Exponential Law for Mappings on Sequentially Locally Convex Topological Vector Spaces and Manifolds

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Abstract: - The exponential law‹ which is an important tool in modern topology and differential calculus, establishes a fundamental isomorphism between function spaces over product domains and iterated mapping spaces. This work studies this principle on the context of sequentially locally convex topological vector spaces and manifolds with rough boundaries. We demonstrate that for a sequentially compact manifold N_2, the map

$$\Phi\colon C^{r,k}(N_1 \times N_2, E) \to C^r(N_1, C^k(N_2, E))$$

is a homeomorphism.

Key-Words: - Generalized differential calculus, Smooth compact-open topology, Infinite-dimensional manifolds, sequentially Locally convex spaces, Exponential law.

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1 Introduction

The exponential law for continuous mapping spaces, often expressed as a homeomorphism

$$C(X \times Y, Z) \cong C(X, C(Y, Z)),$$

has affected diverse areas of mathematics, from algebraic topology to geometric analysis and Lie theory.

The origins of the exponential law lie in mid-20th-century topology. Seminal work, [1], introduced the compact-open topology to equip continuous mapping space with a natural structure. These ideas were later formalized in [2], which established conditions for the canonical map $C(X \times Y, Z) \to C(X, C(Y, Z))$ to become a homeomorphism. Concurrently, applications in homotopy theory, particularly for fibrations and duality, were explored in [3].

For differentiable mapping spaces, foundational contributions emerged in differential topology and global analysis. Systematic studies of smooth maps between manifolds, [4], highlighted their role in geometric problems such as embeddings and isotopies. Challenges in infinite-dimensional calculus, motivated the development of frameworks like the Nash-Moser theorem, [5]. Subsequent advances, [6], restored the exponential law through convenient calculus.

In algebraic topology and Lie theory, mapping spaces play a pivotal role. Early studies of continuous mapping spaces such as loop spaces, [7], implicitly relied on exponential adjunctions. Differentiable analogs, including spaces of smooth paths. The realization that diffeomorphism groups $\mathrm{Diff}(N)$ and loop groups $C^\infty(S^1, G)$ inherit Lie group structures emerged from [8], [9], which employed C^r- and smooth topologies. Unification of these perspectives, [10], demonstrated that infinite-dimensional convenient vector Lie groups satisfy the exponential law. The exponential law for partially differentiable functions over valued fields was established, [11], generalizing classical results to ultrametric settings. This framework enabled two key applications: the density of locally polynomial functions in spaces of partially differentiable functions (resolving an open problem by Enno Nagel) and a novel proof characterizing C^r-functions on $(\mathbb{Z}_p)^n$ via decay properties of Mahler expansions. For mappings on Cartesian products (spaces formed by combining two sets $U \times V$, where U and V are locally convex subsets) the exponential law was established in [12]. This law formalized a canonical correspondence between the differentiable space $C^{r,s}(U \times V, F)$, and the iterated function space $C^r(U, C^s(V, F))$. This work extends this principle to the context of sequentially locally convex topological vector spaces and manifolds with rough boundaries.

2 Preliminaries

This chapter provides an overview of some fundamental concepts and foundational material concerning differential calculus in locally convex topological vector spaces (for more details see, [13], [14], [15], [16]), establishing the necessary theoretical framework for subsequent developments.

Throughout this work,“ $\mathbb{K}$ denotes either the field of real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$. All vector spaces are assumed to be $\mathbb{K}$-vector spaces, and all linear maps are $\mathbb{K}$-linear unless explicitly stated otherwise”. We adopt the standard notation:

$$\mathbb{N}_0 := \{0, 1, 2, \ldots\}, \quad \mathbb{N} := \{1, 2, 3, \ldots\}.$$

Definition 1. *[16] “Let L, E be locally convex topological vector spaces, and let $U \subseteq L$ be open subsets. For integer $r \in \mathbb{N}_0 \cup \{\infty\}$ and $m \in \mathbb{N}_0$ satisfying $m \leq r$, the following holds:*

A mapping $f : U \to E$ is termed C^r-map if, for every $x \in U$ and directions $w_1, \ldots, w_m \in L$, the iterated directional derivatives

$$d^{(m)} f(x, w_1, \ldots, w_m) = (D_{w_m} \circ \cdots \circ D_{w_1})\, f(x)$$

exist and induce continuous maps

$$d^{(m)} f : U \times L^m \to E$$

The case $r = \infty$ corresponds to smooth mappings, with $df := d^{(1)} f$ denoting the differential”.

