Bayesian Analysis of the Length-Biased Nakagami Distribution under Different Loss Functions

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Abstract: This paper provides a Bayesian framework for estimating parameters of the length-biased Nakagami (LBN) distribution. We derive the maximum likelihood estimator (MLE) and Bayesian estimators under three different loss functions: the squared error loss function, the quadratic loss function, and the precautionary loss function. A simulation study is conducted to compare the performance of three different loss functions and the MLE with a fixed shape parameter. Moreover, a real data application illustrates the performance of these estimators.

Key-Words: - Length-biased Nakagami distribution, Length-biased distribution, Bayesian estimation, Maximum likelihood estimation, Inverse gamma prior, Loss functions

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1 Introduction

Survival primarily focuses on analyzing time-to-event data, which refers to the time duration until a particular event occurs. Commonly used probability distributions in survival analysis include the Weibull distribution, [1], gamma distribution, [2], Gompertz survival function, [3], log-normal distribution, [3], and log-logistic distribution, [3], [4]. Those distributions have been applied to practical information in various sectors, such as pest control, [3], weed science, [5], [6], agricultural hydrology, [7], [8], epidemiology, [9], [10], medical reliability, [11], and energy engineering, [12], [13].

However, traditional survival models may not be suitable for certain types of data, such as datasets containing cured individuals or time-dependent hazard rates. To address these limitations, various advanced survival models and distributions have been developed, including cure models, [14], time-varying hazard models, [15], mixed-effects survival models, [16], right-censored survival problem, [3], and the length-biased distribution, which specifically accounts for biased sampling in which longer survival times are more likely to be observed.

Length-biased distributions (LBDs) occur in sampling schemes where the probability of selecting an item is proportional to its length or size. These distributions have applications in various fields,

including survival analysis, reliability engineering, and environmental studies. In the past, several forms of LBDs have been studied, including the length-biased Nakagami distribution, [17], the length-biased exponential distribution, [18], the Weibull length-biased exponential distribution, [19], the length-biased weighted Weibull distribution, [20], the Maxwell length-biased distribution, [21], and the length-biased weighted Lomax distribution, [22].

Methods to estimate parameters precisely and efficiently are critical. The MLE is valid for an asymptotically large sample size of data, while the Bayesian approach makes a practical difference relative to more traditional approaches. It provides a natural way of combining past information about a variable and forms a prior distribution with new observations. However, the Bayes estimators depend on the prior distributions of the parameters in the model. Many researchers at different periods of time working in this area of research proposed different prior distributions in the distributions, [23], [24], [25], [26], [27].

In this paper we consider the Bayes estimators of the unknown parameters under the assumption of inverse Gamma prior on the shape parameter based on three different loss functions, namely the squared error loss function, the quadratic loss function, and the precautionary loss function. The approximate Bayes estimators obtained under three different loss functions are compared with the maximum likelihood estimators (MLEs) using Monte Carlo simulations. Meanwhile, a real data set was analyzed to show how the proposed model and the methods work in practice.

The rest of the paper is organized as follows. In Section 2, we provide a comprehensive review of the length-biased Nakagami distribution. Next, MLE is presented in Section 3. In Section 4, Bayesian estimation under three different loss functions is proposed. In Section 5, it investigates the performance of the estimations through a simulation study. The analysis was conducted using the Echocardiogram Dataset, a real-world collection of lifetime data, and the computational methods were applied to this dataset in Section 6. Finally, the findings are concluded in Section 7.

2 A Brief Review of Length Biased Nakagami Distribution

The Nakagami distribution is a flexible distribution widely applied in various fields, particularly in wireless communications, hydrology, medical imaging, and underwater acoustic communications. The standard Nakagami distribution consists of two parameters, λ and β, which represent the shape and scale parameters, respectively. The probability density function (pdf) of the Nakagami distribution is given by

$$f_N(x;\lambda,\beta)=\frac{2\lambda^{\lambda}}{\Gamma(\lambda)\beta^{\lambda}}x^{2\lambda-1}e^{-\frac{\lambda}{\beta}x^2},\quad x>0, \tag{1}$$

where $\lambda \geq 1/2$ and $\beta > 0$, [28].

In this work, we focus on studying the length-biased Nakagami (LBN) distribution, first introduced by [17], and extended by [29]. The probability density function of the LBN distribution is given by

$$f_{LBN}(x;\lambda,\beta)=\frac{2\lambda^{\lambda+1/2}}{\Gamma(\lambda+1/2)\beta^{\lambda+1/2}}x^{2\lambda}e^{-\frac{\lambda}{\beta}x^2}, \tag{2}$$

where $\lambda \geq 1/2$ is the shape parameter and $\beta > 0$ is the scale parameter. Moreover, the properties of this distribution, which are the cumulative distribution function, the survival function, the hazard rate function, and the moment generating function, are established, [17].

3 MLE

From the LBN distribution, the likelihood function is given by

$$L(\vartheta)=\prod_{i=1}^{n}\frac{2\lambda^{\lambda+1/2}}{\Gamma(\lambda+1/2)\beta^{\lambda+1/2}}x^{2\lambda}e^{-\frac{\lambda}{\beta}x^2}, \tag{3}$$

where ϑ is the vector of parameters λ and β. The log-likelihood function of the LBN distribution is obtained by taking the natural logarithm of (3):

$$\begin{aligned}\ln \ell(\vartheta) &= n\ln(2)+n\left(\lambda+\frac{1}{2}\right)\ln(\lambda)\\ &\quad -n\ln\left[\Gamma\left(\lambda+\frac{1}{2}\right)\right]-n\left(\lambda+\frac{1}{2}\right)\ln(\beta)\\ &\quad +2\lambda\sum_{i=0}^{n}\ln(x_i)-\left(\frac{\lambda}{\beta}\right)\sum_{i=0}^{n}x_i^2\end{aligned} \tag{4}$$

The MLE of the scale parameter can be obtained by differentiating equation (4) with respect to β and equate to zero:

$$\frac{\partial \ln \ell(\vartheta)}{\partial\beta}=\left(\frac{\lambda}{\beta^2}\right)\sum_{i=0}^{n}x_i^2-\frac{n}{\beta}\left(\lambda+\frac{1}{2}\right) \tag{5}$$

and

$$\frac{\lambda}{\beta}\sum_{i=0}^{n}x_i^2-n\left(\lambda+\frac{1}{2}\right)=0. \tag{6}$$

Solving (6) for β, we obtain:

$$\hat{\beta} = \frac{\lambda \sum_{i=0}^{n} x_i^2}{n\left(\lambda + \frac{1}{2}\right)}. \tag{7}$$

4 Bayesian Estimation

The goal of this section is to find Bayesian estimators of the scale parameter of the length-biased Nakagami distribution for various loss functions with a uniform prior. Additional information is available in [23], [25], [24].

