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Radon-Nikodym Derivative Formula of a Shifted Gaussian Random Field under General Shift

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Abstract: Gaussian process considered in this paper is the Slepian fiel which plays a crucial role in testing for constancy of parameters of a linear regression with spatial observations. The Slepian fiel appeared as the limit process of the sequence of moving rectangle processes of recursive residuals. In this work we establish a sufficien condition for the existence of Radon-Nikodym (R-N) derivative of a shifted two parameters Slepian field More precisely we determine an infinit dimensional space of shift functions as well as the corresponding norm in which the derivative of the shifted two parameters Slepian fiel exists. The exact formula of the derivative which is commonly called Cameron-Martin formula is also derived. The result is obtained by means of the distributional property of the Slepian random field.

Key–Words: - Slepian random field, Wiener integral, Cameron-Martin formula, moving sums process, reproducing kernel, Gaussian random field, Radon-Nikodym derivative, model check, and limit process.

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1 Introduction

As mentioned in the literature on testing for the adequacy of a regression function in linear regression with spatial observations using the partial sums method, the limiting distribution of the sequence of the partial sums processes of the residuals when the model is far from adequate has been given by a shifted centered Gaussian random field [1]. The power of the test has been represented by some boundary-crossing probabilities whose behavior can be conveniently studied by investigating the R-N derivative of the limit process. This means the formula of the derivative plays an important role in this context, [2]. To understand the problem, let

$$\mathcal{Z}_{\mathbf{D}} := \{Z(\mathbf{t}) : \mathbf{t} \in \mathbf{D} \subset \mathcal{R}^d\}$$

be a centered Gaussian random fiel with sample paths in the set of continuous functions on $\mathbf{D}$, denoted by $\mathcal{C}(\mathbf{D})$. Let us consider a shifted version of the Gaussian random fiel $\mathcal{Z}_{\mathbf{D}}$ along h denoted by

$$(\mathcal{Z}+h)_{\mathbf{D}} := \{Z(\mathbf{t}) + h(\mathbf{t}) : \mathbf{t} \in \mathbf{D}\},$$

where h is a real-valued function in $\mathcal{C}(\mathbf{D})$. As it has been documented in the literatures, the conditions under which the concrete formula of a derivative $\frac{d\mathbf{P}^{(\mathcal{Z}+h)_{\mathbf{D}}}}{d\mathbf{P}^{\mathcal{Z}_{\mathbf{D}}}}$ exists, [3], [4], and [5], where $\mathbf{P}^{\mathcal{Z}_{\mathbf{D}}}$ and $\mathbf{P}^{(\mathcal{Z}+h)_{\mathbf{D}}}$ denote the probability distributions of $\mathcal{Z}_{\mathbf{D}}$ and $(\mathcal{Z}+h)_{\mathbf{D}}$, respectively.

In case the random fiel is given by the standard Brownian sheet, which is also frequently called the Wiener-Censtov process, the Cameron-Martin formula of the derivative is well known. Let

$$\mathcal{W}_{\mathbf{I}^d} := \left\{W(\mathbf{t}) : \mathbf{t} \in \mathbf{I}^d = [0,1]^d\right\}$$

be the d-variate standard Brownian sheets indexed by the points in the unit closed rectangle $\mathbf{I}^d$ with sample paths in $\mathcal{C}(\mathbf{I}^d)$. It has been shown that $\mathbf{P}^{(\mathcal{W}+h)_{\mathbf{I}^d}}$ is absolutely continuous with respect to $\mathbf{P}^{\mathcal{W}_{\mathbf{I}^d}}$, if and only if h is an admissible shift in the sense h becomes an element of the kernel of $\mathcal{W}_{\mathbf{I}^d}$, [3]. Additionally, the Cameron-Martin formula of the density is given by

$$\frac{d\mathbf{P}^{(\mathcal{W}+h)_{\mathbf{I}^d}}}{d\mathbf{P}^{\mathcal{W}_{\mathbf{I}^d}}} = \exp\left\{-\frac{1}{2}\|h\|^2_{\mathcal{H}_{\mathcal{W}_{\mathbf{I}^d}}} + \int_{\mathbf{I}^d} h'(\mathbf{t})dx(\mathbf{t})\right\},$$

for $\mathbf{P}^{\mathcal{W}_{\mathbf{I}^d}}$ – almost all $x \in \mathcal{C}(\mathbf{I}^d)$, where $\|\cdot\|_{\mathcal{H}_{\mathcal{W}_{\mathbf{I}^d}}}$ is the norm in the kernel $\mathcal{H}_{\mathcal{W}_{\mathbf{I}^d}}$ of $\mathcal{W}_{\mathbf{I}^d}$ and h' is a function of bounded variation on $\mathbf{I}^d$, such that

$$\int_{[\mathbf{0},\mathbf{t}]} h'(\mathbf{u})d\mathbf{u} = h(\mathbf{t}),\ \forall \mathbf{t} := (t_1,\dots,t_d) \in \mathbf{I}^d,$$

with $[0,\mathbf{t}] := \Pi_{i=1}^d [0,t_i]$. This Cameron-Martin formula can be used, for example, in calculating the rate of decay of the power of an unbiased test in linear regression to its pre signed size, [4], [5], and [2], and for estimating the boundary-crossing probabilities, [6].

By considering the reason described above, it is of great interest to investigate the Cameron-Martin formula for other centered Gaussian random field that have frequently appeared in test problems. To this end, Theorem 5.1 of [3], can be used as a framework, see also [4], and [5]. The abstract Cameron-Martin theorem is as follows. Let $\mathcal{H}_{\mathcal{Z}_{\mathbf{D}}} \subset \mathcal{C}(\mathbf{D})$ be the kernel of $\mathcal{Z}_{\mathbf{D}}$ furnished with the norm $\|\cdot\|_{\mathcal{H}_{\mathcal{Z}_{\mathbf{D}}}}$. Then $\mathbf{P}^{(\mathcal{Z}+h)_{\mathbf{D}}}$ is absolutely continuous with respect to $\mathbf{P}^{\mathcal{Z}_{\mathbf{D}}}$, if and only if $h \in \mathcal{H}_{\mathcal{Z}_{\mathbf{D}}}$. Furthermore, if $h \in \mathcal{H}_{\mathcal{Z}_{\mathbf{D}}}$, then for $\mathbf{P}^{\mathcal{Z}_{\mathbf{D}}}-$ almost all $x \in \mathcal{C}(\mathbf{D})$,

$$\frac{d\mathbf{P}^{(\mathcal{Z}+h)_{\mathbf{D}}}}{d\mathbf{P}^{\mathcal{Z}_{\mathbf{D}}}}(x) = \exp\left\{-\frac{1}{2}\|h\|^2_{\mathcal{H}_{\mathcal{Z}_{\mathbf{D}}}} + \mathbf{A}(h,x)\right\} \quad (1)$$

where $\mathbf{A}$ is a linear functional define in $\mathcal{H}_{\mathcal{Z}_{\mathbf{D}}} \times \mathcal{C}(\mathbf{D})$ satisfying the property

$$\mathbf{E}(Z(\mathbf{t})\mathbf{A}(h,x)) = h(\mathbf{t}), \ \forall \mathbf{t} \in \mathbf{D}.$$

Equation 1 suggests that to be able to obtain the formula of the R-N derivative of $\mathbf{P}^{(\mathcal{Z}+h)_{\mathbf{D}}}$, with respect to $\mathbf{P}^{\mathcal{Z}_{\mathbf{D}}}$, the structure of $\mathcal{H}_{\mathcal{Z}_{\mathbf{D}}}$ and the definitio of the associated norm $\|\cdot\|_{\mathcal{H}_{\mathcal{Z}_{\mathbf{D}}}}$ together with the formula of the linear functional $\mathbf{A}(h,x)$ must be concretely determined.

