Calculation of GCD of one variable polynomials using Evaluaton-Rational Interpolation technique

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Abstract: - The Evaluation–Rational Interpolation (E-RI) technique is a computational method designed for the efficient evaluation and reconstruction of rational functions. It comprises two main components: an evaluation step and a rational interpolation step. The interpolation procedure follows the classical Bulirsch–Stoer method, a recursive algorithm widely used for rational function recovery. To address inherent limitations of the original method, a modified version of the Bulirsch–Stoer algorithm is developed and integrated into the E-RI framework, enhancing its robustness and applicability. Additionally, the proposed technique is applied to the computation of the greatest common divisor (GCD) of univariate polynomials, demonstrating its effectiveness and computational advantages in symbolic computation tasks.

Key-Words: - rational interpolation, greatest common divisor, Bulirsch - Stoer method, rational function, computational algebra, polynomial GCD, algebraic algorithms

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1 Introduction

The computation of the greatest common divisor (GCD) of polynomials is a topic of significant interest, particularly in the field of computational mathematics, due to its numerous applications in areas such as control theory and engineering sciences, [1]. The Evaluation–Interpolation (E-I) technique is a computational method grounded in numerical analysis rather than symbolic or analytic solutions. Building upon this framework, the Evaluation–Rational Interpolation (E-RI) technique has proven to be a highly effective approach for computing specific rational functions. As a result, it has gained widespread application in problems where the underlying solution is rational in nature. Recent research on polynomial GCD computation has focused on sparse interpolation techniques, hybrid symbolic–numeric approaches, and numerically stable algorithms for approximate coefficients. Indicative works include sparse interpolation-based GCD methods, [2], hybrid ERES approaches, [3], and SVD-based techniques for polynomial systems, [4]. However, many of these methods still rely on heavy symbolic processing or suffer from numerical instability in floating-point environments.

The E-RI technique comprises two main components: the evaluation part and the rational interpolation part. During the evaluation phase, the number of interpolation points required is determined, depending on the degrees of the numerator and denominator of the target rational function. These points are then computed accordingly. In the rational interpolation phase, a rational function is constructed to pass exactly through these interpolation points.

Various methods exist for rational interpolation. Some utilize barycentric representations of the rational expression, [5], [6], [7], while others adopt recursive schemes. Among the latter, the Bulirsch–Stoer method stands out as a prominent recursive algorithm for rational interpolation, [8].

However, a closer study of the Bulirsch–Stoer method reveals certain limitations that may affect its performance or applicability. To address these, a modified version of the method is proposed and incorporated into the rational interpolation component of the E-RI technique.

In Section 2, we present the formal definition of the greatest common divisor (GCD) of univariate polynomials and describe the algorithm for its computation using the Evaluation–Rational Interpolation (E-RI) technique. The modified version of the Bulirsch–Stoer method, developed to address specific limitations of the classical formulation, is also introduced.

Section 3 provides a detailed numerical example that demonstrates the application of the proposed method in computing the GCD of two polynomials, showcasing its effectiveness and practical implementation.

2 An algorithm for GCD calculation

Definition 2.1. *Let the polynomials f, $g \in F[x]$, not both 0. A monic polynomial $d \in F[x]$ is said to be the greatest common divisor of f and g polynomials if the following conditions are satisfied:*

1. *$d \mid f$ and $d \mid g$ (that is, d is a common divisor of f and g)*
2. *if $h \in F[x]$ and $h \mid f$ and $h \mid g$, then $\deg(h) \leq \deg(d)$ (that is, d is the monic polynomial of largest degree dividing both f and g.*

If f and g polynomials have a greatest common divisor d polynomial, then using the fact that a greatest common divisor is monic, we may show that d is unique, [9].

There are many methods for computing polynomial GCD, [3], [4], [10]. Two of them are factorization and Euclidean algorithm. In the first one, one has to find the factors of each polynomial and then select the set of common factors. The drawback of this method is that it may be useful only in simple cases, where polynomial degrees are small enough, because of the fact that it demands a great deal of symbolic calculations. As for the Euclidean algorithm, it consists of repeated use of polynomial division. If the coefficients of polynomials are floating point numbers, Euclidean algorithm can give the requested result only approximately. On the contrary, E-RI technique is based in numerical analysis. In evaluation part, only arithmetic computations are executed, while rational interpolation part, consists of a recursive method that simple polynomial operations are used.