Definition 2. *[16] “ A subset $U \subseteq L$ is termed* locally convex *if every $x \in U$ admits a convex neighborhood W within U.*

Definition 3. *Let $U \subseteq L$ be a locally convex subset with dense interior. A mapping $f : U \to E$ is classified as a C^r-map if:*

- *The restriction $f|_{U^\circ} : U^\circ \to E$ is C^r-differentiable.*
- *Each derivative $d^{(m)}(f|_{U^\circ}) : U^\circ \times L_1^m \to E$ extends uniquely to a continuous map $d^{(m)} f : U \times L_1^m \to E$”.*

Remark 1. *[13] “ Let L_1, L_2, E be locally convex spaces with $U \subseteq L_1$ and $V \subseteq L_2$ open. A continuous map $f : U \times V \to E$ is C^1-differentiable if and only if:*

$$d^{(1,0)} f(x, y; h_1) := D_{(h_1,0)} f(x, y),$$
$$d^{(0,1)} f(x, y; h_2) := D_{(0,h_2)} f(x, y)$$

exist continuously for all $(x, y) \in U \times V$, $h_1 \in L_1$, $h_2 \in L_2$. In this case:

$$df\,((x, y), (h_1, h_2)) = d^{(1,0)} f(x, y, h_1) + d^{(0,1)} f(x, y, h_2)''.$$

Remark 2. *[13] The following properties of C^r-maps are fundamental:*

- *For each m, $d^{(m)} f(x, \bullet) : L^m \to E$ is symmetric and m-linear.*
- *A map $f : U \to E$ is C^{r+1} if and only if f is C^1 and $df : U \times L \to E$ is C^r.*
- *C^r-maps are closed under composition.*

Definition 4. *[16] “ Let L, E be locally convex spaces and $U \subseteq L$ be a locally convex subset.*

- *The space $C(U, E)$ of continuous maps is endowed with the compact-open topology.*
- *The space $C^r(U, E)$ of C^r-maps is topologized by the* compact-open C^r-topology*, defined as the initial topology induced by the family:*

$$\left(d^{(m)}(\bullet)\right)_{m=0}^{r} : C^r(U, E) \to \prod_{m=0}^{r} C(U \times L^m, E),$$

$$f \mapsto \left(d^{(m)} f\right)''.$$

3 Exponential Law on Sequentially Spaces

Definition 5. *[12] “Let L_1, L_2, and E be locally convex spaces, with $U \subseteq L_1$ and $V \subseteq L_2$ open subsets. For integers $r, k \in \mathbb{N}_0 \cup \{\infty\}$, a mapping $f : U \times V \to E$ is termed a $C^{r,k}$-map if for all indices $m, n \in \mathbb{N}_0$ satisfying $m \leq r$ and $n \leq k$, the following conditions hold:*

1. *The iterated directional derivative*

$$d^{(m,n)} f(x, y, w_1, \ldots, w_m, v_1, \ldots, v_n) := (D_{(w_m,0)} \circ \cdots \circ D_{(w_1,0)} \circ D_{(0,v_n)} \circ \cdots \circ D_{(0,v_1)}) f(x, y)$$

exists for all $x \in U$, $y \in V$, $w_1, \ldots, w_m \in L_1$, and $v_1, \ldots, v_n \in L_2$.

2. *The induced map*

$$d^{(m,n)} f : U \times V \times L_1^m \times L_2^n \to E$$

defined by

$$(x, y, w_1, \ldots, w_m, v_1, \ldots, v_n) \mapsto d^{(m,n)} f(x, y, w_1, \ldots, w_m, v_1, \ldots, v_n)$$

is continuous”.