We notice that for the case of the d-variate standard Brownian sheets $\mathcal{W}_{\mathbf{I}^d}$, the kernel of $\mathbf{P}^{\mathcal{W}_{\mathbf{I}^d}}$ is a linear subspace $\mathcal{H}_{\mathcal{W}_{\mathbf{I}^d}}$, given by

$$\mathcal{H}_{\mathcal{W}_{\mathbf{I}^d}} = \left\{h \mid \exists h' \in \mathcal{L}^2, h(\mathbf{t}) = \int_{[\mathbf{0},\mathbf{t}]} h'(\mathbf{u})d\mathbf{u}\right\}. \quad (2)$$

The associated norm in $\mathcal{H}_{\mathcal{W}_{\mathbf{I}^d}}$ is define as

$$\|h\|^2_{\mathcal{H}_{\mathcal{W}}} := \int_{\mathbf{I}^d} |h'(\mathbf{u})|^2 d\mathbf{u} = \|h'\|^2_{\mathcal{L}^2},$$

where $\mathcal{L}^2$ is the space of square integrable functions with respect to the Lebesque measure on $\mathbf{I}^d$. Correspondingly, the linear functional $\mathbf{A}(h,x)$ is define by

$$\mathbf{A}(h,x) := \int_{\mathbf{I}^d} h'(\mathbf{u})dx(\mathbf{u}),$$

for $h \in \mathcal{H}_{\mathcal{W}}$ and for $\mathbf{P}^{\mathcal{W}_{\mathbf{I}^d}}-$ almost all $x \in \mathcal{C}(\mathbf{I}^d)$. We notice that the last integral is define in term of Wiener integral, so that we get the following result

$$\begin{aligned}\mathbf{E}\left(W(\mathbf{t})\mathbf{A}(h,W)\right) &= \mathbf{E}\left(W(\mathbf{t})\int_{\mathbf{I}^d} h'(\mathbf{u})dW(\mathbf{u})\right) \\ &= \int_{[\mathbf{0},\mathbf{t}]} h'(\mathbf{u})d\mathbf{u} = h(\mathbf{t}). \quad (3)\end{aligned}$$

The organization of the paper is as follows. In Section 2 we discuss multi-parameter Slepian random fiel and its relation to other well-known centered Gaussian fields such as the standard Brownian sheets. The formula for the density of the shifted two-parameter Slepian random field will be proved in Section 3

2 The d-Parameter Slepian Field

A centered Gaussian random fiel

$$\mathcal{S} := \left\{S(\mathbf{t}),\ \mathbf{t} = (t_1,\ldots,t_d)^\top \in \mathcal{R}^d\right\}$$

is called d-parameter Slepian random field if it has a covariance function define by

$$\mathbf{C}_{\mathcal{S}}(\mathbf{t},\mathbf{t}') := \Pi_{i=1}^d (1-|t_i - t_i'|)^+,$$

for $\mathbf{t} = (t_1,\ldots,t_d)^\top$ and $\mathbf{t}' = (t_1',\ldots,t_d')^\top$ in $\mathcal{R}^d$, where $u^+ := \max\{0,u\}$, for $u \in \mathcal{R}$. The d-parameter Slepian random fiel is a generalization of the one-parameter Slepian process define on $\mathcal{R}$, which has been introduced and studied in many literatures, [7], [8], [9], [10], and [11]. In particular, [12] established the upper bound for a boundary-crossing probability involving the one-parameter Slepian process. Afterward, [13] investigated small ball probability for two-parameter Slepian random field

One convenient method of definin the d-parameter Slepian random fiel is by means of a difference operator. For every vector of positive constants $\mathbf{a} = (a_1,\ldots,a_d)^\top \in \mathcal{R}^d$ and every real-valued function F on $\mathcal{R}^d$, let $\mathbf{\Delta}_{a_i}F$ be define by

$$\begin{aligned}&\mathbf{\Delta}_{a_i}F := \\ &F(t_1,\ldots,t_i,\ldots,t_d) - F(t_1,\ldots,t_i - a_i,\ldots,t_d).\end{aligned}$$

For a fi ed i, the quantity $\mathbf{\Delta}_{a_i}F$ measures the difference of F in the i-th component of $\mathbf{t}$ in amount of a_i. Then by referring to [14], the composition $\mathbf{\Delta}_{a_1} \circ \cdots \circ \mathbf{\Delta}_{a_d}$, when it is applied to a function F on $\mathcal{R}^d$, will defin the increment of F on the closed rectangle $[\mathbf{t}-\mathbf{a},\mathbf{t}] := \Pi_{i=1}^d [t_i - a_i, t_i]$, given by

$$(\mathbf{\Delta}_{a_1} \circ \cdots \circ \mathbf{\Delta}_{a_d})(F) = \mathbf{\Delta}_{a_1}(\cdots(\mathbf{\Delta}_{a_d}(F))\ldots).$$

As an example, let us consider the case of $d=2$. Then the increment of F on the closed rectangle $[t_1-a_1,t_1] \times [t_2-a_2,t_2]$ is given by

$$\begin{aligned}&\mathbf{\Delta}_{a_1}(\mathbf{\Delta}_{a_2}(F(t_1,t_2))) \\ &= \mathbf{\Delta}_{a_1}(F(t_1,t_2) - F(t_1,t_2-a_2)) \\ &= F(t_1,t_2) - F(t_1,t_2-a_2) \\ &-F(t_1-a_1,t_2) + F(t_1-a_1,t_2-a_2).\end{aligned}$$

Let $\mathcal{W}_{[\mathbf{0},\mathbf{b}]}$ be the d-parameter standard Brownian sheet with continuous paths in the d-dimensional closed rectangle $[\mathbf{0},\mathbf{b}]$, where $0 < b_i$, for $i = 1,\ldots,d$. The process $\mathcal{W}_{[\mathbf{0},\mathbf{b}]}$ is also called multi-parameters Wiener process, [15] and [16]). Next by using the difference operator we defin the following stochastic process

$$\begin{aligned}&(\mathbf{\Delta}_{a_1}\circ\cdots\circ\mathbf{\Delta}_{a_d})(W)_{[\mathbf{a},\mathbf{b}]}\\&=\{(\mathbf{\Delta}_{a_1}(\cdots(\mathbf{\Delta}_{a_d}(W(\mathbf{t}))\ldots)):\ \mathbf{t}\in[\mathbf{a},\mathbf{b}]\},\end{aligned}$$

where $[\mathbf{a},\mathbf{b}] := \Pi_{i=1}^d[a_i,b_i]$, with $a_i < b_i$, for $i = 1,\ldots,d$. It is clear that for every $\mathbf{t}\in[\mathbf{a},\mathbf{b}]$, the process $(\mathbf{\Delta}_{a_1}(\cdots(\mathbf{\Delta}_{a_d}(W(\mathbf{t}))\ldots))$ is exactly the increment of the process $\mathcal{W}_{[\mathbf{a},\mathbf{b}]}$ on the closed rectangle $[\mathbf{t}-\mathbf{a},\mathbf{t}]$. Hence, $(\mathbf{\Delta}_{a_1}\circ\cdots\circ\mathbf{\Delta}_{a_d})(W)_{[\mathbf{a},\mathbf{b}]}$ is a Gaussian random fiel with the property

$$\mathbf{E}\left((\mathbf{\Delta}_{a_1}\circ\cdots\circ\mathbf{\Delta}_{a_d})(W(\mathbf{t}))\right)=0$$

and the covariance

$$\begin{aligned}&\mathcal{C}\left((\mathbf{\Delta}_{a_1}\circ\cdots\circ\mathbf{\Delta}_{a_d})(W(\mathbf{t}))\right.\\&\left.,\ (\mathbf{\Delta}_{a_1}\circ\cdots\circ\mathbf{\Delta}_{a_d})(W(\mathbf{s}))\right)\\&=\lambda^d\left([\mathbf{t}-\mathbf{a},\mathbf{t}]\cap[\mathbf{s}-\mathbf{a},\mathbf{s}]\right)\\&=\Pi_{i=1}^d(a_i-|t_i-s_i|)^+,\end{aligned}\tag{4}$$

for every $\mathbf{t} = (t_1,\ldots,t_d)^\top$ and $\mathbf{s} = (s_1,\ldots,s_d)^\top$ in $[\mathbf{a},\mathbf{b}]$, where λ^d is the Lebesque measure on $[\mathbf{a},\mathbf{b}]$. Obviously, the d-parameters Slepian random fiel

$$\mathcal{S}_{[\mathbf{1},\mathbf{b}]} := \{S(\mathbf{t}):\ \mathbf{t}\in[\mathbf{1},\mathbf{b}]\}$$

coincides in distribution with the process

$$(\mathbf{\Delta}_1\circ\cdots\circ\mathbf{\Delta}_1)(W)_{[\mathbf{1},\mathbf{b}]},$$

since the last also constitutes a centered Gaussian random fiel owning the same covariance function as that of $\mathcal{S}_{[\mathbf{1},\mathbf{b}]}$. In particular, for $s_i = t_i + u_i$, with $0 < u_i$, for $i = 1,\ldots,d$, we have

$$\begin{aligned}\mathcal{C}((\mathbf{\Delta}_1\circ\cdots\circ\mathbf{\Delta}_1)(W(\mathbf{t})),(\mathbf{\Delta}_1\circ\cdots\circ\mathbf{\Delta}_1)(W(\mathbf{s})))\\=\Pi_{i=1}^d(1-u_i)^+.\end{aligned}$$

