Integral Representations of Companion Fibonacci and Pell Numbers

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Abstract: This paper investigates one-parameter generalizations of the Fibonacci and Pell numbers that preserve the recurrence relations under arbitrary initial conditions. We present integral representations for the companion k-Pell numbers associated with the k-Fibonacci and k-Lucas numbers, as well as for those associated with the k-Pell and k-Pell-Lucas numbers. Our results extend to all companion numbers of generalized Fibonacci and Pell sequences.

Key-Words: k-Fibonacci number, k-Lucas number, k-Pell number, k-Pell-Lucas number, integral representation

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1 Introduction

Recall that the Fibonacci numbers F_n are defined by

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2,$$

with $F_0 = 0$ and $F_1 = 1$. The Lucas numbers L_n are defined by

$$L_n = L_{n-1} + L_{n-2}, \quad n \geq 2,$$

with $L_0 = 2$ and $L_1 = 1$. Pell numbers are like Fibonacci and Lucas numbers, defined by

$$P_n = 2P_{n-1} + P_{n-2}, \quad n \geq 2,$$

with $P_0 = 0$ and $P_1 = 1$. The Pell-Lucas numbers Q_n are defined by

$$Q_n = 2Q_{n-1} + Q_{n-2}, \quad n \geq 2,$$

with $Q_0 = 2$ and $Q_1 = 2$.

Many generalizations of Fibonacci, Lucas, Pell, and Pell-Lucas numbers are defined in different ways; for instance, [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23]. In 2007, [8], introduced a generalization of the Fibonacci numbers, which is called *the k-Fibonacci number*, denoted by $F_{k,n}$ and defined by

$$F_{k,n} = kF_{k,n-1} + F_{k,n-2}, \quad n \geq 2,$$

with $F_{k,0} = 0$ and $F_{k,1} = 1$, where k is a positive integer and n is a non-negative integer. In 2011, [10], introduced a generalization of Lucas numbers, which is called *the k-Lucas number*, denoted by $L_{k,n}$ and defined by

$$L_{k,n} = kL_{k,n-1} + L_{k,n-2}, \quad n \geq 2,$$

with $L_{k,0} = 2$ and $L_{k,1} = k$. Notice that the Fibonacci, Lucas, Pell, and Pell–Lucas numbers are 1-Fibonacci, 1-Lucas, 2-Fibonacci, and 2-Lucas numbers, respectively. In 2017, [11], studied a generalization of the k-Fibonacci and k-Lucas numbers, which is called *the companion k-Fibonacci number*, denoted by $U^{a,b}_{k,n}$ and defined by

$$U^{a,b}_{k,n} = kU^{a,b}_{k,n-1} + U^{a,b}_{k,n-2}, \quad n \geq 2, \qquad (1)$$

with $U^{a,b}_{k,0} = a$ and $U^{a,b}_{k,1} = b$, where a and b are non-negative integers. For $k = 1, 2$, the companion 1-Fibonacci (Gibonacci) and companion 2-Fibonacci numbers are denoted by $G^{a,b}_n$ and $H^{a,b}_n$, respectively. Indeed, $F_{k,n} = U^{0,1}_{k,n}$, $L_{k,n} = U^{2,k}_{k,n}$, $G^{a,b}_n = U^{a,b}_{1,n}$, and $H^{a,b}_n = U^{a,b}_{2,n}$. The initial terms of $\{U_{k,n}\}$ are presented as in Table 1 (APPENDIX).

In 2013, [12], introduced a generalization of Pell numbers, which is called *the k-Pell number*, denoted by $P_{k,n}$ and defined by

$$P_{k,n} = 2P_{k,n-1} + kP_{k,n-2}, \quad n \geq 2,$$

with $P_{k,0} = 0$ and $P_{k,1} = 1$. Subsequently, [13], introduced a generalization of Pell-Lucas numbers which is called *the k-Pell-Lucas number*, denoted by $Q_{k,n}$ and defined by

$$Q_{k,n} = 2Q_{k,n-1} + kQ_{k,n-2}, \quad n \geq 2,$$

with $Q_{k,0} = 2$ and $Q_{k,1} = 2$. We know that $F_n = \frac{1}{2^{n-1}} P_{4,n}$, $L_n = \frac{1}{2^n} Q_{4,n}$, $P_n = P_{2,n}$ and $Q_n = Q_{2,n}$.

According to (1), we begin by introducing a generalization of the k-Pell numbers, which is called *the companion k-Pell number*, denoted by $V_{k,n}^{a,b}$ and defined by

$$V_{k,n}^{a,b} = 2V_{k,n-1}^{a,b} + kV_{k,n-2}^{a,b}, \quad n \geq 2, \tag{2}$$

with $V_{k,0}^{a,b} = a$ and $V_{k,1}^{a,b} = b$, where a and b are non-negative integers. It is well-known that $P_{k,n} = V_{k,n}^{0,1}$, $Q_{k,n} = V_{k,n}^{2,2}$, and $H_{k,n} = V_{1,n}^{a,b}$. The initial terms of $\{V_{k,n}\}$ are presented as in Table 2 (APPENDIX).