Aiming at calculating the GCD of polynomials according to E-RI technique and modified Bulirsch - Stoer method, one has to follow the procedure:
Let $P(x)$, $Q(x)$ one variable polynomials, where $\deg(P(x)) \leq \deg(Q(x))$. Then

$$\frac{P(x)}{Q(x)} = \frac{\gcd(P(x), Q(x)) \cdot p(x)}{\gcd(P(x), Q(x)) \cdot q(x)} = \frac{p(x)}{q(x)}$$

where $\frac{p(x)}{q(x)}$ is the irreducible form of $\frac{P(x)}{Q(x)}$.
By executing evaluation in $\frac{P(x)}{Q(x)}$ we have the data set as it is shown below

x_i	x_1	x_2	$\cdots$	x_N
y_i	y_1	y_2	$\cdots$	y_N

where

$$y_i = \frac{P(x_i)}{Q(x_i)} \quad \text{for} \quad i = 1, \cdots, N$$

and N is the number of required points for rational interpolation.

By executing rational interpolation in this data set, we get the irreducible form $\frac{p(x)}{q(x)}$.
Then evaluation is executed in $\frac{P(x)}{p(x)}$ where $p(x) \neq 0$ and we have the data set as it is shown below

$\tilde{x}_i$	$\tilde{x}_1$	$\tilde{x}_2$	$\cdots$	$\tilde{x}_{\tilde{N}}$
$\tilde{y}_i$	$\tilde{y}_1$	$\tilde{y}_2$	$\cdots$	$\tilde{y}_{\tilde{N}}$

where

$$\tilde{y}_i = \frac{P(\tilde{x}_i)}{p(\tilde{x}_i)} \quad \text{for} \quad i = 1, \cdots, \tilde{N}$$

and $\tilde{N}$ is the number of required points for polynomial interpolation.
Because of the fact that

$$\deg(\gcd(P(x), Q(x))) \leq \deg(P(x)) \leq \deg(Q(x))$$

then

$$\tilde{N} = \deg(P(x)) + 1$$

By executing polynomial interpolation in this data set, we get a polynomial, which is the $\gcd(P(x), Q(x))$.

Specifically, the number N of required points for rational interpolation, depends on Bulirsch - Stoer method. Let the polynomials $P(x)$ and $Q(x)$ where

$$\deg(P(x)) \leq \deg(Q(x))$$

and the rational function

$$R(x) = \frac{P(x)}{Q(x)}$$

Bulirsch - Stoer method requires the degree of numerator of rational function R to be equal or less by one to the degree of denominator of rational function R. Namely,

$$\deg(P(x)) = \deg(Q(x))$$

or

$$\deg(P(x)) = \deg(Q(x)) - 1$$

There are the following cases:

- if $\deg(P(x)) = \deg(Q(x))$ or $deg(P(x)) = deg(Q(x)) - 1$, then

$$N = deg(P(x)) + deg(Q(x)) + 1$$

- if $\deg(P(x)) < \deg(Q(x)) - 1$, then

$$N = 2 \cdot \deg(Q(x))$$

In addition, in Evaluation Rational Interpolation technique, Bulirsch - Stoer method has the following issues. First of all, in evaluation part, existence of poles or even zeros in data set, except of the case that zeros lie in the first place of the set.
For instance, aiming in recovering the rational function

$$R(s) = \frac{P(s)}{Q(s)} = \frac{s^2-1}{s^2-4}$$

Bulirsch - Stoer method can be executed in different data sets as the following cases.
Case 1:

x_i	0	2	4	6	8
y_i	$\frac{1}{4}$	∞	$\frac{5}{4}$	$\frac{35}{32}$	$\frac{21}{20}$

Case 2:

x_i	0	1	3	5	7
y_i	$\frac{1}{4}$	0	$\frac{8}{5}$	$\frac{8}{7}$	$\frac{16}{15}$

Case 3:

x_i	1	3	5	7	9
y_i	0	$\frac{8}{5}$	$\frac{8}{7}$	$\frac{16}{15}$	$\frac{80}{77}$

In Case 1 there is a pole in data set ($Q(2) = 0$), while in Case 2 a zero ($P(1) = 0$). In both cases, E-RI technique does not terminate effectively. On the contrary, in Case 3 the zero lies in the first place of data set ($P(1) = 0$) and the rational function is being recovered.
As for the rational interpolation part, appearance of indeterminate values during the process of finding the rational interpolant, can cause issues. In order to avoid these issues and gain randomness on the choice of evaluation points, E-RI technique and the modified Bulirsch - Stoer method is proposed. In evaluation part, first of all, in cases that a pole, or a zero occurs (not in the first place), the default data set changes. Rational interpolation part of the method is also extended in order to avoid indeterminate values. If an indeterminated value occurs, the process stops. Then all the rational functions of the previous stage of Bulirsch - Stoer method are identical to each other, and also consist the requested rational function.