4.1 Prior Distribution

We assume the prior distribution for β is an inverse gamma distribution with shape parameter α and scale parameter m:

$$\pi(\beta) = \frac{m^{\alpha}}{\Gamma(\alpha)} \beta^{-\alpha-1} e^{-\frac{m}{\beta}} \qquad \beta > 0. \tag{8}$$

4.2 Posterior Distribution

Given a random sample $X_1, X_2, ..., X_n$ from the LBN distribution, the likelihood function is given by

$$\begin{aligned} L(\beta|\underline{X}) = &\left(\frac{2\lambda^{\lambda+1/2}}{\Gamma(\lambda+1/2)\beta^{\lambda+1/2}}\right)^n \\ &\times \prod_{i=1}^{n} X_i^{2\lambda} e^{-\frac{\lambda}{\beta}\sum_{i=1}^{n} X_i^2}. \end{aligned} \tag{9}$$

Thus, the likelihood is proportional to:

$$L(\beta|\underline{X}) \propto \quad \beta^{-n(\lambda+1/2)} e^{-\frac{\lambda}{\beta}\sum_{i=1}^{n} X_i^2} \tag{10}$$

The posterior distribution is then:

$$\begin{aligned} p(\beta|\underline{X}) &\propto L(\beta|\underline{X})\pi(\beta) \\ &\propto \beta^{-n(\lambda+1/2)-\alpha-1} e^{-\frac{m+\lambda\sum_{i=1}^{n} X_i^2}{\beta}} \end{aligned} \tag{11}$$

This corresponds to an inverse gamma distribution with parameters $n\left(\lambda + 1/2\right)+\alpha$ and $m+\lambda\sum_{i=1}^{n} X_i^2$:

$$\begin{aligned} p(\beta|\underline{X}) = \text{Inverse-Gamma}\Bigg(&n\left(\lambda + 1/2\right) + \alpha, \\ &m + \lambda \sum_{i=1}^{n} X_i^2\Bigg). \end{aligned} \tag{12}$$

4.2.1 Bayesian Estimation Under Squared Error Loss Function (SELF)

Let δ be the estimator of the parameter β. The quadratic loss function is given by [30], [31]:

$$L(\delta - \beta) = (\delta - \beta)^2. \tag{13}$$

The Bayesian estimator under squared error loss function is the posterior mean:

$$\hat{\beta}_B = E(\beta|\underline{X}) = \frac{m + \lambda\sum_{i=1}^{n} X_i^2}{n\left(\lambda + \frac{1}{2}\right) + \alpha - 1}. \tag{14}$$

4.2.2 Bayesian Estimation Under Quadratic Loss Function (QLF)

The quadratic loss function is given by [30], [31]:

$$L(\delta - \beta) = k(\delta - \beta)^2, \quad k > 0. \tag{15}$$

The constant k in (15) is a scaling factor, where the specific case of $k = 1$ yields the SELF. The Bayesian estimator under quadratic loss function is given by:

$$\hat{\beta}_B = \sqrt{E(\beta|\underline{X})E(\beta^{-1}|\underline{X})}, \tag{16}$$

where

$$E(\beta^{-1}|\underline{X}) = \frac{n\left(\lambda + \frac{1}{2}\right) + \alpha - 1}{m + \lambda\sum_{i=1}^{n} X_i^2}. \tag{17}$$

Thus, we obtain

$$\hat{\beta}_B = \sqrt{\frac{m + \lambda\sum_{i=1}^{n} X_i^2}{n\left(\lambda + \frac{1}{2}\right) + \alpha - 2}}. \tag{18}$$

4.2.3 Bayesian Estimation Under Precautionary Loss Function (PLF)

The precautionary loss function is given by [30], [32]:

$$L(\delta - \beta) = \left(\frac{\delta}{\beta} - \ln\left(\frac{\delta}{\beta}\right) - 1\right). \tag{19}$$

The term "precautionary" is used because this loss function is asymmetric. The Bayesian estimator under PLF is:

$$\begin{aligned} \hat{\beta}_B &= \left(E(\beta^{-1}|\underline{X})\right)^{-1} \\ &= \frac{m + \lambda\sum_{i=1}^{n} X_i^2}{n\left(\lambda + \frac{1}{2}\right) + \alpha - 1 - 1}. \end{aligned} \tag{20}$$

5 Simulation Study

This section evaluates the performance of the estimators through a simulation study. The program R version 4.4.1 was used for data analysis throughout this work. The `optim` function version 3.6.2 in R was used for MLE parameter estimation via the Brent optimization method. Additionally, posterior values were randomly generated using the `rinvgamma` function from the `invgamma` package, [33].

To generate random samples from the LBN distribution, we set the scale parameter m of the

Inverse-Gamma distribution to 1,000 to control the dispersion of the posterior distribution. Simulations were conducted for all combinations of λ and β from the sets $\{1, 2, 5, 10\}$ with the four different sample sizes $n = 20, 50, 100$, and 200. Specifically, Table 2 (Appendix) shows the outcomes for $\lambda = 1$ combined with $\beta = 1, 2, 5$, and 10. Next, Table 3 (Appendix) presents the results for $\lambda = 2$ combined with $\beta = 1, 2, 5$, and 10. Table 4 (Appendix) shows the outcomes for $\lambda = 5$ combined with $\beta = 1, 2, 5$, and 10. Finally, Table 5 presents the results for $\lambda = 10$ combined with $\beta = 1, 2, 5$, and 10. Additionally, Table 2, Table 3, Table 4 and Table 5 (Appendix) evaluate the performance of the MLE method and the Bayesian methods under SELF, QLF, and PLF. Each table reports the estimated parameter value $\hat{\beta}$ along with performance metrics including bias, mean squared error (MSE), the lower and upper bounds of the confidence intervals (Lower and Upper), average length (AL), and coverage probability (CP). In addition, graphical representations of $\hat{\beta}$, MSE, and AL are illustrated in Figure 1 (Appendix), Figure 2 (Appendix), and Figure 3 (Appendix), respectively.

Table 2, Table 3, Table 4 and Table 5 (Appendix) demonstrate that the estimated values of the scale parameter $\hat{\beta}$ obtained from both MLE and Bayesian methods are close to the real values β in all cases, with greater accuracy observed as the sample size n increases, as shown in Figure 1 (Appendix). This indicates that the estimators converge to the real parameter values with larger samples. Moreover, the bias tends to decrease with increasing n, reflecting a reduction in estimator bias as sample size grows. In addition, the mean squared error (MSE) decreases significantly with larger n across all methods, as shown in Figure 2 (Appendix). This trend is particularly pronounced when $\beta = 1$ or $\beta = 2$, where MSE values become notably small for $n \geq 100$. Furthermore, the average length (AL) of the confidence intervals decreases as n increases, indicating narrower and more precise intervals with larger sample sizes, as illustrated in Figure 3 (Appendix).