The following describes the relationship between companion k-Pell numbers and Gibonacci numbers:

Theorem 1. *Let a, b and n be non-negative integers. Then $G_n^{a,b} = \frac{1}{2^n} V_{4,n}^{a,2b}$.*

Proof. We see that

$$G_0 = a = \frac{1}{2^0} V_{4,0}^{a,2b}$$

and

$$G_1 = b = \frac{1}{2}(2b) = \frac{1}{2} V_{4,1}^{a,2b}.$$

Assume that $G_\ell^{a,b} = \frac{1}{2^\ell} V_{4,\ell}^{a,2b}$, where $1 < \ell < n$. Then

$$\begin{aligned} G_n^{a,b} &= G_{n-1}^{a,b} + G_{n-2}^{a,b} \\ &= \frac{1}{2^{n-1}} V_{4,n-1}^{a,2b} + \frac{1}{2^{n-2}} V_{4,n-2}^{a,2b} \\ &= \frac{1}{2^n} \left(2V_{4,n-1}^{a,2b} + 4V_{4,n-2}^{a,2b} \right) \\ &= \frac{1}{2^n} V_{4,n}^{a,2b}. \end{aligned}$$

This completes the proof by strong mathematical induction. □

Integral representations of the sequences of numbers have been investigated by many researchers; see, [24], [25], [26], [27], [28], [29], [30], [31], [32], [33], [34], [35], [36], [37], [38], [39], [40], [41], [42], [43]. In 2022, the integral representations for the Fibonacci and Lucas numbers are

$$F_{\ell n} = \frac{nF_\ell}{2^n} \int_{-1}^{1} (L_\ell + \sqrt{5} F_\ell x)^{n-1} dx$$

and

$$\begin{aligned} L_{\ell n} = \frac{1}{2^n} \int_{-1}^{1} &(L_\ell + \sqrt{5} F_\ell x)^{n-1} \\ &\times (L_\ell + \sqrt{5}(n+1) F_\ell x) dx, \end{aligned}$$

where ℓ and n are non-negative integers; see, [41], [42]. Moreover, [24], presented an integration formulas involving Fibonacci and Lucas numbers and combinations of trigonometric functions. In [33], the second author presented the integral representations for Pell and Pell-Lucas numbers in the following:

$$P_{\ell n} = \frac{nP_\ell}{2^n} \int_{-1}^{1} (Q_\ell + 2\sqrt{2} P_\ell x)^{n-1} dx$$

and

$$\begin{aligned} Q_{\ell n} = \frac{1}{2^n} \int_{-1}^{1} &(Q_\ell + 2\sqrt{2} P_\ell x)^{n-1} \\ &\times (Q_\ell + 2\sqrt{2}(n+1) P_\ell x) dx, \end{aligned}$$

where ℓ and n are non-negative integers. Integral representations for the k-Fibonacci, k-Lucas, k-Pell, and k-Pell-Lucas numbers are then provided by the authors, [34], [35].

In this paper, using on the integral representations established in [34], [35], we obtain new integral representations for the companion k-Fibonacci and k-Pell numbers.

2 The Companion k-Fibonacci Numbers

In order to obtain new integral representations for the companion k-Fibonacci numbers using relations between the k-Fibonacci and k-Lucas numbers, we require the following results.

Theorem 2 (Binet's formulas)**.** *Let k and n be non-negative integers with $k \neq 0$. Then*

$$U_{k,n}^{a,b} = \frac{2b - ak + a\Delta_k}{2\Delta_k} \varphi_k^n + \frac{ak - 2b + a\Delta_k}{2\Delta_k} \frac{(-1)^n}{\varphi_k^n},$$

where $\Delta_k = \sqrt{k^2+4}$ and $\varphi_k = \frac{1}{2}(k + \Delta_k)$.

Proof. A characteristic equation of the recurrence relation (1) is

$$r^2 - kr - 1 = 0.$$

Since $\Delta_k^2 = k^2 + 4 > 0$ for $k \geq 1$, we have the roots

$$r_1 = \frac{k + \Delta_k}{2} = \varphi_k$$

and

$$r_2 = \frac{k - \Delta_k}{2} = \frac{-2}{(k + \Delta_k)} = \frac{-1}{\varphi_k}.$$

Note that $r_1 + r_2 = k$, $r_1 - r_2 = \Delta_k$, and $r_1 r_2 = -1$. Therefore,

$$U_{k,n}^{a,b} = \alpha r_1^n + \beta r_2^n = \alpha \varphi_k^n + \beta \frac{(-1)^n}{\varphi_k^n}$$

where α and β are the coefficients. Since $U_{k,0}^{a,b} = a$ and $U_{k,1}^{a,b} = b$, we get

$$\alpha + \beta = a \quad \text{and} \quad \alpha r_1 + \beta r_2 = b.$$

It can be shown that

$$\alpha = \frac{2b - ak + a\Delta_k}{2\Delta_k} \quad \text{and} \quad \beta = \frac{ak - 2b + a\Delta_k}{2\Delta_k}.$$

Therefore,

$$U_{k,n}^{a,b} = \frac{2b - ak + a\Delta_k}{2\Delta_k}\varphi_k^n + \frac{ak - 2b + a\Delta_k}{2\Delta_k}\frac{(-1)^n}{\varphi_k^n}.$$

This completes the proof. □

Corollary 3 ([9, Proposition 2], [10, Theorem 2.2]). *Let k and n be non-negative integers with $k \neq 0$. Then*

(i) $F_{k,n} = \frac{1}{\Delta_k}\left(\varphi_k^n - \frac{(-1)^n}{\varphi_k^n}\right)$;

(ii) $L_{k,n} = \varphi_k^n + \frac{(-1)^n}{\varphi_k^n}$,

where $\Delta_k = \sqrt{k^2+4}$ *and* $\varphi_k = \frac{1}{2}(k + \Delta_k)$.

