

$$\bar{x} = \beta\lambda^{\frac{1}{\gamma}} B\left(1-\frac{1}{\gamma}, \beta+\frac{1}{\gamma}\right),$$

$$\bar{t} = \alpha\lambda^{\frac{1}{\gamma}} B\left(1-\frac{1}{\gamma}, \alpha+\frac{1}{\gamma}\right)$$

$$and \quad \bar{z} = \theta\lambda^{\frac{1}{\gamma}} B\left(1-\frac{1}{\gamma}, \theta+\frac{1}{\gamma}\right)$$

The MOM estimator for the first system (1+1) cascade where $x_{\omega_i};\ i = 1,2,\dots,n$ be a strength random samples from $\beta_{\omega_i}$ with the sample size $n\,;\omega = 1,2$.

$$\hat{\beta}_{\omega_i MOM} = \frac{\bar{x}}{\lambda^{\frac{1}{\gamma}} B\left(1-\frac{1}{\gamma}, \beta_{\omega_i}+\frac{1}{\gamma}\right)} \qquad (16)$$

In addition to, a stress random samples from size m taken to each $T\sim Dag(\alpha,\lambda,\gamma)$ and $Z\sim Dag(\theta,\lambda,\gamma)$ respectively, and the MOM estimators with unknown parameters $\alpha,\theta$ will be as follows:

$$\hat{\alpha}_{MOM} = \frac{\bar{t}}{\lambda^{\frac{1}{\gamma}} B\left(1-\frac{1}{\gamma}, \alpha+\frac{1}{\gamma}\right)} \qquad (17)$$

and,

$$\hat{\theta}_{MOM} = \frac{\bar{t}}{\lambda^{\frac{1}{\gamma}} B\left(1-\frac{1}{\gamma}, \theta+\frac{1}{\gamma}\right)} \qquad (18)$$

by substitution (16) , (17) and (18) in the equations (5) and (6), can be the MOM estimator for $R_{11}$ obtained as;

$$\hat{R}_{1MOM} = \frac{\hat{\beta}_{1MOM}}{\hat{\alpha}_{MOM}+\hat{\beta}_{1MOM}} - \frac{\hat{\beta}_{1MOM}}{\hat{\alpha}_{MOM}+\hat{\theta}_{MOM}+\hat{\beta}_{1MOM}}$$

And,

$$\hat{R}_{2MOM} = \left(1 - \hat{R}_{1MOM}\right)\left(\frac{\hat{\beta}_{2MOM}}{\hat{\alpha}_{MOM}+\hat{\beta}_{2MOM}} - \frac{\hat{\beta}_{2MOM}}{\hat{\alpha}_{MOM}+\hat{\theta}_{MOM}+\hat{\beta}_{2MOM}}\right)$$

We can get the reliability expression for cascade (1 + 1) system obtained as:

$$\hat{R}_{11_{MOM}} = \hat{R}_{1_{MOM}} + \hat{R}_{2_{MOM}}$$

In the second system (2+1) cascade can be obtained the MOM estimator for $R_{21}$ by substitution (16), (17) and (18) in the equations (8), (9) and (10), as follow:

$$\hat{R}_{1MOM} = \left(\frac{\hat{\beta}_{1MOM}}{\hat{\alpha}_{MOM}+\hat{\beta}_{1MOM}} - \frac{\hat{\beta}_{1MOM}}{\hat{\alpha}_{MOM}+\hat{\theta}_{MOM}+\hat{\beta}_{1MOM}}\right)\left(\frac{\hat{\beta}_{2MOM}}{\hat{\alpha}_{MOM}+\hat{\beta}_{2MOM}} - \frac{\hat{\beta}_{2MOM}}{\hat{\alpha}_{MOM}+\hat{\theta}_{MOM}+\hat{\beta}_{2MOM}}\right)$$

$$\hat{R}_{2MOM} = (1-c_1)\hat{R}_{1MOM} \quad and \quad \hat{R}_{3MOM} = (1-c_2)\hat{R}_{1MOM}$$

Finally, the reliability expression for cascade (2 + 1) system obtained as;

$$\hat{R}_{21MOM} = \hat{R}_{1MOM} + \hat{R}_{2MOM} + \hat{R}_{3MOM}$$

# 4 Experimental Study of the Estimators Comparison

In this section, it used a simulation study for achieve the best performance to the reliability cascade system (1+1) and (2+1) with unknown parameters following the Dagum distribution of two different estimators from the maximum likelihood methods and method of moment, we compared the results using the mean square error (MSE), so that a simulation study by replicate (1000) times to get of independent samples for different sizes (n= 15,30,90), also parameters and reliability values for five different experiments.