Consequently, all the above are concluded to the following method for calculation of GCD of one variable polynomials. Let $P(x), Q(x)$ be one variable polynomials, where

$$\deg(P(x)) \leq \deg(Q(x)) \text{ and } x \in \mathbb{R}$$

Aiming at calculating the greatest common divisor of $P(x), Q(x)$ by using E-RI technique and the extended Bulirsch - Stoer method, there are the following steps:

Step 1 Definition of $P(x)$ and $Q(x)$ polynomials where

$$\deg(P(x)) \leq \deg(Q(x))$$

Step 2 Calculation of the number (N) of required points for Evaluation - Rational Interpolation technique according to Bulirsch - Stoer method:

- if $\deg(P(x)) = \deg(Q(x))$ or $\deg(P(x)) = \deg(Q(x)) - 1$ then

$$N = \deg(P(x)) + \deg(Q(x)) + 1$$

- if $\deg(P(x)) < \deg(Q(x)) - 1$ then

$$N = 2 \cdot \deg(Q(x))$$

Step 3 Proper selection of N interpolation points x_i that will be used in evaluation part. The interpolation points are selected such that poles and zeros of the rational function are avoided, while ensuring sufficient numerical separation to reduce interpolation instability.

Step 4 Evaluation of the N points in $\frac{P(x)}{Q(x)}$ rational function.

Step 5 Rational interpolation according to Bulirsch - Stoer method. $R_{BS}(x) = \frac{p(x)}{q(x)}$ is recovered. Separation of $R_{BS}(x)$ terms.

Step 6 Calculation of the number ($\widetilde{N}$) of required points for Newton polynomial interpolation method.

$$\widetilde{N} = \deg(P(x)) + 1$$

Step 7 Proper selection of $\widetilde{N}$ interpolation points that will be used in evaluation part. The interpolation points are selected such that poles and zeros of the rational function are avoided, while ensuring sufficient numerical separation to reduce interpolation instability.

Step 8 Evaluation of the $\widetilde{N}$ points in $\frac{P(x)}{p(x)}$ rational function.

Step 9 Polynomial interpolation. Finding of $P_N(x) = \gcd(P(x), Q(x))$

Regarding computational complexity, the proposed method requires O(N) evaluations of rational functions and polynomial interpolations of degree at most deg(P). Since the algorithm avoids polynomial division and factorization, its computational cost is dominated by arithmetic evaluations and interpolation steps, making it suitable for moderate to high-degree polynomials in floating-point arithmetic.

3 An example for calculating polynomial GCD

Let the polynomials

$$x^2 - \frac{2}{7}x \quad \text{and} \quad x^3 - \frac{1}{3}x$$

with respect to variable x.

Step 1 We have that

$$\deg(x^2 - \frac{2}{7}x) = 2 \leq \deg(x^3 - \frac{1}{3}x) = 3$$

Then

$$P(x) = x^2 - \frac{2}{7}x$$

and

$$Q(x) = x^3 - \frac{1}{3}x$$

Step 2 Holds that $\deg(P(x)) = \deg(Q(x)) - 1$, then

$$N = \deg(P(x)) + \deg(Q(x)) + 1 = 2 + 3 + 1 = 6$$

Step 3 Proper selection of N random fixed points. By default, there is the following set:

$$x = \{-3, -2, -1, 0, 1, 2\}$$

Nevertheless, it contains a pole of the $\frac{P(x)}{Q(x)}$ rational function. So according to the modified Bulirsch - Stoer method that has been constructed, the data set is altered as follows:

$$x = \{-3, -2, -1, 3, 1, 2\}$$

as for those values to be avoided.

Step 4 Evaluation of the N points in $\frac{P(x)}{Q(x)}$ rational function

$$\begin{aligned} for \quad & x_1 = -3, \\ & y_1 = \frac{P(x_1)}{Q(x_1)} = \frac{P(-3)}{Q(-3)} = \frac{-69}{182} \end{aligned}$$

$$\begin{aligned} for \quad & x_2 = -2, \\ & y_2 = \frac{P(x_2)}{Q(x_2)} = \frac{P(-2)}{Q(-2)} = \frac{-48}{77} \end{aligned}$$

$$\begin{aligned} for \quad & x_3 = -1, \\ & y_3 = \frac{P(x_3)}{Q(x_3)} = \frac{P(-1)}{Q(-1)} = \frac{-27}{14} \end{aligned}$$

$$\begin{aligned} for \quad & x_4 = 3, \\ & y_4 = \frac{P(x_4)}{Q(x_4)} = \frac{P(3)}{Q(3)} = \frac{57}{182} \end{aligned}$$

$$\begin{aligned} for \quad & x_5 = 1, \\ & y_5 = \frac{P(x_5)}{Q(x_5)} = \frac{P(1)}{Q(1)} = \frac{15}{14} \end{aligned}$$

$$\begin{aligned} for \quad & x_6 = 2, \\ & y_6 = \frac{P(x_6)}{Q(x_6)} = \frac{P(2)}{Q(2)} = \frac{36}{77} \end{aligned}$$