Proof. Setting $(a, b) \in \{(0, 1), (2, k)\}$ in Theorem 2, this completes the proof. □

Corollary 4. *Let n be a non-negative integer. Then*

(i) $G_n^{a,b} = \frac{2b-a+\sqrt{5}a}{2\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n + \frac{a-2b-\sqrt{5}a}{2\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n$;

(ii) $H_n^{a,b} = \frac{b-a+\sqrt{2}a}{2\sqrt{2}}(1+\sqrt{2})^n + \frac{a-b-\sqrt{2}a}{2\sqrt{2}}(1-\sqrt{2})^n$.

Proof. Notice that $U_{1,n}^{a,b} = G_n^{a,b}$, $U_{2,n}^{a,b} = H_n^{a,b}$, $\Delta_1 = \sqrt{5}$, $\Delta_2 = 2\sqrt{2}$, $\varphi_1 = \frac{1+\sqrt{5}}{2}$, and $\varphi_2 = 1 + \sqrt{2}$. Setting $k = 1, 2$ in Theorem 2, the proof is finished. □

Lemma 5 ([35, Lemma 1]). *Let k and n be non-negative integers with $k \neq 0$. Then*

(i) $L_{k,n} + \Delta_k F_{k,n} = 2\varphi_k^n$;

(ii) $L_{k,n} - \Delta_k F_{k,n} = 2\frac{(-1)^n}{\varphi_k^n}$;

(iii) $L_{k,n}^2 - \Delta_k^2 F_{k,n}^2 = 4(-1)^n$,

where $\Delta_k = \sqrt{k^2+4}$ *and* $\varphi_k = \frac{1}{2}(k + \Delta_k)$.

The companion k-Fibonacci numbers, in relation to the k-Fibonacci and k-Lucas numbers, are given as follows:

Theorem 6. *Let k and n be non-negative integers with $k \neq 0$. Then*

$$U_{k,n}^{a,b} = \frac{a}{2}L_{k,n} + \frac{2b - ak}{2}F_{k,n}.$$

Proof. From Theorem 2, (i) and (ii) of Lemma 5, we obtain

$$\begin{aligned}
U_{k,n}^{a,b} &= \frac{2b - ak + a\Delta_k}{2\Delta_k}\varphi_k^n + \frac{ak - 2b + a\Delta_k}{2\Delta_k}\frac{(-1)^n}{\varphi_k^n} \\
&= \left(\frac{2b - ak + a\Delta_k}{2\Delta_k}\right)\left(\frac{L_{k,n} + \Delta_k F_{k,n}}{2}\right) \\
&\quad + \left(\frac{ak - 2b + a\Delta_k}{2\Delta_k}\right)\left(\frac{L_{k,n} - \Delta_k F_{k,n}}{2}\right) \\
&= \frac{a}{2}L_{k,n} + \frac{2b - ak}{2}F_{k,n}.
\end{aligned}$$

This completes this proof. □

Setting $k = 1, 2$, we have the following:

Corollary 7. *Let n be a non-negative integer. Then*

(i) $G_n^{a,b} = \frac{a}{2}L_n + \frac{2b-a}{2}F_n$;

(ii) $H_n^{a,b} = \frac{a}{2}Q_n + (b - a)P_n$.

The following theorem is presented in [35].

Theorem 8 ([35, Theorems 5 and 12]). *Let k, ℓ and n be non-negative integers with $k \neq 0$. Then*

$$F_{k,\ell n} = \frac{nF_{k,\ell}}{2^n}\int_{-1}^{1}(L_{k,\ell} + \Delta_k F_{k,\ell}x)^{n-1}dx,$$

and

$$\begin{aligned}
L_{k,\ell n} = \frac{1}{2^n}\int_{-1}^{1}&(L_{k,\ell} + \Delta_k F_{k,\ell}x)^{n-1} \\
&\times (L_{k,\ell} + (n+1)\Delta_k F_{k,\ell}x)dx,
\end{aligned}$$

where $\Delta_k = \sqrt{k^2+4}$.

We now present new integral representations for the companion k-Fibonacci numbers as follows:

Theorem 9. *Let k, ℓ and n be non-negative integers with $k \neq 0$. Then*

$$\begin{aligned}
U_{k,\ell n}^{a,b} &= \frac{1}{2^{n+1}}\int_{-1}^{1}(L_{k,\ell} + \Delta_k F_{k,\ell}x)^{n-1} \\
&\times \left(aL_{k,\ell} + (2b - ak)nF_{k,\ell} + a(n+1)\Delta_k F_{k,\ell}x\right)dx,
\end{aligned}$$

where $\Delta_k = \sqrt{k^2+4}$.