Definition 6. *Let L_1, L_2, and E be locally convex spaces, and let $U \subseteq L_1$ and $V \subseteq L_2$ be locally convex subsets with dense interiors. A mapping $f : U \times V \to E$ is classified as a $C^{r,k}$-map if:*

1. *The restriction $f|_{U^\circ \times V^\circ} : U^\circ \times V^\circ \to E$ is $C^{r,k}$-differentiable.*

2. *For all $m \leq r$ and $n \leq k$, the restricted derivatives*
$$d^{(m,n)}(f|_{U^\circ \times V^\circ})\colon U^\circ \times V^\circ \times L_1^m \times L_2^n \to E$$
admit continuous extensions
$$d^{(m,n)}f\colon U \times V \times L_1^m \times L_2^n \to E.$$

Lemma 1. *[12] "Let L_1, L_2, and E be locally convex spaces, with $U \subseteq L_1$ and $V \subseteq L_2$ locally convex subsets possessing dense interiors. For $r, k \in \mathbb{N}_0 \cup \{\infty\}$, a mapping $f\colon U \times V \to E$ is $C^{r,k}$-differentiable if and only if the following conditions are satisfied:*

1. *For each fixed $x \in U$, the section $f_x; := f(x, \bullet)\colon V \to E$ is C^k-differentiable.*
2. *For every $y \in V$, $n \leq k$, and directions $v_1, \ldots, v_n \in L_2$, the map*
$$d^{(n)}f_\bullet(y, v_1, \ldots, v_n)\colon U \to E, \quad x \mapsto d^{(n)}f_x(y, v_1, \ldots, v_n)$$
is C^r-differentiable.
3. *The mixed derivative operator*
$$d^{(m,n)}f\colon U \times V \times L_1^m \times L_2^n \to E$$
defined by
$$(x, y, w_1, \ldots, w_m, v_1, \ldots, v_n) \mapsto d^{(m)}\left(d^{(n)}f_\bullet(y, v_1, \ldots, v_n)\right)(x, w_1, \ldots, w_m)$$
is continuous for all $m \leq r$ and $n \leq k$".

Theorem 1. *Let L_1, L_2, and E be locally convex spaces, $U \subseteq L_1$ and $V \subseteq L_2$ be locally convex subsets with dense interiors, and $r, k \in \mathbb{N}_0 \cup \{\infty\}$. Assume V is sequentially compact. Then:*
$$\Phi\colon C^{r,k}(U \times V, E) \to C^r(U, C^k(V, E)), \quad f \mapsto f^\vee$$
is a homeomorphism. Furthermore, if $g\colon U \to C^k(V, E)$ is C^r-differentiable, then:
$$g^\wedge\colon U \times V \to E, \quad g^\wedge(x, y) := g(x)(y)$$
is $C^{r,k}$-differentiable.

Proof. To establish the result, it suffices to verify the latter claim. Let $g \in C^r(U, C^k(V, E))$. The map $g^\wedge$ is $C^{r,k}$-differentiable, implying $g = (g^\wedge)^\vee = \Phi(g^\wedge)$. Surjectivity of Φ follows, and by [12] (Theorem 49), Φ constitutes a topological vector space isomorphism.

Observe that $g^\wedge(x, y) = \varepsilon(g(x), y)$, where the evaluation map:
$$\varepsilon\colon C^k(V, E) \times V \to E, \quad (\gamma, y) \mapsto \gamma(y)$$
is $C^{\infty,k}$-differentiable, [12] (Proposition 42). Consequently, by [12] (Lemma 43), $g^\wedge$ is $C^{r,k}$-map. □

4 Exponential Law for Mappings on Manifolds

Definition 7. *[12] "A* manifold with rough boundary *modelled on a locally convex space L is a Hausdorff topological space N equipped with an atlas of smoothly compatible homeomorphisms $\phi\colon U_\phi \to V_\phi$, where each U_ϕ is open in N and $V_\phi \subseteq L$ is a locally convex subset with dense interior. Specific cases include:*