The application of the one-parameter Slepian process, particularly in statistical inference was pioneered by [17]. He showed that the one-parameter Slepian process, given by

$$\frac{1}{\sqrt{h}}(\mathbf{\Delta}_hB)_{[h,1]} := \left\{\frac{1}{\sqrt{h}}(\mathbf{\Delta}_hB(t)):\ t\in[h,1]\right\},$$

have appeared as the limit process of the scan statistic, where $\{B(t):\ t\in[0,b]\}$ is the standard Brownian motion with continuous sample paths in the closed interval $[0,b]$. Another application of the Slepian process in statistics, especially in the problem of parameter change detection in linear regression models can be found in the literature of parameter change detection, [18]. They define a test based on the moving sum processes of the recursive residuals. When the number of the residuals goes to infinit , they showed that the limit of the moving sum processes was obtained as the Slepian process. In this paper, we consider the d-parameter Slepian random field define by

$$\begin{aligned}&\frac{(\mathbf{\Delta}_{h_1}\circ\cdots\circ\mathbf{\Delta}_{h_d})(W)_{[\mathbf{h},1]}}{\sqrt{\Pi_{i=1}^d h_i}}\\&=\left\{\frac{(\mathbf{\Delta}_{h_1}\circ\cdots\circ\mathbf{\Delta}_{h_d})(W(\mathbf{t}))}{\sqrt{\Pi_{i=1}^d h_i}}:\ \mathbf{t}\in[\mathbf{h},1]\right\},\end{aligned}$$

with the covariance function given by

$$\begin{aligned}\mathbf{C}(\mathbf{t},\mathbf{t}+\mathbf{u}) &= \Pi_{i=1}^d\frac{1}{h_i}(h_i-u_i)^+\\&= \Pi_{i=1}^d\left(1-\frac{u_i}{h_i}\right)^+.\end{aligned}$$

It is worth mentioning that by applying a suitable scaling in the coordinate point $\mathbf{t} = (t_1,\ldots,t_d)^\top$, the d-parameter Slepian random fiel $(\mathbf{\Delta}_1\circ\cdots\circ\mathbf{\Delta}_1)(W)_{[\mathbf{1},\mathbf{b}]}$ can be transferred to the d-parameter Slepian random fiel

$$\frac{(\mathbf{\Delta}_{h_1}\circ\cdots\circ\mathbf{\Delta}_{h_d})(W)_{[\mathbf{h},1]}}{\sqrt{\Pi_{i=1}^d h_i}}$$

by setting $h_i = \frac{1}{b_i}$, for $i = 1,\ldots,d$. More precisely, it is true that $(\mathbf{\Delta}_1\circ\cdots\circ\mathbf{\Delta}_1)(W)_{[\mathbf{1},\mathbf{b}]}$ coincides in distribution with the following d-parameter random field

$$\begin{aligned}&\frac{(\mathbf{\Delta}_{\frac{1}{b_1}}\circ\cdots\circ\mathbf{\Delta}_{\frac{1}{b_d}})(W)_{[\frac{\mathbf{1}}{\mathbf{b}},\mathbf{1}]}}{\sqrt{\frac{1}{\Pi_{i=1}^d b_i}}}\\&=\left\{\frac{(\mathbf{\Delta}_{\frac{1}{b_1}}\circ\cdots\circ\mathbf{\Delta}_{\frac{1}{b_d}})(W(t_1,\ldots,t_d))}{\sqrt{\frac{1}{\Pi_{i=1}^d b_i}}}\right\},\end{aligned}$$

for $t_i\in[\frac{1}{b_i},1], i = 1,\ldots,d$, where

$$\left[\frac{\mathbf{1}}{\mathbf{b}},\mathbf{1}\right] := \Pi_{i=1}^d\left[\frac{1}{b_i},1\right].$$

The covariance function of this process is given by

$$\begin{aligned}\mathbf{C}(\mathbf{t},\mathbf{t}+\mathbf{u}) &= (\Pi_{i=1}^d b_i)\Pi_{i=1}^d\left(\frac{1}{b_i}-u_i\right)^+\\&= \Pi_{i=1}^d\left(1-\frac{u_i}{h_i}\right)^+\end{aligned}$$

which is equal with that of

$$\frac{1}{\sqrt{\Pi_{i=1}^{d} h_i}}(\mathbf{\Delta}_{h_1} \circ \cdots \circ \mathbf{\Delta}_{h_d})(W)_{[\mathbf{h},1]}.$$

By referring to Equation 1, to be able to obtain the Cameron-Martin formula for the derivative $\frac{d\mathbf{P}^{(\mathcal{S}+G)}}{d\mathbf{P}^{\mathcal{S}}}(x)$, we have to defin the kernel of the d-parameter random fiel $\mathcal{S}$, the norm furnishing the kernel, and the functional $\mathbf{A}(G, \mathcal{S})$ that satisfie $\mathbf{E}(\mathcal{S}(\mathbf{t})\mathbf{A}(G, \mathcal{S})) = G(\mathbf{t})$, for every admissible shift function G, where the expectation is computed based on the probability distribution of the random fiel $\mathcal{S}$, [19].

3 Main Results

In this section we restrict the consideration to the case of 2-parameter Slepian random fiel $\mathcal{S}_{h_1h_2}$, given by

$$\left\{\frac{(\mathbf{\Delta}_{h_1} \circ \mathbf{\Delta}_{h_2})(W(t_1,t_2))}{\sqrt{h_1h_2}}|(t_1,t_2) \in [h_1,1]\times[h_2,1]\right\}$$

by the reason it is the most frequently encountered situation in the practice and also by technical reasons, where W is the standard 2-parameter Brownian sheet with sample paths in $\mathcal{C}(\mathbf{I}^2)$. The real constants h_1 and h_2 are fi ed, where $0 < h_1, h_2 \leq 1$. By Proposition 4.1 of [3], the kernel of $\mathcal{S}_{h_1h_2}$ can be derived by considering the linearity and the continuity of the difference operator $\mathbf{\Delta}_{h_1} \circ \mathbf{\Delta}_{h_2}$ on $\mathcal{C}(\mathbf{I}^2)$. Hence, the kernel of $\mathcal{S}_{h_1h_2}$ is given by

$$\mathcal{H}_{\mathcal{S}_{h_1h_2}} = \frac{1}{\sqrt{h_1h_2}}(\mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}(\mathcal{H}_{\mathcal{W}_{\mathbf{I}^2}}))).$$

Since $\mathcal{H}_{\mathcal{W}_{\mathbf{I}^2}} = \{e_f : \mathbf{I}^2 \to \mathcal{R}\}$, where

$$e_f(t_1,t_2) = \int_0^{t_2}\int_0^{t_1} f(u,v)dudv,$$

for some $f \in \mathcal{L}^2(\mathbf{I}^2)$, then we have

$$\mathcal{H}_{\mathcal{S}_{h_1h_2}} = \left\{\frac{1}{\sqrt{h_1h_2}}(\mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}(e_f)))|f \in \mathcal{L}^2(\mathbf{I}^2)\right\}.$$

Clearly, every element of $\mathcal{H}_{\mathcal{S}_{h_1h_2}}$ is a real function on $[h_1,1]\times[h_2,1] \to \mathcal{R}$. By some calculations, we get the following results

$$\begin{aligned}
&\mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}e_f)(t_1,t_2)\\
&= e_f(t_1,t_2) - e_f(t_1-h_1,t_2) - e_f(t_1,t_2-h_2)\\
&+e_f(t_1-h_1,t_2-h_2)\\
&= e_f(h_1,h_2) + e_{\mathbf{\Delta}_{h_1}(f|_{h_2})}(t_1) + e_{\mathbf{\Delta}_{h_2}(f|_{h_1})}(t_2)\\
&+e_{\mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}f)}(t_1,t_2), \qquad (5)
\end{aligned}$$

for $(t_1,t_2) \in [h_1,1]\times[h_2,1]$, where

$$f\,|_{h_1} := \int_0^{h_1} f(u,v)du \;and\; f\,|_{h_2} := \int_0^{h_2} f(u,v)dv.$$

Clearly, both $f\,|_{h_2}$ and $f\,|_{h_1}$ are integrable functions of one variable on the unit interval $[0,1]$, so that the difference functions $\mathbf{\Delta}_{h_1}(f\,|_{h_2})$, as well as $\mathbf{\Delta}_{h_2}(f\,|_{h_1})$ are in $\mathcal{L}^2([h_1,1])$ and $\mathcal{L}^2([h_2,1])$, respectively, implying the following results

$$\int_{h_1}^1 \mathbf{\Delta}_{h_1}(f\,|_{h_2})(u)d\mathcal{S}_{h_1,h_2}(u,v) = 0$$

and

$$\int_{h_2}^1 \mathbf{\Delta}_{h_2}(f\,|_{h_1})(v)d\mathcal{S}_{h_1,h_2}(v) = 0$$

By this reason we defin the kernel of $\mathcal{S}_{h_1h_2}$ by

$$\mathcal{H}_{\mathcal{S}_{h_1h_2}} = \left\{\frac{e_f(h_1,h_2)}{\sqrt{h_1h_2}} + e_{\frac{\mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}f)}{\sqrt{h_1h_2}}}|f \in \mathcal{L}^2(\mathbf{I}^2)\right\},$$

where for some $f \in \mathcal{L}^2(\mathbf{I}^2)$,

$$\frac{\mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}f)}{\sqrt{h_1h_2}} \in \mathcal{L}^2([h_1,1]\times[h_2,1])$$

and

$$\frac{e_f(h_1,h_2}{\sqrt{h_1h_2}} \in \mathcal{R}.$$

The linear operator $\mathbf{\Delta}_{h_1} \circ \mathbf{\Delta}_{h_2}$ is not injective on $\mathcal{H}_{\mathcal{W}\mathbf{I}^d}$, causing the calculation of the norm in $\mathcal{H}_{\mathcal{S}_{h_1h_2}}$ becomes much more difficul than the calculation of the norm in $\mathcal{H}_{\mathcal{W}_{\mathbf{I}^2}}$. However, Remark 4.1 of [3] and Equation 2 above can be applied as an auxiliary result. We then get the following equation:

$$\begin{aligned}
&\|\mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}(e_f))\|^2_{\mathcal{H}_{\mathcal{S}_{h_1h_2}}}\\
&= \inf_{e_q \in \mathcal{H}_{\mathcal{W}_{\mathbf{I}^2}} : \mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}(e_f)) = \mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}(e_q))} \|e_q\|^2_{\mathcal{H}_{\mathcal{W}_{\mathbf{I}^2}}}\\
&= \inf_{q : \mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}(e_f)) = \mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}(e_q))} \|q\|^2_{\mathcal{L}^2(\mathbf{I}^2)}, \qquad (6)
\end{aligned}$$

for some function $e_f \in \mathcal{H}_{\mathcal{W}_{\mathbf{I}^2}}$.