Proof. From Theorem 6, we obtain

$$U_{k,\ell n}^{a,b} = \frac{a}{2}L_{k,\ell n} + \frac{2b - ak}{2}F_{k,\ell n}.$$

Applying Theorem 8, we get

$$\begin{aligned}
&U_{k,\ell n}^{a,b} \\
&= \frac{a}{2}\Bigg(\frac{1}{2^n}\int_{-1}^{1}(L_{k,\ell}+\Delta_k F_{k,\ell}x)^{n-1} \\
&\qquad \times (L_{k,\ell}+(n+1)\Delta_k F_{k,\ell}x)dx\Bigg) \\
&\quad + \frac{2b-ak}{2}\left(\frac{nF_{k,\ell}}{2^n}\int_{-1}^{1}(L_{k,\ell}+\Delta_k F_{k,\ell}x)^{n-1}dx\right) \\
&= \frac{1}{2^{n+1}}\int_{-1}^{1}(L_{k,\ell}+\Delta_k F_{k,\ell}x)^{n-1} \\
&\times (aL_{k,\ell}+(2b-ak)nF_{k,\ell}+a(n+1)\Delta_k F_{k,\ell}x)\,dx.
\end{aligned}$$

This completes the proof. □

Remark 10. As in Theorems 6 and 9, we have the following results.

(i) If $a=0$, then $U_{k,n}^{0,b}=bF_{k,n}$ and

$$U_{k,\ell n}^{0,b}=\frac{bnF_{k,\ell}}{2^n}\int_{-1}^{1}(L_{k,\ell}+\Delta_k Fk,\ell x)^{n-1}dx.$$

(ii) If $b=\frac{ak}{2}$, then $U_{k,n}^{a,ak/2}=\frac{a}{2}L_{k,n}$ and

$$\begin{aligned}U_{k,\ell n}^{a,ak/2}=\frac{a}{2^{n+1}}\int_{-1}^{1}&(L_{k,\ell}+\Delta_k F_{k,\ell}x)^{n-1}\\ &\times (L_{k,\ell}+(n+1)\Delta_k F_{k,\ell}x)dx.\end{aligned}$$

Setting $k=1$, we have the following results.

Theorem 11. *Let ℓ and n be non-negative integers. Then*

$$\begin{aligned}G_{\ell n}^{a,b}&=\frac{1}{2^{n+1}}\int_{-1}^{1}(L_\ell+\sqrt{5}F_\ell x)^{n-1}\\ &\times\left(aL_\ell+(2b-a)nF_\ell+a(n+1)\sqrt{5}F_\ell x\right)dx.\end{aligned}$$

Corollary 12 ([41], Theorem 2.1)**.** *Let ℓ and n be non-negative integers. Then*

$$F_{\ell n}=\frac{nF_\ell}{2^n}\int_{-1}^{1}(L_\ell+\sqrt{5}F_\ell x)^{n-1}dx.$$

Corollary 13 ([41], Theorem 2.2)**.** *Let ℓ and n be non-negative integers. Then*

$$\begin{aligned}L_{\ell n}=\frac{1}{2^n}\int_{-1}^{1}&(L_\ell+\sqrt{5}F_\ell x)^{n-1}\\ &\times(L_\ell+\sqrt{5}(n+1)F_\ell x)dx.\end{aligned}$$

Setting $k=2$, we have the following results.

Theorem 14. *Let ℓ and n be non-negative integers. Then*

$$\begin{aligned}&H_{\ell n}^{a,b}=\frac{1}{2^{n+1}}\int_{-1}^{1}(Q_\ell+2\sqrt{2}P_\ell x)^{n-1}\\ &\times\left(aQ_\ell+2(b-a)nP_\ell+a(n+1)2\sqrt{2}P_\ell x\right)dx.\end{aligned}$$

Corollary 15 ([33], Theorem 3.1)**.** *Let ℓ and n be non-negative integers. Then*

$$P_{\ell n}=\frac{nP_\ell}{2^n}\int_{-1}^{1}(Q_\ell+2\sqrt{2}P_\ell x)^{n-1}dx.$$

Corollary 16 ([33], Theorem 3.4)**.** *Let ℓ and n be non-negative integers. Then*

$$\begin{aligned}Q_{\ell n}=\frac{1}{2^n}\int_{-1}^{1}&\times(Q_\ell+\sqrt{5}P_\ell x)^{n-1}\\ &(Q_\ell+\sqrt{8}(n+1)P_\ell x)dx.\end{aligned}$$

Remark 17. By applying Theorems 16 and 20 in [35] in conjunction with Theorem 6, we obtain the integral representations of the companion k-Fibonacci numbers $U_{k,\ell n+r}^{a,b}$.

3 The Companion k-Pell Numbers

In this section, we derive new integral representations for the companion k-Pell numbers using established relations between the k-Pell and k-Pell-Lucas numbers.

Theorem 18 (Binet's formulas)**.** *Let k and n be non-negative integers with $k\neq 0$. Then*

$$V_{k,n}^{a,b}=\frac{2b-2a+a\Lambda_k}{2\Lambda_k}\sigma_k^n+\frac{2a-2b+a\Lambda_k}{2\Lambda_k}\frac{(-k)^n}{\sigma_k^n},$$

where $\Lambda_k=2\sqrt{k+1}$ and $\sigma_k=1+\frac{\Lambda_k}{2}$.