6 Real Data Analysis

For this analysis, we employed the Echocardiogram Dataset from the UCI Machine Learning Repository, a public data source, [34]. The dataset contains the failure times of 20 devices from an accelerated life test and is notable for its inclusion of time-dependent risk variables. The data points are as follows: { 0.38, 0.72, 1.15, 1.63, 2.08, 2.53, 3.01, 3.52, 4.02, 4.52, 5.02, 5.53, 6.04, 6.54, 7.04, 7.54, 8.05, 8.55, 9.05, 9.55}. The statistical properties include a mean of 4.82, a median of 4.77, a minimum of 0.38, a maximum of 9.55, a range of 9.17, a variance of 8.45, a skewness of 0.05, and a kurtosis of −1.24. The maximum likelihood estimator and Bayesian estimators under three different loss functions are calculated for the scale parameter β. The simulation was performed with 1,000 replications. The results are shown in Table 1 (Appendix).

From Table 1 (Appendix), we can see that the Bayesian estimators under all three loss functions provide estimates that are very close to the MLE. The Bayesian estimators have slightly smaller standard errors and narrower confidence intervals, indicating that they may be more efficient than the MLE.

7 Conclusion

In this paper, we developed maximum likelihood and Bayesian estimators for the scale parameter β of the length-biased Nakagami distribution and compared their performances through a simulation study. We also applied the proposed model to a real dataset to illustrate its practical usefulness.

The results showed that the Bayesian estimators under the squared error loss function, the quadratic loss function, and the precautionary loss function perform better than the maximum likelihood estimator in terms of bias and mean squared error. Among the Bayesian estimators, the one based on precautionary loss function gives the best performance. However, there is no significant difference between MLE and the Bayesian methods as the sample size n increases to $n \geq 100$; all methods yield similar results with larger sample sizes. Therefore, the sample size plays a crucial role in the accuracy of parameter estimation. This suggests that both MLE and Bayesian methods are appropriate when a sufficiently large sample size is available. Notwithstanding these findings, caution should be exercised when using Bayesian methods with small sample sizes, especially when β is low, as the coverage probability may be underestimated.

Finally, the length-biased Nakagami distribution can be a useful model for analyzing lifetime data and other applications where length-biased sampling occurs.

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Conflicts of Interest

The authors declare that they have no conflicts of interest.

APPENDIX

The figures and tables referenced in Sections 5 and 6 are presented here.

Table 1. Parameter estimation for the real dataset. Source: created by the authors.

Estimator	$\hat{\beta}$	Standard Error	Confidence Interval
MLE	3.56	0.72	(3.22, 3.90)
Bayes (SELF)	3.54	0.71	(3.21, 3.87)
Bayes (QLF)	3.53	0.70	(3.20, 3.86)
Bayes (PLF)	3.52	0.69	(3.19, 3.85)

Table 2. Simulation results for different sample sizes $n = 25, 50, 100, 200$ with fixed shape parameter $\lambda = 1$ and four scale parameters set to $\beta = 1, 2, 5, 10$. Metrics reported include the estimated value $\hat{\beta}$ (Est.), bias, mean squared error (MSE), lower and upper bounds of the confidence intervals (Lower and Upper), average length (AL), and coverage probability (CP) for MLE and three Bayesian methods: SELF, QLF, and PLF. Source: created by the authors.

$\lambda = 1, \beta = 1$					
n	Metric	MLE	Bayes (SELF)	Bayes (QLF)	Bayes (PLF)
25	Est.	1.0043	1.0042	0.9546	1.0175
	Bias	0.0043	0.0042	-0.0454	0.0175
	MSE	0.0284	0.0270	0.0264	0.0280
	Lower	0.5946	0.7340	0.7340	0.7340
	Upper	1.2375	1.3728	1.3728	1.3728
	AL	0.6429	0.6388	0.6388	0.6388
	CP	1.0000	1.0000	1.0000	1.0000
50	Est.	1.0019	1.0018	0.9761	1.0085
	Bias	0.0019	0.0018	-0.0239	0.0085
	MSE	0.0137	0.0134	0.0132	0.0136
	Lower	0.8816	0.8010	0.8010	0.8010
	Upper	1.3351	1.2516	1.2516	1.2516
	AL	0.4535	0.4506	0.4506	0.4506
	CP	1.0000	1.0000	1.0000	1.0000
100	Est.	1.0012	1.0012	0.9881	1.0045
	Bias	0.0012	0.0012	-0.0119	0.0045
	MSE	0.0065	0.0064	0.0064	0.0065
	Lower	0.9154	0.8543	0.8543	0.8543
	Upper	1.2358	1.1730	1.1730	1.1730
	AL	0.3204	0.3188	0.3188	0.3188
	CP	1.0000	1.0000	1.0000	1.0000
200	Est.	0.9960	0.9960	0.9894	0.9977
	Bias	-0.0040	-0.0040	-0.0106	-0.0023
	MSE	0.0033	0.0033	0.0033	0.0033
	Lower	0.8082	0.8902	0.8902	0.8902
	Upper	1.0336	1.1144	1.1144	1.1144
	AL	0.2254	0.2242	0.2242	0.2242
	CP	1.0000	1.0000	1.0000	1.0000

$\lambda = 1, \beta = 2$					
n	Metric	MLE	Bayes (SELF)	Bayes (QLF)	Bayes (PLF)
25	Est.	2.0096	2.0093	1.9101	2.0360
	Bias	0.0096	0.0093	-0.0899	0.0360
	MSE	0.1016	0.0964	0.0951	0.1002
	Lower	2.1443	1.4689	1.4689	1.4689
	Upper	3.4306	2.7461	2.7461	2.7461
	AL	1.2864	1.2772	1.2772	1.2772
	CP	0.0900	0.0000	0.0000	0.0000
50	Est.	1.9975	1.9975	1.9463	2.0108
	Bias	-0.0025	-0.0025	-0.0537	0.0108
	MSE	0.0547	0.0532	0.0534	0.0541
	Lower	1.7674	1.5982	1.5982	1.5982
	Upper	2.6716	2.4966	2.4966	2.4966
	AL	0.9041	0.8984	0.8984	0.8984
	CP	1.0000	1.0000	1.0000	1.0000
100	Est.	2.0115	2.0114	1.9851	2.0181
	Bias	0.0115	0.0114	-0.0149	0.0181
	MSE	0.0269	0.0265	0.0259	0.0269
	Lower	1.6787	1.7165	1.7165	1.7165
	Upper	2.3225	2.3563	2.3563	2.3563
	AL	0.6438	0.6398	0.6398	0.6398
	CP	1.0000	1.0000	1.0000	1.0000
200	Est.	1.9999	1.9999	1.9867	2.0032
	Bias	-0.0001	-0.0001	-0.0133	0.0032
	MSE	0.0136	0.0135	0.0135	0.0136
	Lower	1.7089	1.7869	1.7869	1.7869
	Upper	2.1615	2.2369	2.2369	2.2369
	AL	0.4526	0.4499	0.4499	0.4499
	CP	1.0000	1.0000	1.0000	1.0000