The following result will be shown crucial in determining the inherent norm in $\mathcal{H}_{\mathcal{S}_{h_1h_2}}$. In fact by Equation 5, for any two functions e_f and e_q in $\mathcal{H}_{\mathcal{W}\mathbf{I}^d} \subset \mathcal{C}(\mathbf{I}^d)$ and arbitrary fi ed $0 < h_1, h_2 \leq 1$, with $f, q \in \mathcal{L}^2(\mathbf{I}^2)$, it holds λ^2-almost everywhere

$$\begin{aligned}
&\mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}(e_f)) = \mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}(e_q)) \Longleftrightarrow\\
&e_f(h_1,h_2) = e_q(h_1,h_2),\\
&\mathbf{\Delta}_{h_1}(f\,|_{h_2}) = \mathbf{\Delta}_{h_1}(q\,|_{h_2}),\\
&\mathbf{\Delta}_{h_2}(f\,|_{h_1}) = \mathbf{\Delta}_{h_2}(q\,|_{h_1}),\; and\\
&\mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}(f(u,v))) = \mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}(q(u,v))), \qquad (7)
\end{aligned}$$

for every $(u,v) \in [h_1,1] \times [h_2,1]$,

In the following we present the Cameron-Martin theorem for 2-parameter Slepian random field For that we need to derive the concrete formula for the norm in $\mathcal{H}_{\mathcal{S}_{h_1h_2}}$ as well as the formula for the operator $\mathbf{A}$ mentioned in (1). By referring to [20], we can state that every function $f \in \mathcal{L}^2(\mathbf{I}^2)$ has a λ^2-almost everywhere representation as follows

$$f(t_1,t_2) = \sum_{i,j=1}^{3} f(t_1,t_2)\mathbf{1}_{\mathbf{I}_i\times\mathbf{J}_j}(t_1,t_2),$$

where $\mathbf{1}_F$ is the indicator of $F \subseteq \mathbf{I}^2$. The symbols $\mathbf{I}_i$ and $\mathbf{J}_j$ in (8) denote intervals on $[0,1]$, define by $\mathbf{I}_1 := [0,1-h_1)$, $\mathbf{I}_2 := [1-h_1,h_1)$, $\mathbf{I}_3 := [h_1,1]$, $\mathbf{J}_1 := [0,1-h_2)$, $\mathbf{J}_2 := [1-h_2,h_2)$ and $\mathbf{J}_3 := [h_2,1]$. However, this representation equivalent to the following orthogonal form

$$f(t_1,t_2) = \sum_{i,j=1}^{3} \left(\gamma_{ij}\mathbf{1}_{\mathbf{I}_i\times\mathbf{J}_j}(t_1,t_2) + b_{ij}(t_1,t_2)\right), \quad (8)$$

where γ_{ij} is a real constant define as below

$$\begin{aligned}
\gamma_{11} &= \frac{s_f(1-h_1,1-h_2)}{L_{11}},\\
\gamma_{21} &= \frac{s_f(h_1,1-h_2)-s_f(1-h_1,1-h_2)}{L_{21}},\\
\gamma_{31} &= \frac{s_f(1,1-h_2)-s_f(h_1,1-h_2)}{L_{31}},\\
\gamma_{12} &= \frac{s_f(1-h_1,h_2)-s_f(1-h_1,1-h_2)}{L_{12}},\\
\gamma_{22} &= \frac{1}{L_{22}}\left[s_f(h_1,h_2)-s_f(1-h_1,h_2)\right.\\
&\qquad \left.-s_f(h_1,1-h_2)+s_f(1-h_1,1-h_2)\right],
\end{aligned}$$

$$\begin{aligned}
\gamma_{32} &= \frac{1}{L_{32}}\left[s_f(1,h_2)-s_f(h_1,h_2)\right.\\
&\qquad \left.-s_f(1,1-h_2)+s_f(h_1,1-h_2)\right],\\
\gamma_{13} &= \frac{s_f(1-h_1,1)-s_f(1-h_1,h_2)}{L_{13}},\\
\gamma_{23} &= \frac{1}{L_{23}}\left[s_f(h_1,1)-s_f(1-h_1,1)\right.\\
&\qquad \left.-s_f(h_1,h_2)+s_f(1-h_1,h_2)\right],\\
\gamma_{33} &= \frac{1}{L_{33}}\left[s_f(1,1)-s_f(h_1,1)\right.\\
&\qquad \left.-s_f(1,h_2)+s_f(h_1,h_2)\right],
\end{aligned}$$

b_{ij} is a real function in $\mathcal{L}^2(\mathbf{I}^2)$ define by

$$\begin{aligned}
b_{11}(t_1,t_2) &= (f(t_1,t_2)-\gamma_{11})\mathbf{1}_{\mathbf{I}_1\times\mathbf{J}_1}(t_1,t_2),\\
b_{21}(t_1,t_2) &= (f(t_1,t_2)-\gamma_{21})\mathbf{1}_{\mathbf{I}_2\times\mathbf{J}_1}(t_1,t_2),\\
b_{31}(t_1,t_2) &= (f(t_1,t_2)-\gamma_{31})\mathbf{1}_{\mathbf{I}_3\times\mathbf{J}_1}(t_1,t_2),\\
b_{12}(t_1,t_2) &= (f(t_1,t_2)-\gamma_{12})\mathbf{1}_{\mathbf{I}_1\times\mathbf{J}_2}(t_1,t_2),\\
b_{22}(t_1,t_2) &= (f(t_1,t_2)-\gamma_{22})\mathbf{1}_{\mathbf{I}_2\times\mathbf{J}_2}(t_1,t_2),\\
b_{32}(t_1,t_2) &= (f(t_1,t_2)-\gamma_{32})\mathbf{1}_{\mathbf{I}_3\times\mathbf{J}_2}(t_1,t_2),\\
b_{13}(t_1,t_2) &= (f(t_1,t_2)-\gamma_{13})\mathbf{1}_{\mathbf{I}_1\times\mathbf{J}_1}(t_1,t_2),\\
b_{23}(t_1,t_2) &= (f(t_1,t_2)-\gamma_{23})\mathbf{1}_{\mathbf{I}_2\times\mathbf{J}_3},\\
b_{33}(t_1,t_2) &= (f(t_1,t_2)-\gamma_{33})\mathbf{1}_{\mathbf{I}_3\times\mathbf{J}_3}.
\end{aligned}$$

and for $i,j = 1,2,3$,

$$L_{ij} := \int_{\mathbf{I}^2} \mathbf{1}_{\mathbf{I}_i\times\mathbf{J}_j}(u,v)dudv,\ i,j \in \{1,2,3\},$$

so that

$$\begin{aligned}
L_{11} &:= (1-h_1)(1-h_2),\\
L_{21} &:= (2h_1-1)(1-h_2),\\
L_{31} &:= (1-h_1)(1-h_2),\\
L_{12} &:= (1-h_1)(2h_2-1),\\
L_{22} &:= (2h_1-1)(2h_2-1),\\
L_{32} &:= (1-h_1)(2h_2-1),\\
L_{13} &:= (1-h_1)(1-h_2),\\
L_{23} &:= (2h_1-1)(1-h_2),\\
L_{33} &:= (1-h_1)(1-h_2).
\end{aligned}$$

Readably, the function b_{ij} satisfie the property

$$\int_{\mathbf{I}^2} b_{ij}(u,v)dudv = 0,\ i,j \in \{1,2,3\}.$$

It is understood that the set

$$\left\{\gamma_{ij}\mathbf{1}_{\mathbf{I}_i\times\mathbf{J}_j} + b_{ij} :\ i,j = 1,2,3\right\}$$

in (8) builds a set of orthogonal functions in $\mathcal{L}^2(\mathbf{I}^2)$. By means of this orthogonal representation we will derive the norm

$$\|\mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}(e_f))\|^2_{\mathcal{H}_{\mathcal{S}_{h_1h_2}}},\ e_f \in \mathcal{H}_{\mathcal{W}_{\mathbf{I}^2}}$$

By some calculations, we get

$$e_f(h_1,h_2) = \gamma_{11}L_{11} + \gamma_{21}L_{21} + \gamma_{12}L_{12} + \gamma_{22}L_{22},$$

and

$$\begin{aligned}
&\mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}(f(t_1,t_2)))\\
&= (\gamma_{33}-\gamma_{13}-\gamma_{31}+\gamma_{11}) + b_{33}(t_1,t_2)\\
&\quad -b_{13}(t_1,t_2) - b_{31}(t_1,t_2) + b_{11}(t_1,t_2), \quad (9)
\end{aligned}$$

for $(t_1,t_2) \in [h_1,1]\times[h_2,1]$. Both results will be considered in establishing the proof of the following two lemmas.

