Interrelations Among Concircular, Conformal and Conharmonic Curvatures in Fifth-Order Recurrent Finsler Spaces

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Abstract: - The significance of this research lies in the generalization of curvature tensors, particularly the concircular curvature tensor, which is studied as a fifth recurrent tensor within the framework of Cartan's fourth curvature tensor. It contributes to the fundamental understanding of geometric properties in Finsler spaces. This paper builds upon new the concircular curvature tensor M_{ijkh} in generaralized fifth recurrent Finsler space that Cartan's fourth curvature tensor K^i_{jkh} in sense of Berwald $(G\mathcal{B}K-5RF_n)$ via Lie derivative. We obtain the relation between the concircular curvature tensor M_{ijkh}, conformal curvature tensor C_{ijkh}, conharmonic curvature tensor L^r_{ikh} and associate curvature tensor R_{ijkh} if the metric tensor $g_{rj}=1$ by Lie - derivative. Also, we show these curvature tensors have the same extension and direction. In addition, the concircular curvature tensor and conharmonic curvature tensor L^i_{jkh} are co-directional. We prove that the concircular curvature tensor M_{ijkh} behaves as fifth recurrent by Lie derivative in the main space.

Key-Words: - Concircular curvature tensor M_{ijkh}, Conformal curvature tensor C_{ijkh}, Conharmonic curvature tensor L^i_{jkh}, Lie-derivative L_v, $G\mathcal{B}K-5RF_n$.

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1 Introduction and Preliminaries

Finsler geometry is a generalization of Riemannian geometry in which the metric depends on both position and direction. This framework allows for the study of more general geometric structures and has important applications in mathematics and physics. In this context, [1] introduced the foundations of Finsler geometry and special Finsler spaces.

In differential geometry, curvature tensors play a fundamental role. Accordingly, the relationships between various curvature tensors in Finsler geometry have been extensively investigated by many authors. For instance, the relations among the W-curvature tensor, conformal curvature tensor, conharmonic curvature tensor, and concircular curvature tensor were studied by [2]. In a related direction, [3] analyzed the inheritance of conharmonic curvature in the spacetime of general relativity.

Furthermore, the theory of Lie derivatives has been systematically developed by [4] and [5]. Subsequently, [6] studied the Lie derivative in recurrent and birecurrent Finsler spaces. In addition, the Lie derivative and its applications in fluid mechanics were investigated in [7] and [8]. Recently, various relations between the Berwald covariant derivative and the Lie derivative of the conharmonic curvature tensor have been obtained in generalized fifth recurrent Finsler spaces, [9].

The metric tensor g_{ij} and the Kronecker delta δ_h^i satisfy the relation, [10] and [11]

$$g_{ij}\,g^{ik} = \delta_j^k = \begin{cases} 1 & if\ \ j = k\ , \\ 0 & if\ \ j \neq k\ . \end{cases} \tag{1}$$

The associate curvature tensor R_{ijkh} of the Cartan's third curvature tensor R_{jkh}^i is given by

$$R_{ijkh} = g_{rj}\,R_{ikh}^r\ . \tag{2}$$

A Finsler space in which the Berwald connection parameter G_{kh}^i does not depend on the directional coefficients y^i is known as an affinely connected space (Berwald space). Therefore, the affinely connected space is defined by one of the equivalent conditions, [11].

$$a)\ \mathcal{B}_k g_{ij} = 0 \qquad \text{and} \qquad b)\ \mathcal{B}_k g^{ij} = 0. \tag{3}$$

Consider the infinitesimal transformation point given by [12]

$$x^{-i} = x^i + v^i(x)\varepsilon, \tag{4}$$

where ε is an infinitesimal point constant and $v^i(x)$ is a contravariant vector field independent of the directional argument and dependent on positional coordinates x^i only and $v^i(x) \neq 0$. The infinitesimal method is a tool that leeds to Lie-derivative.

A Lie - derivative evaluates the rate of change of a vector field or a tensor field along the flow of another vector or tensor field. The Lie-derivative of a vector field x^i in the sense of Berwald is given by [6].

$$L_v x^i = v^j \mathcal{B}_j\, x^i - x^j \mathcal{B}_j v^i + (\dot{\partial}_j\, x^i)\mathcal{B}_s v^j y^s. \tag{5}$$

The Lie-derivative of a general mixed tensor field T_{jkh}^i expressed in the form [13]

$$\begin{aligned} L_v T_{jkh}^i = v^m \mathcal{B}_m T_{jkh}^i - T_{jkh}^m \mathcal{B}_m v^i + T_{mkh}^i \mathcal{B}_j v^m \\ + T_{jmh}^i \mathcal{B}_k v^m + T_{jkm}^i \mathcal{B}_h v^m \\ + \dot{\partial}_m T_{jkh}^i \mathcal{B}_r v^m y^r. \end{aligned} \tag{6}$$

The Lie - derivative of the metric tensors g_{ij} is vanishing, [14].

$$L_v g_{ij} = 0. \tag{7}$$

The Berwald covariant derivative of the non-zero contravariant vector field v^m, vanishes identically [7], i.e.