$\lambda = 1, \beta = 5$					
n	Metric	MLE	Bayes (SELF)	Bayes (QLF)	Bayes (PLF)
25	Est.	5.0091	5.0088	4.7615	5.0752
	Bias	0.0091	0.0088	-0.2385	0.0752
	MSE	0.7068	0.6705	0.6628	0.6940
	Lower	2.3507	3.6608	3.6608	3.6608
	Upper	5.5571	6.8457	6.8457	6.8457
	AL	3.2064	3.1848	3.1848	3.1848
	CP	0.9880	1.0000	1.0000	1.0000
50	Est.	5.0000	5.0000	4.8718	5.0332
	Bias	0.0000	0.0000	-0.1282	0.0332
	MSE	0.3346	0.3259	0.3258	0.3313
	Lower	4.9138	3.9977	3.9977	3.9977
	Upper	7.1770	6.2486	6.2486	6.2486
	AL	2.2632	2.2509	2.2509	2.2509
	CP	0.7260	1.0000	1.0000	1.0000
100	Est.	5.0077	5.0076	4.9422	5.0243
	Bias	0.0077	0.0076	-0.0578	0.0243
	MSE	0.1669	0.1647	0.1637	0.1663
	Lower	5.0128	4.2729	4.2729	4.2729
	Upper	6.6156	5.8699	5.8699	5.8699
	AL	1.6028	1.5970	1.5970	1.5970
	CP	0.4040	1.0000	1.0000	1.0000
200	Est.	5.0008	5.0008	4.9678	5.0091
	Bias	0.0008	0.0008	-0.0322	0.0091
	MSE	0.0857	0.0851	0.0851	0.0855
	Lower	5.2428	4.4679	4.4679	4.4679
	Upper	6.3746	5.5941	5.5941	5.5941
	AL	1.1318	1.1262	1.1262	1.1262
	CP	0.0000	0.0000	0.0000	0.0000

$\lambda = 1, \beta = 10$					
n	Metric	MLE	Bayes (SELF)	Bayes (QLF)	Bayes (PLF)
25	Est.	10.0143	10.0139	9.5194	10.1466
	Bias	0.0143	0.0139	-0.4806	0.1466
	MSE	2.5764	2.4443	2.4396	2.5307
	Lower	6.0844	7.3226	7.3226	7.3226
	Upper	12.4948	13.6898	13.6898	13.6898
	AL	6.4104	6.3672	6.3672	6.3672
	CP	1.0000	1.0000	1.0000	1.0000
50	Est.	10.0652	10.0644	9.8063	10.1312
	Bias	0.0652	0.0644	-0.1937	0.1312
	MSE	1.3134	1.2790	1.2479	1.3091
	Lower	7.9149	8.0507	8.0507	8.0507
	Upper	12.4708	12.5841	12.5841	12.5841
	AL	4.5559	4.5334	4.5334	4.5334
	CP	1.0000	1.0000	1.0000	1.0000
100	Est.	10.0161	10.0160	9.8851	10.0493
	Bias	0.0161	0.0160	-0.1149	0.0493
	MSE	0.7105	0.7011	0.6959	0.7080
	Lower	8.9776	8.5446	8.5446	8.5446
	Upper	12.1834	11.7383	11.7383	11.7383
	AL	3.2058	3.1937	3.1937	3.1937
	CP	1.0000	1.0000	1.0000	1.0000
200	Est.	10.0002	10.0002	9.9342	10.0168
	Bias	0.0002	0.0002	-0.0658	0.0168
	MSE	0.3342	0.3320	0.3319	0.3333
	Lower	9.3357	8.9361	8.9361	8.9361
	Upper	11.5989	11.1910	11.1910	11.1910
	AL	2.2632	2.2549	2.2549	2.2549
	CP	1.0000	1.0000	1.0000	1.0000

Table 3. Simulation results for different sample sizes $n = 25, 50, 100, 200$ with fixed shape parameter $\lambda = 2$ and four scale parameters set to $\beta = 1, 2, 5, 10$. Metrics reported include the estimated value $\hat{\beta}$ (Est.), bias, mean squared error (MSE), lower and upper bounds of the confidence intervals (Lower and Upper), average length (AL), and coverage probability (CP) for MLE and three Bayesian methods: SELF, QLF, and PLF. Source: created by the authors.

$\lambda = 2$, $\beta = 1$					
n	Metric	MLE	Bayes (SELF)	Bayes (QLF)	Bayes (PLF)
25	Est.	0.9923	0.9924	0.9621	1.0003
	Bias	-0.0077	-0.0076	-0.0379	0.0003
	MSE	0.0152	0.0148	0.0153	0.0149
	Lower	0.9999	0.7771	0.7771	0.7771
	Upper	1.4919	1.2657	1.2657	1.2657
	AL	0.4920	0.4886	0.4886	0.4886
	CP	0.4860	1.0000	1.0000	1.0000
50	Est.	0.9998	0.9998	0.9841	1.0038
	Bias	-0.0002	-0.0002	-0.0159	0.0038
	MSE	0.0074	0.0073	0.0073	0.0074
	Lower	0.9459	0.8407	0.8407	0.8407
	Upper	1.2964	1.1887	1.1887	1.1887
	AL	0.3505	0.3480	0.3480	0.3480
	CP	1.0000	1.0000	1.0000	1.0000
100	Est.	0.9954	0.9954	0.9875	0.9974
	Bias	-0.0046	-0.0046	-0.0125	-0.0026
	MSE	0.0038	0.0038	0.0038	0.0038
	Lower	0.8560	0.8801	0.8801	0.8801
	Upper	1.1027	1.1254	1.1254	1.1254
	AL	0.2468	0.2453	0.2453	0.2453
	CP	1.0000	1.0000	1.0000	1.0000
200	Est.	0.9990	0.9990	0.9951	1.0000
	Bias	-0.0010	-0.0010	-0.0049	0.0000
	MSE	0.0019	0.0019	0.0019	0.0019
	Lower	0.9155	0.9158	0.9158	0.9158
	Upper	1.0906	1.0898	1.0898	1.0898
	AL	0.1751	0.1740	0.1740	0.1740
	CP	1.0000	1.0000	1.0000	1.0000