Proof. A characteristic equation of the recurrence relation (2) is

$$r^2-2r-k=0.$$

Since $\Lambda_k^2=4(k+1)>0$ for $k\geq 1$, we have the roots

$$r_1=1+\frac{\Lambda_k}{2}=\sigma_k \quad\text{and}\quad r_2=1-\frac{\Lambda_k}{2}=\frac{-k}{\sigma_k}.$$

Note that $r_1+r_2=2$, $r_1-r_2=\Lambda_k$, and $r_1r_2=-k$. Then

$$V_{k,n}^{a,b}=\alpha r_1^n+\beta r_2^n=\alpha\sigma_k^n+\beta\frac{(-k)^n}{\sigma_k^n}$$

where α and β are the coefficients. Since $V_{k,0}^{a,b} = a$ and $V_{k,1}^{a,b} = b$, we get

$$\alpha + \beta = a \quad \text{and} \quad \alpha r_1 + \beta r_2 = b.$$

It can be shown that

$$\alpha = \frac{2b - 2a + a\Lambda_k}{2\Lambda_k} \quad \text{and} \quad \beta = \frac{2a - 2b + a\Lambda_k}{2\Lambda_k}.$$

Therefore,

$$V_{k,n}^{a,b} = \frac{2b - 2a + a\Lambda_k}{2\Lambda_k}\sigma_k^n + \frac{2a - 2b + a\Lambda_k}{2\Lambda_k}\frac{(-k)^n}{\sigma_k^n}$$

has been proved. □

Corollary 19 ([12, Proposition 1], [13, Proposition 1]). *Let k and n be non-negative integers with $k \neq 0$. Then*

(i) $P_{k,n} = \frac{1}{\Lambda_k}\left(\sigma_k^n - \frac{(-k)^n}{\sigma_k^n}\right)$;

(ii) $Q_{k,n} = \sigma_k^n + \frac{(-k)^n}{\sigma_k^n}$,

where $\Lambda_k = 2\sqrt{k+1}$ and $\sigma_k = 1 + \frac{\Lambda_k}{2}$.

Proof. Setting $(a, b) \in \{(0, 1), (2, k)\}$ in Theorem 18, this completes the proof. □

Remark 20. By setting $k = 1$ and $k = 4$ in Theorem 18, we obtain Corollary 4. Indeed,

(i) Notice that $H_n^{a,b} = V_{1,n}^{a,b}$, $\Lambda_1 = 2\sqrt{2}$, and $\sigma_1 = \frac{1+\sqrt{2}}{2}$. Setting $k = 1$ in Theorem 18, we have $H_n^{a,b} = \frac{b-a+\sqrt{2}a}{2\sqrt{2}}(1+\sqrt{2})^n + \frac{a-b-\sqrt{2}a}{2\sqrt{2}}(1-\sqrt{2})^n$.

(ii) We see that $\Lambda_4 = 2\sqrt{5}$, and $\sigma_4 = 1+\sqrt{5}$. Setting $k = 4$ in Theorem 18,

$$\begin{aligned}
V_{4,n}^{a,2b} &= \frac{2(2b) - 2a + a\Lambda_4}{2\Lambda_4}\sigma_4^n \\
&\quad + \frac{2a - 2(2b) + a\Lambda_4}{2\Lambda_4}\frac{(-4)^n}{\sigma_4^n} \\
&= \frac{4b - 2a + 2\sqrt{5}a}{4\sqrt{5}}(1+\sqrt{5})^n \\
&\quad + \frac{2a - 4b + 2\sqrt{5}a}{4\sqrt{5}}\frac{(-4)^n}{(1+\sqrt{5})^n} \\
&= \frac{2b - a + \sqrt{5}a}{2\sqrt{5}}(1+\sqrt{5})^n \\
&\quad + \frac{a - 2b + \sqrt{5}a}{2\sqrt{5}}(1-\sqrt{5})^n.
\end{aligned}$$

From Theorem 1, we get $G_n^{a,b} = \frac{1}{2^n}V_{4,n}^{a,2b}$ and so $G_n^{a,b} = \frac{2b-a+\sqrt{5}a}{2\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n + \frac{a-2b-\sqrt{5}a}{2\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n$ holds.

Lemma 21 ([34, Lemma 2.1]). *Let k and n be non-negative integers with $k \neq 0$. Then*

(i) $Q_{k,n} + \Lambda_k P_{k,n} = 2\sigma_k^n$;

(ii) $Q_{k,n} - \Lambda_k P_{k,n} = 2\frac{(-k)^n}{\sigma_k^n}$;

(iii) $Q_{k,n}^2 - \Lambda_k^2 P_{k,n}^2 = 4(-k)^n$,

where $\Lambda_k = 2\sqrt{k+1}$ and $\sigma_k = 1 + \frac{\Lambda_k}{2}$.

The following presents the companion k-Pell numbers in relation to the k-Pell and k-Pell-Lucas numbers.