$$\mathcal{B}_j v^m = 0. \tag{8}$$

A conformal curvature tensor C_{ijkh} known as a Weyl conformal curvature tensor is defined as [2], [11].

$$\begin{aligned} R_{ijkh} = C_{ijkh} \\ + \frac{1}{2}\left(g_{ik}R_{jh} + g_{jh}R_{ik} - g_{ih}R_{jk} - g_{jk}R_{ih}\right) \\ + \tfrac{R}{6}\left(g_{ih}g_{jk} - g_{ik}g_{jh}\right). \end{aligned} \tag{9}$$

A conharmonic curvature tensor L_{jkh}^i is defined as

$$\begin{aligned} L_{jkh}^i = R_{jkh}^i \\ - \tfrac{1}{2}\left(g_{jk}R_h^i + \delta_h^i R_{jk} - \delta_k^i R_{jh} - \ g_{jh}R_k^i\right). \end{aligned} \tag{10}$$

A concircular curvature tensor M_{ijkh} is defined as [2]

$$M_{ijkh} = R_{ijkh} - \tfrac{R}{12}\left(g_{jk}g_{ih} - g_{jh}g_{ik}\right). \tag{11}$$

Let us explore a generalized $\mathcal{B}K$-fifth recurrent Finsler space satisfying the relation [15]

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R_{jfkh} = a_{sqlnm} R_{jfkh}, \tag{12}$$

if and only if

$$\begin{aligned} b_{sqlnm}\left(g_{hf}g_{jk} - g_{kf}g_{jh}\right) \\ -2b_{qlnm}y^r\mathcal{B}_r\left(g_{hf}C_{jks} - g_{kf}C_{jhs}\right) \\ -\ c_{sqlnm}\left(g_{hf}C_{jkn} - g_{kf}C_{jhn}\right) \\ -d_{sqlnm}\left(g_{hf}C_{jkl} - g_{kf}C_{jhl}\right) \\ -e_{sqlnm}\left(g_{hf}C_{jkq} - g_{kf}C_{jhq}\right) \\ +\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(C_{jft}H_{kh}^t) \\ -a_{sqlnm}(C_{jft}H_{kh}^t) = 0. \end{aligned} \tag{13}$$

This paper discusses the relationships between the concircular curvature tensor M_{ijkh}, conformal curvature tensor C_{ijkh}, conharmonic curvature tensor L_{jkh}^i and associate curvature tensor R_{ijkh} in the generalized fifth recurrent Finsler space for Cartan's fourth curvature tensor K_{jkh}^i in the sense of Berwald via Lie-derivative.

2 Main Results

This paper discusses the concircular curvature tensor M_{ijkh} that behaves as a fifth recurrent. We study its relationship with other curvature tensors in a generalized fifth recurrent Finsler space for Cartan's fourth curvature tensor K^i_{jkh} in the sense of Berwald by the Lie - derivative.

Theorem 2.1 In $G\mathcal{B}K-5RF_n$, the Berwald covariant derivative of the fifth order for the concircular curvature tensor M_{ijkh} and the associate curvature tensor R_{ijkh} have the same extension and direction.

Proof. Transvecting Eq. (11) by the non-zero tensor g^{jm} and using Eq. (1), we get

$$M_{ijkh} = R_{ijkh}. \tag{14}$$

Taking the Lie - derivative of both sides for Eq. (14) and using Eqs. (6) and (8), we get

$$\mathcal{B}_m M_{ijkh} = \mathcal{B}_m R_{ijkh}.$$

Taking the Berwald covariant derivative of fourth order for the above equation with respect to x^n, x^l, x^q and x^s, respectively, we get

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m M_{ijkh} = \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R_{ijkh}. \tag{15}$$

The above equation gives the required expression.

Theorem 2.2 In $G\mathcal{B}K-5RF_n$, the Berwald covariant derivative of the fifth order for the concircular curvature tensor M_{ijkh} and the conharmonic curvature tensor$L^r_{ikh}$is co-directional.

Proof. Using Eq. (2) in Eq. (14), we get

$$M_{ijkh} = g_{rj}\, R^r_{ikh}\,.$$

Taking the Lie - derivative of both sides of above equation and using Eqs. (6), (7) and (8), we get

$$\mathcal{B}_m M_{ijkh} = g_{rj}\, \mathcal{B}_m R^r_{ikh}\,.$$

Taking the Berwald covariant derivative of fourth order for the above equation with respect to x^n, x^l, x^q and x^s, respectively. Then using Eq. [(3)a], we get

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m M_{ijkh} = g_{rj}\, \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R^r_{ikh}\,. \tag{16}$$

Transvecting Eq. (10) by the non-zero tensor g^{jm} and using Eq. (1), we get

$$L^i_{jkh} = R^i_{jkh}\,. \tag{17}$$

Using above equation in Eq. (16), we get

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m M_{ijkh} = g_{rj}\, \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m L^r_{ikh}\,. \tag{18}$$

The above equation gives the result required.

Theorem 2.3 In $G\mathcal{B}K-5RF_n$, the Lie-derivative is distributive on the addition of the fifth order Berwald covariant derivative for the conharmonic curvature tensor L^i_{jkh} and the concircular curvature tensor M_{ijkh}.