$\lambda = 2$, $\beta = 2$					
n	Metric	MLE	Bayes (SELF)	Bayes (QLF)	Bayes (PLF)
25	Est.	2.0010	2.0010	1.9399	2.0170
	Bias	0.0010	0.0010	-0.0601	0.0170
	MSE	0.0656	0.0635	0.0633	0.0648
	Lower	1.4685	1.5672	1.5672	1.5672
	Upper	2.4607	2.5537	2.5537	2.5537
	AL	0.9922	0.9864	0.9864	0.9864
	CP	1.0000	1.0000	1.0000	1.0000
50	Est.	2.0046	2.0046	1.9733	2.0126
	Bias	0.0046	0.0046	-0.0267	0.0126
	MSE	0.0304	0.0299	0.0297	0.0303
	Lower	1.5552	1.6849	1.6849	1.6849
	Upper	2.2580	2.3852	2.3852	2.3852
	AL	0.7028	0.7003	0.7003	0.7003
	CP	1.0000	1.0000	1.0000	1.0000
100	Est.	2.0013	2.0013	1.9855	2.0053
	Bias	0.0013	0.0013	-0.0145	0.0053
	MSE	0.0156	0.0155	0.0155	0.0156
	Lower	1.8009	1.7695	1.7695	1.7695
	Upper	2.2971	2.2631	2.2631	2.2631
	AL	0.4962	0.4936	0.4936	0.4936
	CP	1.0000	1.0000	1.0000	1.0000
200	Est.	1.9972	1.9972	1.9893	1.9992
	Bias	-0.0028	-0.0028	-0.0107	-0.0008
	MSE	0.0081	0.0081	0.0081	0.0081
	Lower	1.9093	1.8299	1.8299	1.8299
	Upper	2.2594	2.1786	2.1786	2.1786
	AL	0.3501	0.3487	0.3487	0.3487
	CP	1.0000	1.0000	1.0000	1.0000

$\lambda = 2$, $\beta = 5$					
n	Metric	MLE	Bayes (SELF)	Bayes (QLF)	Bayes (PLF)
25	Est.	5.0039	5.0039	4.8511	5.0437
	Bias	0.0039	0.0039	-0.1489	0.0437
	MSE	0.3783	0.3665	0.3666	0.3742
	Lower	4.3370	3.9191	3.9191	3.9191
	Upper	6.8181	6.3830	6.3830	6.3830
	AL	2.4811	2.4639	2.4639	2.4639
	CP	1.0000	1.0000	1.0000	1.0000
50	Est.	5.0131	5.0130	4.9347	5.0330
	Bias	0.0131	0.0130	-0.0653	0.0330
	MSE	0.1988	0.1956	0.1937	0.1981
	Lower	4.4346	4.2132	4.2132	4.2132
	Upper	6.1923	5.9625	5.9625	5.9625
	AL	1.7576	1.7492	1.7492	1.7492
	CP	1.0000	1.0000	1.0000	1.0000
100	Est.	5.0048	5.0047	4.9652	5.0147
	Bias	0.0048	0.0047	-0.0348	0.0147
	MSE	0.1059	0.1051	0.1046	0.1057
	Lower	5.0986	4.4246	4.4246	4.4246
	Upper	6.3394	5.6599	5.6599	5.6599
	AL	1.2408	1.2353	1.2353	1.2353
	CP	0.0100	0.0000	0.0000	0.0000
200	Est.	5.0050	5.0050	4.9851	5.0100
	Bias	0.0050	0.0050	-0.0149	0.0100
	MSE	0.0504	0.0502	0.0500	0.0503
	Lower	4.7556	4.5869	4.5869	4.5869
	Upper	5.6330	5.4607	5.4607	5.4607
	AL	0.8774	0.8738	0.8738	0.8738
	CP	1.0000	1.0000	1.0000	1.0000

$\lambda = 2$, $\beta = 10$					
n	Metric	MLE	Bayes (SELF)	Bayes (QLF)	Bayes (PLF)
25	Est.	9.9403	9.9413	9.6377	10.0205
	Bias	-0.0597	-0.0587	-0.3623	0.0205
	MSE	1.5486	1.5002	1.5380	1.5211
	Lower	8.1380	7.7874	7.7874	7.7874
	Upper	13.0668	12.6864	12.6864	12.6864
	AL	4.9288	4.8990	4.8990	4.8990
	CP	1.0000	1.0000	1.0000	1.0000
50	Est.	9.9530	9.9533	9.7978	9.9931
	Bias	-0.0470	-0.0467	-0.2022	-0.0069
	MSE	0.7418	0.7301	0.7462	0.7337
	Lower	8.1766	8.3660	8.3660	8.3660
	Upper	11.6662	11.8367	11.8367	11.8367
	AL	3.4896	3.4707	3.4707	3.4707
	CP	1.0000	1.0000	1.0000	1.0000
100	Est.	9.9680	9.9682	9.8894	9.9881
	Bias	-0.0320	-0.0318	-0.1106	-0.0119
	MSE	0.3809	0.3779	0.3832	0.3785
	Lower	10.0450	8.8126	8.8126	8.8126
	Upper	12.5162	11.2744	11.2744	11.2744
	AL	2.4713	2.4618	2.4618	2.4618
	CP	0.2770	1.0000	1.0000	1.0000
200	Est.	10.0171	10.0171	9.9773	10.0271
	Bias	0.0171	0.0171	-0.0227	0.0271
	MSE	0.2095	0.2086	0.2072	0.2095
	Lower	9.0085	9.1820	9.1820	9.1820
	Upper	10.7645	10.9292	10.9292	10.9292
	AL	1.7560	1.7472	1.7472	1.7472
	CP	1.0000	1.0000	1.0000	1.0000

Table 4. Simulation results for different sample sizes $n = 25, 50, 100, 200$ with fixed shape parameter $\lambda = 5$ and four scale parameters set to $\beta = 1, 2, 5, 10$. Metrics reported include the estimated value $\hat{\beta}$ (Est.), bias, mean squared error (MSE), lower and upper bounds of the confidence intervals (Lower and Upper), average length (AL), and coverage probability (CP) for MLE and three Bayesian methods: SELF, QLF, and PLF. Source: created by the authors.