In the Table 1 (Appendix), the parameter values for the five experiments were chosen to be appropriate for the data based on subject information, the researchers' experience and vision for choosing these values, as well as calculating reliability values, are listed as shown.

The performance of the two systems reliability estimators was compared, and then simulations were performed using MATLAB 2020 through the following:

- To generate the random value, we use according to the inverse function formula as following:

$$x = \lambda^{\frac{1}{\gamma}}\left((F(x))^{\frac{-1}{\beta}} - 1\right)^{\frac{-1}{\gamma}}$$

- Mean is calculated using the equation: $= \frac{\sum_{i=}^{N}\hat{R}_i}{N}$

Although there are many criteria for determining the best estimation between estimators different, the best and most widely used is the mean square error (MSE) criterion to comparison of estimation methods by$MSE(\hat{R}) = \frac{\sum_{i=1}^{N}(\hat{R}_i - R)^2}{N}$ ; so N the number of repetitions for each experiment equal to 1000.

Table 2 shows that the MSE values for all five experiments decreases as the sample size n is increased for both MLE and MOM. We observed the best performance for MSE for $R_{11}$ reliability values is the MLE estimate compared to MOM.

Table 2. The results of simulation $R_{11}$

| Exp. | n | ML | | MO | | Best |
|---|---|---|---|---|---|---|
| | | Mean | MSE | Mean | MSE | |
| 1 | 15 | 0.3349 | 0.0070 | 0.3320 | 0.0137 | ML |
| | 30 | 0.3305 | 0.0037 | 0.3312 | 0.0086 | ML |
| | 90 | 0.3314 | 0.0013 | 0.3286 | 0.0037 | ML |
| 2 | 15 | 0.4730 | 0.0067 | 0.4641 | 0.0155 | ML |
| | 30 | 0.4721 | 0.0033 | 0.4659 | 0.0083 | ML |
| | 90 | 0.4792 | 0.0011 | 0.4771 | 0.0036 | ML |
| 3 | 15 | 0.1789 | 0.0040 | 0.1778 | 0.0081 | ML |
| | 30 | 0.1788 | 0.0022 | 0.1764 | 0.0048 | ML |
| | 90 | 0.1760 | 0.0007 | 0.1789 | 0.0022 | ML |
| 4 | 15 | 0.3336 | 0.0073 | 0.3295 | 0.0203 | ML |
| | 30 | 0.3295 | 0.0037 | 0.3192 | 0.0149 | ML |
| | 90 | 0.3308 | 0.0013 | 0.3289 | 0.0083 | ML |
| 5 | 15 | 0.4714 | 0.0069 | 0.4556 | 0.0265 | ML |
| | 30 | 0.4758 | 0.0033 | 0.4578 | 0.0194 | ML |
| | 90 | 0.4762 | 0.0012 | 0.4632 | 0.0111 | ML |

*Source: created by the authors*

Table 3. The results of simulation $R_{21}$

| Exp. | n | ML | | MO | | Best |
|---|---|---|---|---|---|---|
| | | Mean | MSE | Mean | MSE | |
| 1 | 15 | 0.0677 | 0.0001 | 0.0508 | 0.0006 | ML |
| | 30 | 0.0531 | 0.0005 | 0.0414 | 0.0012 | ML |
| | 90 | 0.0683 | 0.0001 | 0.0630 | 0.0002 | ML |
| 2 | 15 | 0.2665 | 0.0065 | 0.2346 | 0.0024 | MO |
| | 30 | 0.1540 | 0.0010 | 0.1579 | 0.0008 | MO |
| | 90 | 0.1589 | 0.0007 | 0.1330 | 0.0028 | ML |
| 3 | 15 | 0.0181 | 0.0001 | 0.0006 | 0.0007 | ML |
| | 30 | 0.0162 | 0.0001 | 0.0123 | 0.0002 | ML |
| | 90 | 0.0401 | 0.0002 | 0.0551 | 0.0007 | ML |
| 4 | 15 | 0.0616 | 0.0002 | 0.0625 | 0.0002 | ML=MO |
| | 30 | 0.0571 | 0.0004 | 0.1029 | 0.0007 | ML |
| | 90 | 0.0598 | 0.0003 | 0.0656 | 0.0001 | MO |
| 5 | 15 | 0.2215 | 0.0013 | 0.2119 | 0.0007 | MO |
| | 30 | 0.1665 | 0.0004 | 0.2062 | 0.0004 | ML=MO |
| | 90 | 0.1935 | 0.0001 | 0.2443 | 0.0034 | ML |

*Source: created by the authors*

Table 3 shows that the MSE values for all five experiments vary as the sample size n is increased for both MLE and MOM and observed the best performance of the MSE for $R_{21}$ reliability values is the MLE estimate compared to MOM. However, there are some experiments where MOM is better than ML in sample 15 and 30 for experiment 2 and in sample 15 for experiment 5 also in sample 90 for experiment 4. There are two cases where MLE and MOM values are equal for both experiments 4 and 5 in sample 15 and 30, respectively.