Lemma 1 Let $\frac{1}{2} \leq h_1, h_2 \leq 1$. Then $\forall f \in \mathcal{L}^2(\mathbf{I}^2)$, it holds true

$$\begin{aligned}
&\|\mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}(e_f))\|^2_{\mathcal{H}_{\mathcal{S}_{h_1h_2}}} = \gamma_{11}^{*2} L_{11} \\
&+\frac{1}{L_{21}}\left[s_q(h_1,h_2) - \gamma_{11}^* L_{11} - \gamma_{12}^* L_{12} - \gamma_{22}^* L_{22}\right]^2 \\
&+(\gamma_{11}^* - \alpha^*)^2 L_{31} + \gamma_{12}^{*2} L_{12} + \gamma_{22}^{*2} L_{22} + \gamma_{13}^{*2} L_{13} \\
&+(\gamma_{13}^* + \alpha^*)^2 L_{33} + \frac{1}{2}\|\mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}(f))\|^2_{\mathcal{L}^2(\mathbf{I}_3\times\mathbf{J}_3)} \\
&+2\alpha^{*2} L_{33} - 2\alpha^* \int_{\mathbf{I}_3\times\mathbf{J}_3} \mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}(f(u,v))dudv
\end{aligned}$$

where

$$\begin{aligned}
\gamma_{11}^* &:= \frac{L_{11} e_q(h_1,h_2) + \alpha L_{31}\left(L_{22} + L_{21} + L_{12}\right)}{B} \\
\gamma_{21}^* &:= \frac{e_q(h_1,h_2) - \gamma_{11}^* L_{11} - \gamma_{12}^* L_{12} - \gamma_{22}^* L_{22}}{L_{21}} \\
\gamma_{12}^* &:= \frac{L_{11} e_q(h_1,h_2) - \alpha L_{11} L_{31} + L_{31} e_q(h_1,h_2)}{B} \\
\gamma_{22}^* &:= \frac{L_{11} e_q(h_1,h_2) - \alpha L_{11} L_{31} + L_{31} e_q(h_1,h_2)}{B} \\
\gamma_{13}^* &:= \frac{-\alpha L_{33}}{(L_{13} + L_{33})}\ \alpha^* := \frac{\gamma_{11}^* L_{31} - \gamma_{13}^* L_{33}}{L_{31} + L_{33}},
\end{aligned}$$

and

$$\begin{aligned}
B := L_{11}&\left(L_{11} + L_{21} + L_{12} + L_{22}\right) \\
&+L_{31}\left(L_{21} + L_{12} + L_{22}\right).
\end{aligned}$$

Proof: In the firs step we consider a family of step functions in $\mathcal{L}^2(\mathbf{I}^2)$ having the following representation

$$\begin{aligned}
q(t_1,t_2) = \gamma_{11}&\mathbf{1}_{\mathbf{I}_1\times\mathbf{J}_1}(t_1,t_2) + \gamma_{21}\mathbf{1}_{\mathbf{I}_2\times\mathbf{J}_1}(t_1,t_2) \\
+(\gamma_{11} - \alpha)&\mathbf{1}_{\mathbf{I}_3\times\mathbf{J}_1}(t_1,t_2) + \gamma_{12}\mathbf{1}_{\mathbf{I}_1\times\mathbf{J}_2}(t_1,t_2) \\
&+\gamma_{22}\mathbf{1}_{\mathbf{I}_2\times\mathbf{J}_2}(t_1,t_2) + \gamma_{32}\mathbf{1}_{\mathbf{I}_3\times\mathbf{J}_2}(t_1,t_2) \\
&+\gamma_{13}\mathbf{1}_{\mathbf{I}_1\times\mathbf{J}_3}(t_1,t_2) + \gamma_{23}\mathbf{1}_{\mathbf{I}_2\times\mathbf{J}_3}(t_1,t_2) \\
&\qquad+(\gamma_{13} + \alpha)\mathbf{1}_{\mathbf{I}_3\times\mathbf{J}_3}(t_1,t_2),
\end{aligned}$$

for real constants $\gamma_{11}, \gamma_{21}, \gamma_{12}, \gamma_{22}, \gamma_{32}, \gamma_{13}, \gamma_{23}$ and α. Let h_1 and h_2 be arbitrary with $1/2 \leq h_1 \leq 1$ and $1/2 \leq h_2 \leq 1$. Then by some algebraic computations, we obtain the following equation

$$e_q(h_1,h_2) = \gamma_{11} L_{11} + \gamma_{21} L_{21} + \gamma_{12} L_{12} + \gamma_{22} L_{22} \quad (10)$$

and

$$\begin{aligned}
&\mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}(q(t_1,t_2))) \\
&= (\gamma_{13} + \alpha) - \gamma_{13} - (\gamma_{11} - \alpha) + \gamma_{11} \\
&= 2\alpha. \qquad (11)
\end{aligned}$$

Equation 10 gives the following solution for γ_{21}:

$$\gamma_{21} = \frac{e_q(h_1,h_2) - \gamma_{11} L_{11} - \gamma_{12} L_{12} - \gamma_{22} L_{22}}{L_{21}}.$$

The definitio of $\mathcal{H}_{\mathcal{S}_{h_1h_2}}$, together with Equation 10 and Equation 11 leads us to the following equation

$$\begin{aligned}
&\mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}(e_q(t_1,t_2))) \\
&= e_q(h_1,h_2) + \int_{h_2}^{t_2}\int_{h_1}^{t_1} \mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}(q(u,v)))dudv \\
&= e_q(h_1,h_2) + 2\alpha(t_1 - h_1)(t_2 - h_2).
\end{aligned}$$

Let us defin a function q_0 in $\mathcal{L}^2(\mathbf{I}^2)$, given by

$$\begin{aligned}
&q_0(t_1,t_2) \\
&= \gamma_{11}^{(0)}\mathbf{1}_{\mathbf{I}_1\times\mathbf{J}_1}(t_1,t_2) + \gamma_{21}^{(0)}\mathbf{1}_{\mathbf{I}_2\times\mathbf{J}_1}(t_1,t_2) \\
&+(\gamma_{11}^{(0)} - \alpha_0)\mathbf{1}_{\mathbf{I}_3\times\mathbf{J}_1}(t_1,t_2) + \gamma_{12}^{(0)}\mathbf{1}_{\mathbf{I}_1\times\mathbf{J}_2}(t_1,t_2) \\
&+\gamma_{22}^{(0)}\mathbf{1}_{\mathbf{I}_2\times\mathbf{J}_2}(t_1,t_2) + \gamma_{32}^{(0)}\mathbf{1}_{\mathbf{I}_3\times\mathbf{J}_2}(t_1,t_2) \\
&+\gamma_{13}^{(0)}\mathbf{1}_{\mathbf{I}_1\times\mathbf{J}_3}(t_1,t_2) + \gamma_{23}^{(0)}\mathbf{1}_{\mathbf{I}_2\times\mathbf{J}_3}(t_1,t_2) \\
&+(\gamma_{13}^{(0)} + \alpha_0)\mathbf{1}_{\mathbf{I}_3\times\mathbf{J}_3}(t_1,t_2),
\end{aligned}$$

such that for $h_1 \leq t_1 \leq 1$ and $h_2 \leq t_2 \leq 1$,

$$\begin{aligned}
&\mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}(e_q(t_1,t_2))) = \mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}(e_{q_0}(t_1,t_2))) \\
&= e_{q_0}(h_1,h_2) + 2\alpha_0(t_1 - h_1)(t_2 - h_2),
\end{aligned}$$