$\lambda = 5$, $\beta = 1$					
n	Metric	MLE	Bayes (SELF)	Bayes (QLF)	Bayes (PLF)
25	Est.	0.9992	0.9992	0.9850	1.0028
	Bias	-0.0008	-0.0008	-0.0150	0.0028
	MSE	0.0071	0.0070	0.0070	0.0071
	Lower	0.9736	0.8467	0.8467	0.8467
	Upper	1.3076	1.1792	1.1792	1.1792
	AL	0.3340	0.3325	0.3325	0.3325
	CP	0.9760	1.0000	1.0000	1.0000
50	Est.	0.9999	0.9999	0.9927	1.0017
	Bias	-0.0001	-0.0001	-0.0073	0.0017
	MSE	0.0034	0.0034	0.0034	0.0034
	Lower	0.8434	0.8892	0.8892	0.8892
	Upper	1.0798	1.1245	1.1245	1.1245
	AL	0.2363	0.2353	0.2353	0.2353
	CP	1.0000	1.0000	1.0000	1.0000
100	Est.	0.9998	0.9998	0.9962	1.0007
	Bias	-0.0002	-0.0002	-0.0038	0.0007
	MSE	0.0019	0.0019	0.0019	0.0019
	Lower	0.9502	0.9202	0.9202	0.9202
	Upper	1.1173	1.0862	1.0862	1.0862
	AL	0.1671	0.1660	0.1660	0.1660
	CP	1.0000	1.0000	1.0000	1.0000
200	Est.	0.9997	0.9997	0.9978	1.0001
	Bias	-0.0003	-0.0003	-0.0022	0.0001
	MSE	0.0009	0.0009	0.0009	0.0009
	Lower	0.8975	0.9426	0.9426	0.9426
	Upper	1.0157	1.0601	1.0601	1.0601
	AL	0.1181	0.1175	0.1175	0.1175
	CP	1.0000	1.0000	1.0000	1.0000

$\lambda = 5$, $\beta = 2$					
n	Metric	MLE	Bayes (SELF)	Bayes (QLF)	Bayes (PLF)
25	Est.	2.0027	2.0027	1.9742	2.0100
	Bias	0.0027	0.0027	-0.0258	0.0100
	MSE	0.0271	0.0267	0.0266	0.0270
	Lower	2.0228	1.6968	1.6968	1.6968
	Upper	2.6923	2.3642	2.3642	2.3642
	AL	0.6695	0.6674	0.6674	0.6674
	CP	0.1990	1.0000	1.0000	1.0000
50	Est.	2.0001	2.0001	1.9857	2.0037
	Bias	0.0001	0.0001	-0.0143	0.0037
	MSE	0.0140	0.0139	0.0139	0.0139
	Lower	1.9961	1.7785	1.7785	1.7785
	Upper	2.4689	2.2487	2.2487	2.2487
	AL	0.4728	0.4703	0.4703	0.4703
	CP	0.6200	1.0000	1.0000	1.0000
100	Est.	2.0025	2.0025	1.9953	2.0044
	Bias	0.0025	0.0025	-0.0047	0.0044
	MSE	0.0072	0.0072	0.0072	0.0072
	Lower	1.8447	1.8426	1.8426	1.8426
	Upper	2.1794	2.1755	2.1755	2.1755
	AL	0.3347	0.3329	0.3329	0.3329
	CP	1.0000	1.0000	1.0000	1.0000
200	Est.	1.9974	1.9974	1.9938	1.9983
	Bias	-0.0026	-0.0026	-0.0062	-0.0017
	MSE	0.0036	0.0035	0.0036	0.0035
	Lower	1.8712	1.8833	1.8833	1.8833
	Upper	2.1073	2.1180	2.1180	2.1180
	AL	0.2361	0.2347	0.2347	0.2347
	CP	1.0000	1.0000	1.0000	1.0000

$\lambda = 5$, $\beta = 5$					
n	Metric	MLE	Bayes (SELF)	Bayes (QLF)	Bayes (PLF)
25	Est.	4.9829	4.9831	4.9121	5.0012
	Bias	-0.0171	-0.0169	-0.0879	0.0012
	MSE	0.1863	0.1836	0.1858	0.1846
	Lower	4.5939	4.2231	4.2231	4.2231
	Upper	6.2597	5.8810	5.8810	5.8810
	AL	1.6658	1.6579	1.6579	1.6579
	CP	1.0000	1.0000	1.0000	1.0000
50	Est.	4.9971	4.9972	4.9612	5.0062
	Bias	-0.0029	-0.0028	-0.0388	0.0062
	MSE	0.0931	0.0924	0.0926	0.0928
	Lower	4.3859	4.4432	4.4432	4.4432
	Upper	5.5671	5.6201	5.6201	5.6201
	AL	1.1812	1.1769	1.1769	1.1769
	CP	1.0000	1.0000	1.0000	1.0000
100	Est.	5.0005	5.0005	4.9824	5.0050
	Bias	0.0005	0.0005	-0.0176	0.0050
	MSE	0.0501	0.0499	0.0499	0.0500
	Lower	4.5629	4.6012	4.6012	4.6012
	Upper	5.3988	5.4337	5.4337	5.4337
	AL	0.8358	0.8326	0.8326	0.8326
	CP	1.0000	1.0000	1.0000	1.0000
200	Est.	5.0029	5.0029	4.9938	5.0052
	Bias	0.0029	0.0029	-0.0062	0.0052
	MSE	0.0223	0.0223	0.0222	0.0223
	Lower	4.5187	4.7175	4.7175	4.7175
	Upper	5.1100	5.3050	5.3050	5.3050
	AL	0.5913	0.5875	0.5875	0.5875
	CP	1.0000	1.0000	1.0000	1.0000

$\lambda = 5$, $\beta = 10$					
n	Metric	MLE	Bayes (SELF)	Bayes (QLF)	Bayes (PLF)
25	Est.	9.9457	9.9461	9.8045	9.9822
	Bias	-0.0543	-0.0539	-0.1955	-0.0178
	MSE	0.7280	0.7176	0.7327	0.7202
	Lower	8.1266	8.4282	8.4282	8.4282
	Upper	11.4514	11.7374	11.7374	11.7374
	AL	3.3248	3.3092	3.3092	3.3092
	CP	1.0000	1.0000	1.0000	1.0000
50	Est.	9.9629	9.9631	9.8914	9.9812
	Bias	-0.0371	-0.0369	-0.1086	-0.0188
	MSE	0.3574	0.3548	0.3602	0.3551
	Lower	8.5037	8.8596	8.8596	8.8596
	Upper	10.8588	11.2051	11.2051	11.2051
	AL	2.3550	2.3455	2.3455	2.3455
	CP	1.0000	1.0000	1.0000	1.0000
100	Est.	10.0002	10.0002	9.9640	10.0093
	Bias	0.0002	0.0002	-0.0360	0.0093
	MSE	0.1664	0.1658	0.1659	0.1662
	Lower	9.1484	9.2007	9.2007	9.2007
	Upper	10.8199	10.8682	10.8682	10.8682
	AL	1.6715	1.6676	1.6676	1.6676
	CP	1.0000	1.0000	1.0000	1.0000
200	Est.	9.9970	9.9970	9.9789	10.0016
	Bias	-0.0030	-0.0030	-0.0211	0.0016
	MSE	0.0957	0.0955	0.0956	0.0956
	Lower	9.7963	9.4265	9.4265	9.4265
	Upper	10.9779	10.6020	10.6020	10.6020
	AL	1.1815	1.1754	1.1754	1.1754
	CP	1.0000	1.0000	1.0000	1.0000