Theorem 22. *Let k and n be non-negative integers with $k \neq 0$. Then*

$$V_{k,n}^{a,b} = \frac{a}{2}Q_{k,n} + (b-a)P_{k,n}.$$

Proof. From Theorem 18, (i) and (ii) of Lemma 21, we get

$$\begin{aligned}
V_{k,n}^{a,b} &= \frac{2b - 2a + a\Lambda_k}{2\Lambda_k}\sigma_k^n + \frac{2a - 2b + a\Lambda_k}{2\Lambda_k}\frac{(-k)^n}{\sigma_k^n} \\
&= \left(\frac{2b - 2a + a\Lambda_k}{2\Lambda_k}\right)\left(\frac{Q_{k,n} + \Lambda_k P_{k,n}}{2}\right) \\
&\quad + \left(\frac{2a - 2b + a\Lambda_k}{2\Lambda_k}\right)\left(\frac{Q_{k,n} - \Lambda_k P_{k,n}}{2}\right) \\
&= \frac{a}{2}Q_{k,n} + (b-a)P_{k,n}.
\end{aligned}$$

This completes this proof. □

Remark 23. By setting $k = 1$ and $k = 4$ in Theorem 22, we obtain Corollary 7. Indeed,

(i) Notice that $H_n^{a,b} = V_{1,n}^{a,b}$, $P_n = P_{1,n}$ and $Q_n = Q_{1,n}$. Setting $k = 1$ in Theorem 22, we have $H_n^{a,b} = \frac{a}{2}Q_n + (b-a)P_n$ holds.

(ii) Notice that $G_n^{a,b} = \frac{1}{2^n}V_{4,n}^{a,2b}$, $F_n = \frac{1}{2^{n-1}}P_{4,n}$ and $L_n = \frac{1}{2^n}Q_{4,n}$. Setting $k = 4$ in Theorem 22, we have

$$\begin{aligned}
G_n^{a,b} &= \frac{1}{2^n}V_{4,n}^{a,2b} \\
&= \frac{1}{2^n}\left(\frac{a}{2}Q_{4,n} + (2b-a)P_{4,n}\right) \\
&= \frac{a}{2}\left(\frac{1}{2^n}Q_{4,n}\right) + \frac{2b-a}{2}\left(\frac{1}{2^{n-1}}P_{4,n}\right) \\
&= \frac{a}{2}L_n + \frac{2b-a}{2}F_n.
\end{aligned}$$

The following theorem is presented in [34].

Theorem 24 ([34, Theorems 3.1 and 3.5])**.** *Let k, ℓ and n be non-negative integers with $k \neq 0$. Then*

$$P_{k,\ell n} = \frac{nP_{k,\ell}}{2^n} \int_{-1}^{1} (Q_{k,\ell} + \Lambda_k \, P_{k,\ell} x)^{n-1} dx$$

and

$$Q_{k,\ell n} = \frac{1}{2^n} \int_{-1}^{1} (Q_{k,\ell} + \Lambda_k \, P_{k,\ell} x)^{n-1} \times (Q_{k,\ell} + (n+1)\Lambda_k P_{k,\ell} x) dx,$$

where $\Lambda_k = 2\sqrt{k+1}$.

The following is the new integral representation for the companion k-Pell numbers:

Theorem 25. *Let k, ℓ and n be non-negative integers with $k \neq 0$. Then*

$$V_{k,\ell n}^{a,b} = \frac{1}{2^{n+1}} \int_{-1}^{1} (Q_{k,\ell} + \Lambda_k P_{k,\ell} x)^{n-1} \times (aQ_{k,\ell} + 2(b-a)n\Lambda_{k,\ell} + a(n+1)\Lambda_k P_{k,\ell} x)\, dx,$$

where $\Lambda_k = 2\sqrt{k+1}$.

Proof. From (22), we obtain

$$V_{k,\ell n}^{a,b} = \frac{a}{2} Q_{k,\ell n} + (b-a) P_{k,\ell n}.$$

Applying Theorem 24, we get

$$\begin{aligned} V_{k,\ell n}^{a,b} &= \frac{a}{2} \left(\frac{1}{2^n} \int_{-1}^{1} (Q_{k,\ell} + \Lambda_k P_{k,\ell} x)^{n-1} \times (Q_{k,\ell} + (n+1)\Lambda_k P_{k,\ell} x) dx \right) \\ &\quad + (b-a) \left(\frac{nP_{k,\ell}}{2^n} \int_{-1}^{1} (Q_{k,\ell} + \Lambda_k P_{k,\ell} x)^{n-1} dx \right) \\ &= \frac{1}{2^{n+1}} \int_{-1}^{1} (Q_{k,\ell} + \Lambda_k Q_{k,\ell} x)^{n-1} \times (aQ_{k,\ell} + 2(b-a)nP_{k,\ell} + a(n+1)\Lambda_k P_{k,\ell} x)\, dx. \end{aligned}$$

This completes the proof. □

Remark 26. As in Theorems 22 and 25, we have the following results.