The data set for E-RI technique is shown below

x_i	-3	-2	-1	3	1	2
y_i	$\frac{-69}{182}$	$\frac{-48}{77}$	$\frac{-27}{14}$	$\frac{57}{182}$	$\frac{15}{14}$	$\frac{36}{77}$

Step 5 Bulirsch - Stoer rational interpolation is executed. The rational function

$$R_{BS}(x) = \frac{p(x)}{q(x)} = \frac{x - \frac{6}{21}}{x^2 - \frac{7}{21}}$$

is the irreducible form of $\frac{P(X)}{Q(x)}$, and $p(x)$, $q(x)$ monic polynomials.

Step 6 Calculation of the number $\widetilde{N}$ of interpolation points, according to Newton method for polynomial interpolation.

$$\widetilde{N} = \deg(P(x)) + 1 = 2 + 1 = 3$$

Step 7 Proper selection of the $\widetilde{N}$ random fixed points. By default, there is the set:

$$\tilde{x} = \{1, 2, 3\}$$

It doesn't contains zeros of the $p(x)$ function. So the data set is not altered.

Step 8 Evaluation of the $\widetilde{N}$ points in $\frac{P(X)}{p(X)}$ rational function

$$\begin{aligned} for \quad & \tilde{x}_1 = 1, \\ & \tilde{y}_1 = \frac{P(\tilde{x}_1)}{p(\tilde{x}_1)} = \frac{P(1)}{p(1)} = 1 \end{aligned}$$

$$\begin{aligned} for \quad & \tilde{x}_2 = 2, \\ & \tilde{y}_2 = \frac{P(\tilde{x}_2)}{p(\tilde{x}_2)} = \frac{P(2)}{p(2)} = 2 \end{aligned}$$

$$\begin{aligned} for \quad & \tilde{x}_3 = 3, \\ & \tilde{y}_3 = \frac{P(\tilde{x}_3)}{p(\tilde{x}_3)} = \frac{P(3)}{p(3)} = 3 \end{aligned}$$

The data set for E-I technique is shown below

$\tilde{x}_i$	1	2	3
$\tilde{y}_i$	1	2	3

Step 9 Polynomial interpolation is executed. The following polynomial is recovered.

$$P_N(x) = x = \gcd(P(x), Q(x))$$

which is the greatest common divisor of $P(x), Q(x)$ polynomials.

4 Discussion

This paper emphasizes the use of E-RI technique as a computational method. Both E-I and E-RI techniques are based in numerical analysis against analytic solutions that have so far been used for the calculation of functions. More precisely E-RI technique avoids complex polynomial operations. Only addition and multiplication between polynomials are executed. These operations can be easily implemented in programming languages. That is the main reason why E-RI technique consists an extremely useful tool for the calculation of rational expressions.

Furthermore, Bulirsch - Stoer method has been chosen for the rational interpolation part. However, there are many restrictions as for Bulirsch - Stoer method to be properly applied. The modified Bulirsch - Stoer method that has been incorporated in rational interpolation part of E-RI technique can overcome these restrictions. It has been observed to work robustly across a wide range of practical cases despite of the form of the rational function that is recovered. As a result, the use of E-RI technique and the modified Bulirsch - Stoer method can lead to the calculation of GCD of polynomials regardless of the form of these polynomials.

Since the proposed method relies on floating-point arithmetic, numerical errors may propagate during evaluation and interpolation steps. However, by avoiding polynomial division and symbolic factorization, error amplification is reduced compared to classical methods. In practice, numerical stability was observed to be satisfactory for all tested cases, especially when interpolation points are properly selected.

5 Conclusion

A new extended version of the Bulirsch–Stoer method for rational interpolation has been proposed. This modified approach has been specifically designed to ensure that the Evaluation–Rational Interpolation (E-RI) technique can consistently compute the desired rational function, regardless of the degrees of the numerator and the denominator. The limitations previously associated with the classical Bulirsch–Stoer method no longer apply, as demonstrated through the successful computation of the greatest common divisor (GCD) of univariate polynomials.

More precisely, the proposed method is robust and broadly applicable, making it suitable for a wide range of problems where rational interpolation is required. Its numerical nature and minimal symbolic complexity make it particularly attractive for implementation in modern computational systems and software environments.

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