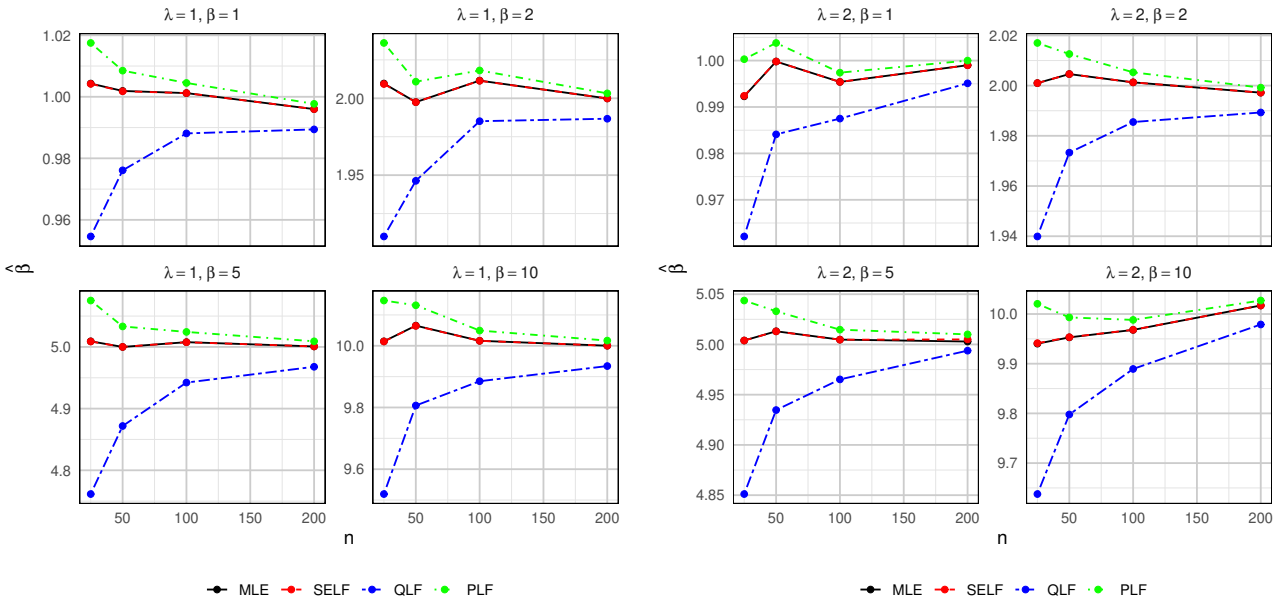
Table 5. Simulation results for different sample sizes $n = 25, 50, 100, 200$ with fixed shape parameter $\lambda = 10$ and four scale parameters set to $\beta = 1, 2, 5, 10$. Metrics reported include the estimated value $\hat{\beta}$ (Est.), bias, mean squared error (MSE), lower and upper bounds of the confidence intervals (Lower and Upper), average length (AL), and coverage probability (CP) for MLE and three Bayesian methods: SELF, QLF, and PLF. Source: created by the authors.

$\lambda = 10$, $\beta = 1$					
n	Metric	MLE	Bayes (SELF)	Bayes (QLF)	Bayes (PLF)
25	Est.	0.9989	0.9989	0.9914	1.0008
	Bias	-0.0011	-0.0011	-0.0086	0.0008
	MSE	0.0037	0.0037	0.0037	0.0037
	Lower	1.0110	0.8858	0.8858	0.8858
	Upper	1.2527	1.1265	1.1265	1.1265
	AL	0.2417	0.2407	0.2407	0.2407
	CP	0.0700	0.0000	0.0000	0.0000
50	Est.	0.9999	0.9999	0.9961	1.0009
	Bias	-0.0001	-0.0001	-0.0039	0.0009
	MSE	0.0019	0.0019	0.0019	0.0019
	Lower	0.9103	0.9184	0.9184	0.9184
	Upper	1.0814	1.0886	1.0886	1.0886
	AL	0.1711	0.1703	0.1703	0.1703
	CP	1.0000	1.0000	1.0000	1.0000
100	Est.	0.9981	0.9981	0.9962	0.9986
	Bias	-0.0019	-0.0019	-0.0038	-0.0014
	MSE	0.0010	0.0010	0.0010	0.0010
	Lower	0.9486	0.9398	0.9398	0.9398
	Upper	1.0694	1.0601	1.0601	1.0601
	AL	0.1207	0.1203	0.1203	0.1203
	CP	1.0000	1.0000	1.0000	1.0000
200	Est.	1.0005	1.0005	0.9995	1.0007
	Bias	0.0005	0.0005	-0.0005	0.0007
	MSE	0.0005	0.0005	0.0005	0.0005
	Lower	0.9794	0.9587	0.9587	0.9587
	Upper	1.0650	1.0440	1.0440	1.0440
	AL	0.0856	0.0853	0.0853	0.0853
	CP	1.0000	1.0000	1.0000	1.0000