# 5 Conclusions

Through experimental and simulation results, the study demonstrated that a performance for the maximum likelihood estimator (ML) outperforms the moment method estimator (MO) in most experiments at different sample sizes (n). The mean and mean squared error (MSE) results for both methods were recorded for estimating the reliability of the (1+1) and (2+1) systems of the P(T<X<Z) model, where the variables T, Z, and X follow a Dagum distribution with different coefficients. Table 2 and Table 3 compare the reliability of the two derived and formulated systems. The results showed that the ML estimator's reliability estimation performance surpasses that of the moment method estimator (MO), except in a few very minor cases, as in Table 3, for experiment (2) at (n=15,30) and experiment (4) (n=90).

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*References:*

[1] R. Sharma,D. Kumar, and P. Kumar. "Fuzzy modeling of system behavior for risk and reliability analysis. International Journal of Systems Science", Vol. 39, No. 6, pp.563-581,2022. https://doi.org/10.1080/00207720701717708

[2] N. S. Karam, Sh. M. Yousif, G. S. Karam and Z. M. Abood, "Cascade Stress-Strength Reliability for P(X<Y<Z) of (1+1) and (2+1) Systems", *AIP Conference Proceedings* 2437-020071, 2022. https://doi.org/10.1063/5.0109248.

[3] N. S. Karam and A. H. Khaleel, "Generalized inverse Rayleigh reliability estimation for the (2+1) cascade model", AIP Conference Proceedings 2123-020046, 2019. https://doi.org/10.1063/1.5116973.

[4] A. H. Khaleel and N. S. Karam, "Estimating the Reliability Function of (2+1) Cascade Model", Baghdad Science Journal, Vol.16, No.2,2019, http://dx.doi.org/10.21123/bsj.2019.16.2.0395

[5] A. H. Khaleel, "Exponential Reliability Estimation of (3+1) Cascade Model", Iraqi Journal of Science MJPS, Vol.8, No.2, 2021,

[6] R. R. Mutkekar and S. B. Munoli, "Estimation of Reliability for Stress-Strength Cascade Model", Scientific Research Publishing. Open Journal of Statistics, Vol.6, pp.873-881, 2016.

[7] C. Dagum, "A New Model for Personal Income Distribution: Specification and Estimation. Economic Applique", Vol. 30, No.3, 413–437,1977.

[8] N. Singh, "On The Estimation of P $(y_1 < x < y_2)$", Communication in Statistics-Theory and Methods, Vol. 19, No, 15, pp.1551-1561,1980.

[9] S. Dey, B. Al-Zahrani and S. Basloom, "Dagum distribution Properties and different methods of estimation", International Journal of Statistics and Probability, Vol.6, No.2, pp.74-92, 2017.

[10] N. S. Karam and A. M. Attia, "Stress-Strength Reliability for P $(\boldsymbol{T} < \boldsymbol{X} < \boldsymbol{Z})$ using Dagum Distribution", Journal of Physics: Conference Series,1879-032004, 2021.

[11] C. Doloi, M. Borah, and J. Gogoi, "Cascade System with Pr(X < Y < Z). " Journal of Informatics and Mathematical Sciences Vol. 5, No. 1, pp.37–47, 2013

[12] A. K. Fulment, P. K. Josephat, and G. S. Rao, "Estimation of Reliability in Multicomponent Stress-Strength Based on Dagum Distribution. Stochastics and Quality Control", Vol. 32 No. 2, pp.77-85, 2017, https://doi.org/10.1515/eqc-2017-0009.

# APPENDIX

Table 1. the values of parameters and Reliability

| Experiment | $\beta_1$ | $\beta_2$ | $\alpha$ | $\theta$ | $\lambda$ | $\gamma$ | $R_{11}$ | $R_{21}$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.8 | 0.6 | 1.4 | 2.2 | 1.5 | 2 | 0.3306 | 0.0760 |
| 2 | 1.3 | 0.9 | 0.7 | 1.5 | 1.5 | 2 | 0.4795 | 0.1857 |
| 3 | 0.3 | 0.4 | 0.9 | 0.7 | 1.5 | 2 | 0.1757 | 0.0278 |
| 4 | 0.8 | 0.6 | 1.4 | 2.2 | 2 | 1.5 | 0.3306 | 0.0760 |
| 5 | 1.3 | 0.9 | 0.7 | 1.5 | 2 | 1.5 | 0.4795 | 0.1857 |

*Source: created by the authors*

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