(i) If $a = 0$, then $V_{k,n}^{0,b} = bP_{k,n}$ and

$$V_{k,\ell n}^{0,b} = \frac{bnP_{k,\ell}}{2^n} \int_{-1}^{1} (Q_{k,\ell} + \Lambda_k Pk, \ell x)^{n-1} dx.$$

(ii) If $a = b$, then $U_{k,n}^{a,a} = \frac{a}{2} Q_{k,n}$ and

$$V_{k,\ell n}^{a,a} = \frac{a}{2^{n+1}} \int_{-1}^{1} (Q_{k,\ell} + \Lambda_k P_{k,\ell} x)^{n-1} \times (Q_{k,\ell} + (n+1)\Lambda_k P_{k,\ell} x) dx.$$

(iii) If $k = 1$, then we have the same Theorem 14 as follows:

$$H_{\ell n}^{a,b} = \frac{1}{2^{n+1}} \int_{-1}^{1} (Q_\ell + 2\sqrt{2} P_\ell x)^{n-1} \times \left(aQ_\ell + 2(b-a)nP_\ell + a(n+1)2\sqrt{2} P_\ell x \right) dx.$$

(iv) If $k = 4$, then we have the same Theorem 11 as follows:

$$G_{\ell n}^{a,b} = \frac{1}{2^{n+1}} \int_{-1}^{1} (L_\ell + \sqrt{5} F_\ell x)^{n-1} \times \left(aL_\ell + (2b-a)nF_\ell + a(n+1)\sqrt{5} F_\ell x \right) dx.$$

Remark 27. By applying Theorems 3.8 and 3.11 in [34], in conjunction with Theorem 22, we obtain the integral representations of the companion k-Pell numbers $V_{k,\ell n+r}^{a,b}$.

4 Conclusions

In this paper, we investigate the companion numbers of the Fibonacci and Pell sequences that preserve the recurrence relations under arbitrary initial conditions. We establish integral representations of the companion k-Pell numbers associated with the k-Fibonacci and k-Lucas numbers, as well as those associated with the k-Pell and k-Pell-Lucas numbers.

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APPENDIX

The initial terms of $\{U_{k,n}\}$ and $\{V_{k,n}\}$ are presented in Table 1 and Table 2, respectively.

Table 1: Initial terms of $\{U_{k,n}\}$

k	a	b	$U_{k,n}$	code in EOIS, [44]
1	0	1	$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \dots$	A000045
1	1	3	$1, 3, 4, 7, 11, 18, 29, 47, 76, 123, \dots$	A000204
1	1	4	$1, 4, 5, 9, 14, 23, 37, 60, 97, 157, \dots$	A000285
1	1	5	$1, 5, 6, 11, 17, 28, 45, 73, 118, 191, \dots$	A022095
1	1	6	$1, 6, 7, 13, 20, 33, 53, 86, 139, 225, \dots$	A022096
1	2	1	$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, \dots$	A000032
1	2	3	$2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \dots$	A020695
2	0	1	$0, 1, 2, 5, 12, 29, 70, 169, 408, 985, \dots$	A000129
2	1	1	$1, 1, 3, 7, 17, 41, 99, 239, 577, 1393, \dots$	A001333
2	1	3	$1, 3, 7, 17, 41, 99, 239, 577, 1393, 3363, \dots$	A078057
2	1	4	$1, 4, 9, 22, 53, 128, 309, 746, 1801, 4348, \dots$	A048654
2	1	5	$1, 5, 11, 27, 65, 157, 379, 915, 2209, 5333, \dots$	A048655
2	2	2	$2, 2, 6, 14, 34, 82, 198, 478, 1154, 2786, \dots$	A002203
2	2	3	$2, 3, 8, 19, 46, 111, 268, 647, 1562, 3771, \dots$	A078343
2	2	4	$2, 4, 10, 24, 58, 140, 338, 816, 1970, 4756, \dots$	A052542
3	0	1	$0, 1, 3, 10, 33, 109, 360, 1189, 3927, 12970, 42837, \dots$	A006190
3	1	1	$1, 1, 4, 13, 43, 142, 469, 1549, 5116, 16897, \dots$	-
3	1	2	$1, 2, 7, 23, 76, 251, 829, 2738, 9043, 29867, \dots$	A052924
3	1	4	$1, 4, 13, 43, 142, 469, 1549, 5116, 16897, 55807, \dots$	A003688
3	1	5	$1, 5, 16, 53, 175, 578, 1909, 6305, 20824, 68777, \dots$	A108300
3	2	1	$2, 1, 5, 16, 53, 175, 578, 1909, 6305, 20824, \dots$	-
3	2	2	$2, 2, 8, 26, 86, 284, 938, 3098, 10232, 33794, \dots$	-
3	2	3	$2, 3, 11, 36, 119, 393, 1298, 4287, 14159, 46764, \dots$	A006497
3	2	5	$2, 5, 17, 56, 185, 611, 2018, 6665, 22013, 72704, \dots$	A180148
3	3	1	$3, 1, 6, 19, 63, 208, 687, 2269, 7494, 24751, \dots$	A189735
3	3	2	$3, 2, 9, 29, 96, 317, 1047, 3458, 11421, 37721, \dots$	-
4	0	1	$0, 1, 4, 17, 72, 305, 1292, 5473, 23184, 98209, \dots$	A001076
4	1	1	$1, 1, 5, 21, 89, 377, 1597, 6765, 28657, 121393, \dots$	A015448
4	1	2	$1, 2, 9, 38, 161, 682, 2889, 12238, 51841, 219602, \dots$	A001077
4	1	3	$1, 3, 13, 55, 233, 987, 4181, 17711, 75025, 317811, \dots$	A033887
5	0	1	$0, 1, 5, 26, 135, 701, 3640, 18901, 98145, 509626 \dots$	-
5	1	1	$1, 1, 6, 31, 161, 836, 4341, 22541, 117046, 607771, \dots$	A015449
5	1	2	$1, 2, 11, 57, 296, 1537, 7981, 41442, 215191, 1117397, \dots$	A164581
5	1	3	$1, 3, 16, 83, 431, 2238, 11621, 60343, 313336, 1627023, \dots$	-
5	1	4	$1, 4, 21, 109, 566, 2939, 15261, 79244, 411481, 2136649, \dots$	A100237
5	1	5	$1, 5, 26, 135, 701, 3640, 18901, 98145, 509626, 2646275, \dots$	A052918