Then by considering (7), we get

$$e_q(h_1,h_2) = e_{q_0}(h_1,h_2);\ \alpha = \alpha_0,$$

so that after the substitution of γ_{21}, we have

$$\begin{aligned}
&q(t_1,t_2) \\
&= \gamma_{11}\mathbf{1}_{\mathbf{I}_1\times\mathbf{J}_1}(t_1,t_2) + (\gamma_{13} + \alpha_0)\mathbf{1}_{\mathbf{I}_3\times\mathbf{J}_3}(t_1,t_2) \\
&+\frac{(e_{q_0}(h_1,h_2) - \gamma_{11} L_{11} - \gamma_{12} L_{12} - \gamma_{22} L_{22})}{L_{21}} \\
&\quad\times\mathbf{1}_{\mathbf{I}_2\times\mathbf{J}_1}(t_1,t_2) \\
&+(\gamma_{11} - \alpha_0)\mathbf{1}_{\mathbf{I}_3\times\mathbf{J}_1}(t_1,t_2) + \gamma_{12}\mathbf{1}_{\mathbf{I}_1\times\mathbf{J}_2}(t_1,t_2) \\
&+\gamma_{22}\mathbf{1}_{\mathbf{I}_2\times\mathbf{J}_2}(t_1,t_2) + \gamma_{32}\mathbf{1}_{\mathbf{I}_3\times\mathbf{J}_2}(t_1,t_2) \\
&+\gamma_{13}\mathbf{1}_{\mathbf{I}_1\times\mathbf{J}_3}(t_1,t_2) + \gamma_{23}\mathbf{1}_{\mathbf{I}_2\times\mathbf{J}_3}(t_1,t_2).
\end{aligned}$$

Hence, the square norm of q in $\mathcal{L}^2(\mathbf{I}^2)$ is given by

$$\begin{aligned}
&\|q\|^2_{\mathcal{L}^2(\mathbf{I}^2)} = \gamma_{11}^2 L_{11} + (\gamma_{11} - \alpha_0)^2 L_{31} + \gamma_{12}^2 L_{12} \\
&+\frac{(e_{q_0}(h_1,h_2) - \gamma_{11} L_{11} - \gamma_{12} L_{12} - \gamma_{22} L_{22})^2}{L_{21}} \\
&+\gamma_{22}^2 L_{22} + \gamma_{32}^2 L_{32} + \gamma_{13}^2 L_{13} + \gamma_{23}^2 L_{23} \\
&+(\gamma_{13} + \alpha_0)^2 L_{33}.
\end{aligned}$$

The minimum condition of the norm $\|q\|^2_{\mathcal{L}^2(\mathbf{I}^2)}$ can be obtained by applying differential method. It can be shown by a computation using a computer program that the minimum value will be attained at the constants γ_{11}^*, γ_{12}^*, γ_{21}^*, γ_{22}^*, γ_{32}^*, γ_{13}^*, γ_{23}^*, which are define as follows

$$\gamma_{11}^* = \frac{L_{11}e_{q_0}(h_1,h_2) + \alpha_0 L_{31}\left(L_{22}+L_{21}+L_{12}\right)}{B},$$
$$\gamma_{21}^* = \frac{e_{q_0}(h_1,h_2) - \gamma_{11}^* L_{11} - \gamma_{12}^* L_{12} - \gamma_{22}^* L_{22}}{L_{21}},$$
$$\gamma_{12}^* = \frac{L_{11}e_{q_0}(h_1,h_2) - \alpha_0 L_{11}L_{31} + L_{31}e_{q_0}(h_1,h_2)}{B},$$
$$\gamma_{22}^* = \frac{L_{11}e_{q_0}(h_1,h_2) - \alpha_0 L_{11}L_{31} + L_{31}e_{q_0}(h_1,h_2)}{B},$$
$$\gamma_{32}^* = 0,\ \gamma_{13}^* = \frac{-\alpha_0 L_{33}}{(L_{13}+L_{33})},\ \gamma_{23}^* = 0.$$

Hence by recalling Equation 6, the following equation is obtained

$$\begin{aligned}
&\|\mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}(e_{q_0}))\|^2_{\mathcal{H}_{\mathcal{S}_{h_1h_2}}} \\
&= \inf_{q\in\mathcal{L}^2(\mathbf{I}^2):\mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}(e_{q_0}))=\mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}(e_q))} \|q\|^2_{\mathcal{L}^2(\mathbf{I}^2)} \\
&= \frac{\left(e_{q_0}(h_1,h_2) - \gamma_{11}^* L_{11} - \gamma_{12}^* L_{12} - \gamma_{22}^* L_{22}\right)^2}{L_{21}} \\
&\quad + \gamma_{11}^{*2} L_{11} + (\gamma_{11}^* - \alpha_0)^2 L_{31} + \gamma_{12}^{*2} L_{12} \\
&\quad + \gamma_{22}^{*2} L_{22} + \gamma_{13}^{*2} L_{13} + (\gamma_{13}^* + \alpha_0)^2 L_{33}.
\end{aligned}$$

Next, we expand the study to a general $f \in \mathcal{L}^2(\mathbf{I}^2)$, which is define by

$$\begin{aligned}
f(t_1,t_2) &= \gamma_{11}\mathbf{1}_{\mathbf{I}_1\times\mathbf{J}_1}(t_1,t_2) + b_{11}(t_1,t_2) \\
&+ \gamma_{21}\mathbf{1}_{\mathbf{I}_2\times\mathbf{J}_1}(t_1,t_2) + b_{21}(t_1,t_2) \\
&+ (\gamma_{11}-\alpha)\mathbf{1}_{\mathbf{I}_3\times\mathbf{J}_1}(t_1,t_2) + b_{31}(t_1,t_2) \\
&+ \gamma_{12}\mathbf{1}_{\mathbf{I}_1\times\mathbf{J}_2}(t_1,t_2) + b_{12}(t_1,t_2) \\
&+ \gamma_{22}\mathbf{1}_{\mathbf{I}_2\times\mathbf{J}_2}(t_1,t_2) + b_{22}(t_1,t_2) \\
&+ \gamma_{32}\mathbf{1}_{\mathbf{I}_3\times\mathbf{J}_2}(t_1,t_2) + b_{32}(t_1,t_2) \\
&+ \gamma_{13}\mathbf{1}_{\mathbf{I}_1\times\mathbf{J}_3}(t_1,t_2) + b_{13}(t_1,t_2) \\
&+ \gamma_{23}\mathbf{1}_{\mathbf{I}_2\times\mathbf{J}_3}(t_1,t_2) + b_{23}(t_1,t_2) \\
&+ (\gamma_{13}+\alpha)\mathbf{1}_{\mathbf{I}_3\times\mathbf{J}_3}(t_1,t_2) + b_{33}(t_1,t_2) \\
&= q(t_1,t_2) + b_{11}(t_1,t_2) + b_{21}(t_1,t_2) \\
&+ b_{31}(t_1,t_2) + b_{12}(t_1,t_2) + b_{22}(t_1,t_2) \\
&+ b_{32}(t_1,t_2) + b_{13}(t_1,t_2) + b_{23}(t_1,t_2) \\
&+ b_{33}(t_1,t_2),\ (t_1,t_2)\in\mathbf{I}^2.
\end{aligned}$$

By considering the linearity of the operator $\mathbf{\Delta}_{h_1}\circ\mathbf{\Delta}_{h_2}$, we obtain

$$\mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}(e_f)) = \mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}(e_{q+\sum_{i=1}^3\sum_{j=1}^3 b_{ij}})).$$

So by recalling Equation 6 and by the preceding consideration, we get:

$$\begin{aligned}
&\|\mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}(e_f))\|^2_{\mathcal{H}_{\mathcal{S}_{h_1h_2}}} \\
&= \min\|q\|^2_{\mathcal{L}^2(\mathbf{I}^2)} + \min\left\{\sum_{i=1}^3\sum_{j=1}^3\|b_{ij}\|^2_{\mathcal{L}^2(\mathbf{I}^2)}\right\} \\
&= \frac{\left(e_q(h_1,h_2) - \gamma_{11}^* L_{11} - \gamma_{12}^* L_{12} - \gamma_{22}^* L_{22}\right)^2}{L_{21}} \\
&+ \gamma_{11}^{*2} L_{11} + (\gamma_{11}^* - \alpha^*)^2 L_{31} \\
&+ \gamma_{12}^{*2} L_{12} + \gamma_{22}^{*2} L_{22} + \gamma_{13}^{*2} L_{13} + (\gamma_{13}^* + \alpha^*)^2 L_{33} \\
&+ \min\left\{\sum_{i=1}^3\sum_{j=1}^3\|b_{ij}\|^2_{\mathcal{L}^2(\mathbf{I}^2)}\right\},
\end{aligned} \tag{12}$$

where γ_{ij}^* in (12) is obtained by replacing q_0 and α_0 with q and α^*, respectively, where

$$\alpha^* = (\gamma_{11}^* L_{31} - \gamma_{13}^* L_{33})/(L_{31}+L_{33}).$$

The minimum in (12) is chosen among all

$$b_{ij} : \mathbf{I}_i \times \mathbf{J}_j \to \mathcal{R}$$

that satisfy the following properties

$$\int_{\mathbf{I}_i\times\mathbf{J}_j} b_{ij}(u,v)\,du\,dv = 0$$

and

$$\begin{aligned}
&\mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}(f(t_1,t_2))) - 2\alpha^* = \\
&b_{33}(t_1,t_2) - b_{13}(t_1,t_2) - b_{31}(t_1,t_2) + b_{11}(t_1,t_2).
\end{aligned}$$