Proof. Adding the conharmonic curvature tensor L^i_{jkh} of both sides of Eq. (14), then taking the fifth order Berwald covariant derivative for the result equation with respect to x^m, x^n, x^l, x^q and x^s, respectively, we get

$$\begin{aligned}&\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m (L^i_{jkh} + M_{ijkh})\\ &= \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m L^i_{jkh} + \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R_{ijkh}.\end{aligned}$$

Taking the Lie -derivative of both sides of the above equation and using Eq. (14), we get

$$\begin{aligned}&L_v[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m (L^i_{jkh} + M_{ijkh})]\\ &= L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m L^i_{jkh})\\ &+ L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m M_{ijkh}).\end{aligned} \tag{19}$$

Thus, the theorem is proved.

Theorem 2.4 In $G\mathcal{B}K-5RF_n$, the relation between the concircular curvature tensor M_{ijkh}, conformal curvature tensor C_{ijkh}, conharmonic curvature tensor L^r_{ikh} and associate curvature tensor R_{ijkh} by the Lie - derivative is given by

$$\begin{aligned}&L_v[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m (2M_{ijkh} + C_{ijkh} + L^r_{ikh})]\\ &= 4L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R_{ijkh}),\end{aligned} \tag{20}$$

if the metric tensor $g_{rj} = 1$.

Proof. Transvecting Eq. (9) by the non-zero tensor $(g^{mh} g^{rk})$ and using Eq. (1), we get

$$C_{ijkh} = R_{ijkh}. \tag{21}$$

Adding the concircular curvature tensor M_{ijkh} of both sides of above equation, then taking the fifth order Berwald covariant derivative for the result equation with respect to x^m, x^n, x^l, x^q and x^s, respectively, we get

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(M_{ijkh} + C_{ijkh}) \\ = \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m M_{ijkh} + \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R_{ijkh}.$$

Taking the Lie -derivative of both sides of the above equation and using Eqs. (14) and (21), we get

$$L_v[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(M_{ijkh} + C_{ijkh})] \\ = 2L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m C_{ijkh}). \tag{22}$$

Similarly, adding the conharmonic curvature tensor L^r_{ikh} on both sides of Eq. (14), then taking the fifth order Berwald covariant derivative for the result equation with respect to x^m, x^n, x^l, x^q and x^s, respectively, we get

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(M_{ijkh} + L^r_{ikh}) \\ = \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R_{ijkh} + \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m L^r_{ikh}.$$

Taking the Lie - derivative of both sides of the above equation and using Eqs. (2) and (17), we get

$$L_v[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(M_{ijkh} + L^r_{ikh})] \\ = L_v[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(g_{rj}\, R^r_{ikh} + R^r_{ikh})].$$

Assuming the metric tensor $g_{rj} = 1$ on the right side of the above equation, the resulting equation can be written as:

$$L_v[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(M_{ijkh} + L^r_{ikh})] \\ = 2L_v[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(g_{rj}\, R^r_{ikh})]. \tag{23}$$

Now, adding Eq. (22) to Eq. (23), using Eqs. (2) and (21), we get Eq. (20). Hence, the proof is complete.

The following corollaries are obtained from the previous theorems. Using Eq. (12) in Eq. (15), we get:

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m M_{ijkh} = a_{sqlnm} R_{ijkh}.$$

Using Eq. (14) in the above equation, we get:

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m M_{ijkh} = a_{sqlnm} M_{ijkh}. \tag{24}$$

Thus, we have
Corollary 2.5 In $G\mathcal{B}K - 5RF_n$, the concircular curvature tensor M_{ijkh} behaves as a fifth recurrent [provided Eq. (13) holds].

Using Eq. (21) in Eq. (15), we get

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m M_{ijkh} = \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m C_{ijkh}. \tag{25}$$

Thus, we have
Corollary 2.6 In $G\mathcal{B}K - 5RF_n$, the fifth order Berwald covariant derivative for the concircular curvature tensor M_{ijkh} and the conformal curvature tensor C_{ijkh} have the same extension and direction.

In view of Eq. (20), we get:

$$L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m M_{ijkh}) \\ = 2L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R_{ijkh}) \tag{26}$$

if

$$L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m C_{ijkh}) \\ = -L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m L^r_{ikh}). \tag{27}$$

Thus, we have

Corollary 2.7 In $G\mathcal{B}K - 5RF_n$, the Lie - derivative of Berwald covariant derivative of the fifth order for the concircular curvature tensor M_{ijkh} is double derivative of Lie of the Berwald covariant derivative of the fifth order for the associate curvature tensor R_{ijkh} if Eq. (27) holds.

3 Conclusion

In this paper, we investigated the relationships between the concircular curvature tensor M_{ijkh}, conformal curvature tensor C_{ijkh}, conharmonic curvature tensor L^r_{ikh} and associate curvature tensor R_{ijkh} in $G\mathcal{B}K - 5RF_n$ using the Lie-derivative. Several theorems have been discussed in the mentioned space.

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