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Table 2: Initial terms of $\{V_{k,n}\}$

k	a	b	$V_{k,n}$	code in EOIS, [44]
1	0	1	$0, 1, 2, 5, 12, 29, 70, 169, 408, 985, \ldots$	A000129
1	1	1	$1, 1, 3, 7, 17, 41, 99, 239, 577, 1393, \ldots$	A001333
1	1	3	$1, 3, 7, 17, 41, 99, 239, 577, 1393, 3363, \ldots$	A078057
1	1	4	$1, 4, 9, 22, 53, 128, 309, 746, 1801, 4348, \ldots$	A048654
1	1	5	$1, 5, 11, 27, 65, 157, 379, 915, 2209, 5333, \ldots$	A048655
1	2	2	$2, 2, 6, 14, 34, 82, 198, 478, 1154, 2786, \ldots$	A002203
2	0	1	$0, 1, 2, 6, 16, 44, 120, 328, 896, 2448, \ldots$	A002605
2	1	1	$1, 1, 4, 10, 28, 76, 208, 568, 1552, 4240, \ldots$	A026150
2	1	3	$1, 3, 8, 22, 60, 164, 448, 1224, 3344, 9136, \ldots$	A028859
2	1	5	$1, 5, 12, 34, 92, 252, 688, 1880, 5136, 14032, \ldots$	-
2	2	2	$2, 2, 8, 20, 56, 152, 416, 1136, 3104, 8480, \ldots$	A080040
2	2	3	$2, 3, 10, 26, 72, 196, 536, 1464, 4000, 10928, \ldots$	-
3	0	1	$0, 1, 2, 7, 20, 61, 182, 547, 1640, 4921, \ldots$	A015518
3	1	1	$1, 1, 5, 13, 41, 121, 365, 1093, 3281, 9841, \ldots$	A046717
3	1	3	$1, 3, 9, 27, 81, 243, 729, 2187, 6561, 19683, \ldots$	A000244
3	1	4	$1, 4, 11, 34, 101, 304, 911, 2734, 8201, 24604, \ldots$	A060925
3	1	5	$1, 5, 13, 41, 121, 365, 1093, 3281, 9841, 29525, \ldots$	A164907
3	2	2	$2, 2, 10, 26, 82, 242, 730, 2186, 6562, \ldots$	A102345
3	2	3	$2, 3, 12, 33, 102, 303, 912, 2733, 8202, 24603, \ldots$	A135522
3	2	4	$2, 4, 14, 40, 122, 364, 1094, 3280, 9842, 29524, \ldots$	A152011
3	2	5	$2, 5, 16, 47, 142, 425, 1276, 3827, 11482, 34445, \ldots$	-
3	2	6	$2, 6, 18, 54, 162, 486, 1458, 4374, 13122, 39366, \ldots$	A008776
4	0	1	$0, 1, 2, 8, 24, 80, 256, 832, 2688, 8704, \ldots$	A085449
4	0	2	$0, 2, 4, 16, 48, 160, 512, 1664, 5376, 17408, \ldots$	A103435
4	1	1	$1, 1, 6, 16, 56, 176, 576, 1856, 6016, 19456, \ldots$	A084057
4	1	2	$1, 2, 8, 24, 80, 256, 832, 2688, 8704, 28160, \ldots$	A063727
4	1	3	$1, 3, 10, 32, 104, 336, 1088, 3520, 11392, 36864, \ldots$	A063782
4	1	4	$1, 4, 12, 40, 128, 416, 1344, 4352, 14080, 45568, \ldots$	A087206
4	1	5	$1, 5, 14, 48, 152, 496, 1600, 5184, 16768, 54272, \ldots$	A134418
4	2	1	$2, 1, 10, 24, 88, 272, 896, 2880, 9344, 30208, \ldots$	-
4	2	2	$2, 2, 12, 32, 112, 352, 1152, 3712, 12032, 38912, \ldots$	A087131
4	2	3	$2, 3, 14, 40, 136, 432, 1408, 4544, 14720, 47616, \ldots$	-
5	0	1	$0, 1, 2, 9, 28, 101, 342, 1189, 4088, 14121, \ldots$	A002532
5	1	1	$1, 1, 7, 19, 73, 241, 847, 2899, 10033, 34561, \ldots$	A002533
5	1	3	$1, 3, 11, 37, 129, 443, 1531, 5277, 18209, 62803, \ldots$	A180168
5	1	4	$1, 4, 13, 46, 157, 544, 1873, 6466, 22297, 76924, \ldots$	-
5	1	5	$1, 5, 15, 55, 185, 645, 2215, 7655, 26385, 91045 \ldots$	A123011

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