The last property follows by Equation 9 and the equation

$$\gamma_{33} - \gamma_{13} - \gamma_{31} + \gamma_{11} = 2\alpha^*.$$

Since for fi ed $(t_1,t_2)\in\mathbf{I}_3\times\mathbf{J}_3$, it holds

$$b_{12}(t_1,t_2) = b_{22}(t_1,t_2) = b_{32}(t_1,t_2) = 0,$$

then

$$\begin{aligned}
&\min\left\{\sum_{i=1}^3\sum_{j=1}^3\|b_{ij}\|^2_{\mathcal{L}^2(\mathbf{I}^2)}\right\} \\
&= \min\Big\{\|b_{33}\|^2_{\mathcal{L}^2(\mathbf{I}^2)} + \|b_{13}\|^2_{\mathcal{L}^2(\mathbf{I}^2)} \\
&+ \|b_{31}\|^2_{\mathcal{L}^2(\mathbf{I}^2)} + \|b_{11}\|^2_{\mathcal{L}^2(\mathbf{I}^2)}\Big\} \\
&= \min\Big\{\|b_{33}(\cdot,\cdot)\|^2_{\mathcal{L}^2(\mathbf{I}_3\times\mathbf{J}_3)} \\
&+ \|b_{13}(\cdot-h_1,\cdot)\|^2_{\mathcal{L}^2(\mathbf{I}_3\times\mathbf{J}_3)} \\
&+ \|b_{31}(\cdot,\cdot-h_2)\|^2_{\mathcal{L}^2(\mathbf{I}_3\times\mathbf{J}_3)} \\
&+ \|b_{11}(\cdot-h_1,\cdot-h_2)\|^2_{\mathcal{L}^2(\mathbf{I}_3\times\mathbf{J}_3)}\Big\}.
\end{aligned}$$

One more simplificatio gives

$$\min\left\{\sum_{i=1}^{3}\sum_{j=1}^{3}\|b_{ij}\|^2_{\mathcal{L}^2(\mathbf{I}^2)}\right\}$$
$$=\int_{\mathbf{I}_3\times\mathbf{J}_3}\min\Big\{|b_{33}(t_1,t_2)|^2+|b_{13}(t_1-h_1,t_2)|^2$$
$$+|b_{31}(t_1,t_2-h_2)|^2$$
$$+|b_{11}(t_1-h_1,t_2-h_2)|^2\Big\}\,dt_1dt_2.$$

We get for $(t_1,t_2)\in\mathbf{I}_3\times\mathbf{J}_3$,

$$\min\Big\{|b_{33}(t_1,t_2)|^2+|b_{13}(t_1-h_1,t_2)|^2$$
$$+|b_{31}(t_1,t_2-h_2)|^2+|b_{11}(t_1-h_1,t_2-h_2)|^2\Big\}$$
$$=\Big|\frac{1}{2}(b_{33}(t_1,t_2)-b_{13}(t_1-h_1,t_2)$$
$$-b_{31}(t_1,t_2-h_2)+b_{11}(t_1-h_1,t_2-h_2))\Big|^2$$
$$+\Big|\frac{1}{2}(-b_{33}(t_1,t_2)+b_{13}(t_1-h_1,t_2)$$
$$+b_{31}(t_1,t_2-h_2)-b_{11}(t_1-h_1,t_2-h_2))\Big|^2$$
$$=\frac{1}{2}\,|b_{33}(t_1,t_2)-b_{13}(t_1-h_1,t_2)$$
$$-b_{31}(t_1,t_2-h_2)+b_{11}(t_1-h_1,t_2-h_2)|^2$$
$$=\frac{1}{2}\,|\boldsymbol{\Delta}_{h_1}(\boldsymbol{\Delta}_{h_2}(f(t_1,t_2))-2\alpha^*|^2.$$

Hence,

$$\min\left\{\sum_{i=1}^{3}\sum_{j=1}^{3}\|b_{ij}\|^2_{\mathcal{L}^2(\mathbf{I}^2)}\right\}$$
$$=\frac{1}{2}\,\|\boldsymbol{\Delta}_{h_1}(\boldsymbol{\Delta}_{h_2}(f))-2\alpha^*\|^2_{\mathcal{L}^2(\mathbf{I}_3\times\mathbf{J}_3)}$$
$$=\frac{1}{2}\,\|\boldsymbol{\Delta}_{h_1}(\boldsymbol{\Delta}_{h_2}(f))\|^2_{\mathcal{L}^2(\mathbf{I}_3\times\mathbf{J}_3)}+2\alpha^{*2}L_{33}$$
$$-2\alpha^*\int_{\mathbf{I}_3\times\mathbf{J}_3}\boldsymbol{\Delta}_{h_1}(\boldsymbol{\Delta}_{h_2}(f(u,v))dudv. \quad (13)$$

Finally, by (12) and (13, we get

$$\|\boldsymbol{\Delta}_{h_1}(\boldsymbol{\Delta}_{h_2}(e_f))\|^2_{\mathcal{H}_{\mathcal{S}_{h_1h_2}}}$$
$$=\frac{(e_q(h_1,h_2)-\gamma^*_{11}L_{11}-\gamma^*_{12}L_{12}-\gamma^*_{22}L_{22})^2}{L_{21}}$$
$$+\gamma^{*2}_{11}L_{11}+(\gamma^*_{11}-\alpha^*)^2L_{31}+\gamma^{*2}_{12}L_{12}$$
$$+\gamma^{*2}_{22}L_{22}+\gamma^{*2}_{13}L_{13}+(\gamma^*_{13}+\alpha^*)^2L_{33}$$
$$+\frac{1}{2}\,\|\boldsymbol{\Delta}_{h_1}(\boldsymbol{\Delta}_{h_2}(f))\|^2_{\mathcal{L}^2(\mathbf{I}_3\times\mathbf{J}_3)}+2\alpha^{*2}L_{33}$$
$$-2\alpha^*\int_{\mathbf{I}_3\times\mathbf{J}_3}\boldsymbol{\Delta}_{h_1}(\boldsymbol{\Delta}_{h_2}(f(u,v))dudv$$

establishing the proof of Lemma 1.

Lemma 2 For $\frac{1}{2}\le h_1,h_2\le 1$, the functional

$$\mathbf{A}:\mathcal{H}_{\mathcal{S}_{h_1h_2}}\times\mathcal{C}([h_1,1]\times[h_2,1])\mapsto\mathcal{R}$$

that satisfie the condition

$$\mathbf{E}\left(\mathcal{S}_{h_1,h_2}(t_1,t_2)\mathbf{A}\left(\frac{\boldsymbol{\Delta}_{h_1}(\boldsymbol{\Delta}_{h_2}(e_f))}{\sqrt{h_1h_2}},\mathcal{S}_{h_1,h_2}\right)\right)$$
$$=\frac{\boldsymbol{\Delta}_{h_1}(\boldsymbol{\Delta}_{h_2}(e_f(t_1,t_2))}{\sqrt{h_1h_2}}$$

is given by

$$\mathbf{A}\left(\frac{1}{\sqrt{h_1h_2}}\boldsymbol{\Delta}_{h_1}(\boldsymbol{\Delta}_{h_2}(e_f)),\mathcal{S}_{h_1,h_2}\right)$$
$$:=\frac{1}{h_1h_2}e_f(h_1,h_2)W(1,1)$$
$$+\int_{h_1}^{1}\int_{0}^{1-h_2}\boldsymbol{\Delta}_{h_1}(\boldsymbol{\Delta}_{h_2}(f(u,v)))dW(u,v)$$
$$+\int_{0}^{1-h_1}\int_{h_2}^{1}\boldsymbol{\Delta}_{h_1}(\boldsymbol{\Delta}_{h_2}(f(u,v)))dW(u,v)$$
$$-\int_{0}^{1-h_1}\int_{0}^{1-h_2}\boldsymbol{\Delta}_{h_1}(\boldsymbol{\Delta}_{h_2}(f(u,v)))dW(u,v)$$
$$+\sqrt{h_1h_2}\int_{h_1}^{1}\int_{h_2}^{1}\boldsymbol{\Delta}_{h_1}(\boldsymbol{\Delta}_{h_2}(f(u,v)))d\mathcal{S}_{h_1,h_2}(u,v)$$

where

$$\int_{h_1}^{1}\int_{h_2}^{1}\boldsymbol{\Delta}_{h_1}(\boldsymbol{\Delta}_{h_2}(f(u,v)))d\mathcal{S}_{h_1,h_2}(u,v)$$
$$=\int_{h_1}^{1}\int_{h_2}^{1}\frac{\boldsymbol{\Delta}_{h_1}(\boldsymbol{\Delta}_{h_2}(f(u,v)))}{\sqrt{h_1h_2}}dW(u,v)$$
$$-\int_{h_1}^{1}\int_{h_2}^{1}\frac{\boldsymbol{\Delta}_{h_1}(\boldsymbol{\Delta}_{h_2}(f(u,v)))}{\sqrt{h_1h_2}}dW(u-h_1,v)$$
$$-\int_{h_1}^{1}\int_{h_2}^{1}\frac{\boldsymbol{\Delta}_{h_1}(\boldsymbol{\Delta}_{h_2}(f(u,v)))}{\sqrt{h_1h_2}}dW(u,v-h_2)$$
$$+\int_{h_1}^{1}\int_{h_2}^{1}\frac{\boldsymbol{\Delta}_{h_1}(\boldsymbol{\Delta}_{h_2}(f(u,v)))}{\sqrt{h_1h_2}}dW(u-h_1,v-h_2).$$