- Ordinary manifolds*: All V_ϕ are open in L.*
- Manifolds with smooth boundary*: Each V_ϕ lies in a closed hyperplane $\lambda^{-1}([0, \infty))$ for some $\lambda \in L'$.*
- Manifolds with corners*: Each V_ϕ is open in $\lambda_1^{-1}([0, \infty)) \cap \cdots \cap \lambda_n^{-1}([0, \infty))$ for linearly independent $\lambda_1, \ldots, \lambda_n \in L'$".*

Definition 8 ($C^{r,k}$-Maps on Product Manifolds)**.** *Let N_1 and N_2 be manifolds (possibly with rough boundary) modelled on locally convex spaces, $r, k \in \mathbb{N}_0 \cup \{\infty\}$, and E a locally convex space. A continuous map $f\colon N_1 \times N_2 \to E$ is called a $C^{r,k}$-map if for all charts $\varphi\colon U_\varphi \to V_\varphi$ of N_1 and $\psi\colon U_\psi \to V_\psi$ of N_2, the composition:*
$$f \circ (\varphi^{-1} \times \psi^{-1})\colon V_\varphi \times V_\psi \to E$$
is $C^{r,k}$-differentiable.

Definition 9. *The space $C^{r,k}(N_1 \times N_2, E)$ is endowed with the initial topology induced by the family of maps:*
$$f \mapsto f \circ (\varphi^{-1} \times \psi^{-1}) \in C^{r,k}(V_\varphi \times V_\psi, E),$$
where φ and ψ range over the maximal atlases of N_1 and N_2, respectively.

Lemma 2. *[17] "Let $\{\theta_j : X_j \to Y_j\}_{j \in J}$ be a family of topological embeddings. Then:*
$$\prod_{j \in J} \theta_j\colon \prod_{j \in J} X_j \to \prod_{j \in J} Y_j, \quad (x_j)_{j \in J} \mapsto (\theta_j(x_j))_{j \in J}$$
is a topological embedding".

Proposition 1. *[17] "Let N_1, N_2 be manifolds modelled on locally convex spaces, and E a locally convex space. Then:*

1. *For every $f \in C^{r,k}(N_1 \times N_2, E)$, the adjoint map $f^\vee \in C^r(N_1, C^k(N_2, E))$.*
2. *The linear map:*
$$\Phi\colon C^{r,k}(N_1 \times N_2, E) \to C^r(N_1, C^k(N_2, E)),$$
$$f \mapsto f^\vee$$
is a topological embedding".

Theorem 2. *Let N_1, N_2 be rough-boundary manifolds modelled on locally convex spaces L_1, L_2, respectively, and E a locally convex space. If N_2 is sequentially compact, then:*

$$\Phi\colon C^{r,k}(N_1 \times N_2, E) \to C^r(N_1, C^k(N_2, E))$$

is a homeomorphism. Furthermore, a map $g\colon N_1 \to C^k(N_2, E)$ is C^r-differentiable if and only if its adjoint:

$$g^\wedge\colon N_1 \times N_2 \to E, \quad (x, y) \mapsto g(x)(y)$$

is $C^{r,k}$-differentiable.

Proof. By Proposition 1, surjectivity of Φ suffices. Let $g \in C^r(N_1, C^k(N_2, E))$. For charts φ on N_1 and ψ on N_2, the map:

$$f \circ (\varphi^{-1} \times \psi^{-1}) = \left(C^k(\psi^{-1}, E) \circ g \circ \varphi^{-1}\right)^\wedge$$

is $C^{r,k}$-differentiable by Theorem 1, as $C^k(\psi^{-1}, E)$ is continuous linear. The k-space property of $V_\varphi \times V_\psi \times L_1 \times L_2$ ensures applicability of the exponential law. □

5 Conclusion

his work establishes the exponential law for $C^{r,k}$-mappings on products of sequentially locally convex spaces and manifolds by constructing a topological isomorphism between mapping spaces and their adjoints. Our results provide foundational tools for studying Lie group structures on non-compact mapping spaces. This work bridges infinite-dimensional differential geometry and modern mathematical physics. Future research may explore applications in non-linear PDEs and symplectic topology or investigating applications in geometric fractional analysis (as in [18], [19], [20]), where sequential compactness arises naturally in configuration spaces.

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