Proof: By considering the linearity of the expectation and by recalling Equation 4, we get

$$\mathbf{E}\left(\mathcal{S}_{h_1,h_2}(t_1,t_2)\mathbf{A}\left(\frac{\mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}(e_f))}{\sqrt{h_1h_2}},\mathcal{S}_{h_1,h_2}\right)\right)$$
$$=\mathbf{E}\left(\frac{\mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}(W(t_1,t_2)))}{\sqrt{h_1h_2}}\frac{e_f(h_1,h_2)W(1,1)}{h_1h_2}\right)$$
$$+\mathbf{E}\left(\frac{\mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}(W(t_1,t_2)))}{\sqrt{h_1h_2}}\times\int_{h_1}^{1}\int_{0}^{1-h_2}\mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}(f(u,v)))dW(u,v)\right)$$
$$+\mathbf{E}\left(\frac{\mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}(W(t_1,t_2)))}{\sqrt{h_1h_2}}\times\int_{0}^{1-h_1}\int_{h_2}^{1}\mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}(f(u,v)))dW(u,v)\right)$$
$$-\mathbf{E}\left(\frac{\mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}(W(t_1,t_2)))}{\sqrt{h_1h_2}}\times\int_{0}^{1-h_1}\int_{0}^{1-h_2}\mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}(f(u,v)))dW(u,v)\right)$$
$$+\mathbf{E}\left(\frac{\mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}(W(t_1,t_2)))}{\sqrt{h_1h_2}}\sqrt{h_1h_2}\times\int_{h_1}^{1}\int_{h_2}^{1}\mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}(f(u,v)))d\mathcal{S}_{h_1,h_2}(u,v)\right)$$
$$=\frac{e_f(h_1,h_2)}{h_1h_2\sqrt{h_1h_2}}(t_1-(t_1-h_1))(t_2-(t_2-h_2))$$
$$+\int_{h_1}^{t_1}\int_{t_2-h_2}^{1-h_2}\frac{\mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}(f(u,v)))}{\sqrt{h_1h_2}}dvdu+\int_{t_1-h_1}^{1-h_1}\int_{h_2}^{t_2}\frac{\mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}(f(u,v)))}{\sqrt{h_1h_2}}dvdu$$
$$-\int_{t_1-h_1}^{1-h_1}\int_{t_2-h_2}^{1-h_2}\frac{\mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}(f(u,v)))}{\sqrt{h_1h_2}}dvdu+\int_{h_1}^{t_1}\int_{h_2}^{t_2}\frac{\mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}(f(u,v)))}{\sqrt{h_1h_2}}dvdu$$
$$-\int_{t_1-h_1}^{1-h_1}\int_{h_2}^{t_2}\frac{\mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}(f(u,v)))}{\sqrt{h_1h_2}}dvdu-\int_{h_1}^{t_1}\int_{t_2-h_2}^{1-h_2}\frac{\mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}(f(u,v)))}{\sqrt{h_1h_2}}dvdu$$
$$+\int_{t_1-h_1}^{1-h_1}\int_{t_2-h_2}^{1-h_2}\frac{\mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}(f(u,v)))}{\sqrt{h_1h_2}}dvdu$$
$$=\frac{e_f(h_1,h_2)}{\sqrt{h_1h_2}}+\frac{1}{\sqrt{h_1h_2}}e_{\mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}(f))}(t_1,t_2)$$
$$=\frac{e_f(h_1,h_2)}{\sqrt{h_1h_2}}+e_{\frac{\mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}(f))}{\sqrt{h_1h_2}}}(t_1,t_2),$$

finishin the proof of the lemma.

By combining Lemma 1 and Lemma 2, we obtain the following Cameron-Martin formula for the Radon-Nikodym derivative of the probability measure

$$\mathbf{P}^{\mathcal{S}_{h_1,h_2}+\frac{e_f(h_1,h_2)}{\sqrt{h_1h_2}}+e_{\frac{\mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}(f))}{\sqrt{h_1,h_2}}}}$$

with respect to $\mathbf{P}^{\mathcal{S}_{h_1,h_2}}$, for $f\in\mathcal{L}^2(\mathbf{I}^2)$.

Theorem 3 The Radon-Nikodym derivative of

$$\mathbf{P}^{\mathcal{S}_{h_1,h_2}+\frac{e_f(h_1,h_2)}{\sqrt{h_1h_2}}+e_{\frac{\mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}(f))}{\sqrt{h_1,h_2}}}}$$

with respect to $\mathbf{P}^{\mathcal{S}_{h_1,h_2}}$ exists, if and only if

$$\frac{e_f(h_1,h_2)}{\sqrt{h_1h_2}}+e_{\frac{1}{\sqrt{h_1,h_2}}\mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}(f))}\in\mathcal{H}_{\mathcal{S}_{h_1,h_2}},$$

for $\frac{1}{2}\leq h_1,h_2\leq 1$. Moreover, if the condition

$$\frac{e_f(h_1,h_2)}{\sqrt{h_1h_2}}+e_{\frac{1}{\sqrt{h_1,h_2}}\mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}(f))}\in\mathcal{H}_{\mathcal{S}_{h_1,h_2}},$$

is fulfilled then

$$\frac{d\mathbf{P}^{\mathcal{S}_{h_1,h_2}+\frac{e_f(h_1,h_2)}{\sqrt{h_1h_2}}+e_{\frac{\mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}(f))}{\sqrt{h_1,h_2}}}}}{d\mathbf{P}^{\mathcal{S}_{h_1,h_2}}}\left\{\frac{\mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}(W))}{\sqrt{h_1h_2}}\right\}$$

$$=\exp\Bigg\{-\frac{1}{2h_1h_2}\Bigg(\gamma_{11}^{*2}L_{11}+\frac{1}{L_{21}}\left[s_q(h_1,h_2)-\gamma_{11}^{*}L_{11}-\gamma_{12}^{*}L_{12}-\gamma_{22}^{*}L_{22}\right]^2$$
$$+(\gamma_{11}^{*}-\alpha^{*})^2L_{31}+\gamma_{12}^{*2}L_{12}+\gamma_{22}^{*2}L_{22}+\gamma_{13}^{*2}L_{13}+(\gamma_{13}^{*}+\alpha^{*})^2L_{33}$$
$$+\frac{1}{2}\left\|\mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}(f))\right\|^2_{\mathcal{L}^2(\mathbf{I}_3\times\mathbf{J}_3)}+2\alpha^{*2}L_{33}-2\alpha^{*}\int_{\mathbf{I}_3\times\mathbf{J}_3}\mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}(f(u,v))dudv\Bigg)\Bigg\}$$

$$\times\exp\Bigg\{\frac{e_f(h_1,h_2)}{h_1h_2}W(1,1)+\int_{h_1}^{1}\int_{0}^{1-h_2}\mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}(f(u,v)))dW(u,v)$$
$$+\int_{0}^{1-h_1}\int_{h_2}^{1}\mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}(f(u,v)))dW(u,v)-\int_{0}^{1-h_1}\int_{0}^{1-h_2}\mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}(f(u,v)))dW(u,v)$$
$$+\sqrt{h_1h_2}\times\int_{h_1}^{1}\int_{h_2}^{1}\mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}(f(u,v)))d\mathcal{S}_{h_1,h_2}(u,v)\Bigg\}.$$

Proof: The result follows by the substitution of the norm

$$\|\mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}(e_f))\|^2_{\mathcal{H}_{\mathcal{S}_{h_1 h_2}}}$$

and the functional

$$\mathbf{A}\left(\frac{\mathbf{\Delta}_{h_1}(\mathbf{\Delta}_{h_2}(e_f))}{\sqrt{h_1 h_2}}, \mathcal{S}_{h_1,h_2}\right)$$

define in Equation 1, by the formulas obtained in Lemma 1 and Lemma 2, respectively. Hence, finish ing the proof of the theorem.

4 Concluding Remrks

In this paper the Cameron-Martin formula of the Radon-Nikodym derivative of the shifted two parameters Slepian random fiel has been established for an infinit dimensional class of shift functions. The formula for the norm of the shift function has been derived by applying an optimization principle in infinit dimensional space. The linear functional characterizing the Cameron-Martin formula has been obtained by making use of the property that the Slepian fiel is equivalent in distribution with the increment of the Wiener field

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