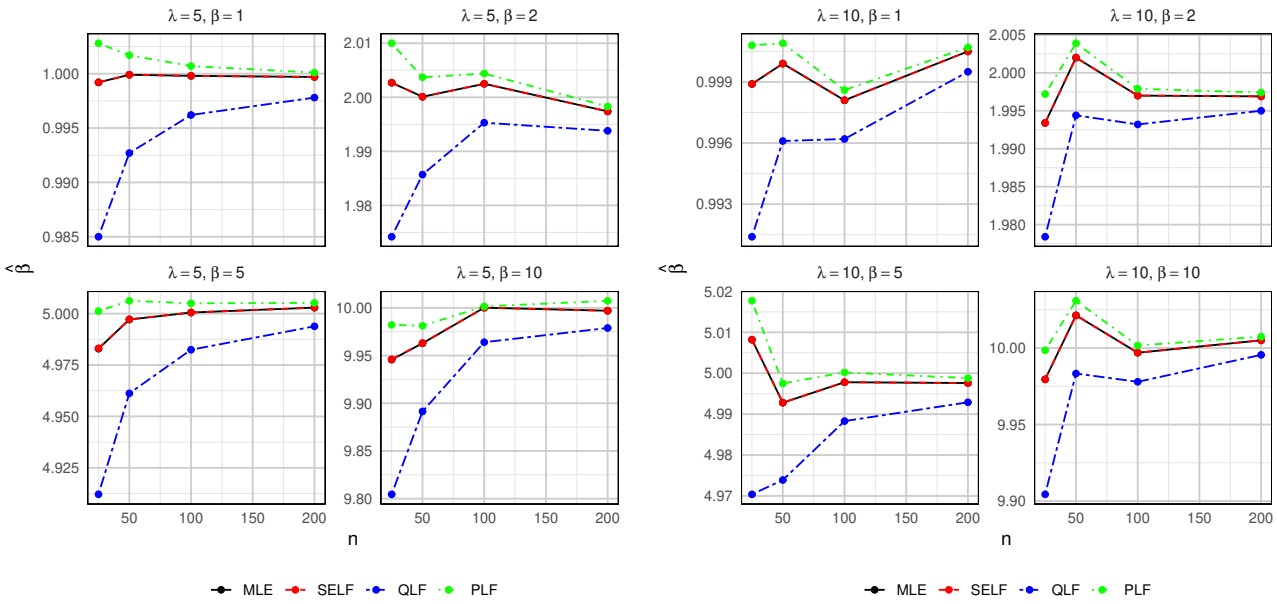
$\lambda = 10$, $\beta = 2$					
n	Metric	MLE	Bayes (SELF)	Bayes (QLF)	Bayes (PLF)
25	Est.	1.9934	1.9934	1.9784	1.9972
	Bias	-0.0066	-0.0066	-0.0216	-0.0028
	MSE	0.0159	0.0158	0.0160	0.0158
	Lower	1.7559	1.7680	1.7680	1.7680
	Upper	2.2382	2.2481	2.2481	2.2481
	AL	0.4823	0.4801	0.4801	0.4801
	CP	1.0000	1.0000	1.0000	1.0000
50	Est.	2.0020	2.0020	1.9944	2.0039
	Bias	0.0020	0.0020	-0.0056	0.0039
	MSE	0.0076	0.0075	0.0075	0.0076
	Lower	1.8894	1.8387	1.8387	1.8387
	Upper	2.2319	2.1795	2.1795	2.1795
	AL	0.3425	0.3408	0.3408	0.3408
	CP	1.0000	1.0000	1.0000	1.0000
100	Est.	1.9970	1.9970	1.9932	1.9979
	Bias	-0.0030	-0.0030	-0.0068	-0.0021
	MSE	0.0037	0.0037	0.0037	0.0037
	Lower	1.8682	1.8803	1.8803	1.8803
	Upper	2.1098	2.1206	2.1206	2.1206
	AL	0.2416	0.2403	0.2403	0.2403
	CP	1.0000	1.0000	1.0000	1.0000
200	Est.	1.9969	1.9969	1.9950	1.9974
	Bias	-0.0031	-0.0031	-0.0050	-0.0026
	MSE	0.0019	0.0019	0.0019	0.0019
	Lower	1.9619	1.9136	1.9136	1.9136
	Upper	2.1327	2.0838	2.0838	2.0838
	AL	0.1708	0.1702	0.1702	0.1702
	CP	1.0000	1.0000	1.0000	1.0000

$\lambda = 10$, $\beta = 5$					
n	Metric	MLE	Bayes (SELF)	Bayes (QLF)	Bayes (PLF)
25	Est.	5.0082	5.0082	4.9704	5.0177
	Bias	0.0082	0.0082	-0.0296	0.0177
	MSE	0.0972	0.0964	0.0958	0.0971
	Lower	4.6446	4.4417	4.4417	4.4417
	Upper	5.8562	5.6484	5.6484	5.6484
	AL	1.2117	1.2067	1.2067	1.2067
	CP	1.0000	1.0000	1.0000	1.0000
50	Est.	4.9928	4.9928	4.9739	4.9975
	Bias	-0.0072	-0.0072	-0.0261	-0.0025
	MSE	0.0506	0.0504	0.0507	0.0505
	Lower	4.5782	4.5855	4.5855	4.5855
	Upper	5.4324	5.4372	5.4372	5.4372
	AL	0.8542	0.8517	0.8517	0.8517
	CP	1.0000	1.0000	1.0000	1.0000
100	Est.	4.9978	4.9978	4.9883	5.0002
	Bias	-0.0022	-0.0022	-0.0117	0.0002
	MSE	0.0226	0.0225	0.0226	0.0225
	Lower	4.5369	4.7057	4.7057	4.7057
	Upper	5.1414	5.3079	5.3079	5.3079
	AL	0.6046	0.6022	0.6022	0.6022
	CP	1.0000	1.0000	1.0000	1.0000
200	Est.	4.9976	4.9976	4.9929	4.9988
	Bias	-0.0024	-0.0024	-0.0071	-0.0012
	MSE	0.0109	0.0109	0.0109	0.0109
	Lower	4.8930	4.7891	4.7891	4.7891
	Upper	5.3205	5.2146	5.2146	5.2146
	AL	0.4275	0.4255	0.4255	0.4255
	CP	1.0000	1.0000	1.0000	1.0000

$\lambda = 10$, $\beta = 10$					
n	Metric	MLE	Bayes (SELF)	Bayes (QLF)	Bayes (PLF)
25	Est.	9.9795	9.9795	9.9044	9.9985
	Bias	-0.0205	-0.0205	-0.0956	-0.0015
	MSE	0.3894	0.3865	0.3894	0.3876
	Lower	9.4668	8.8511	8.8511	8.8511
	Upper	11.8813	11.2530	11.2530	11.2530
	AL	2.4145	2.4019	2.4019	2.4019
	CP	1.0000	1.0000	1.0000	1.0000
50	Est.	10.0213	10.0213	9.9833	10.0308
	Bias	0.0213	0.0213	-0.0167	0.0308
	MSE	0.1914	0.1906	0.1890	0.1915
	Lower	9.1205	9.2044	9.2044	9.2044
	Upper	10.8349	10.9103	10.9103	10.9103
	AL	1.7144	1.7060	1.7060	1.7060
	CP	1.0000	1.0000	1.0000	1.0000
100	Est.	9.9969	9.9969	9.9779	10.0016
	Bias	-0.0031	-0.0031	-0.0221	0.0016
	MSE	0.0917	0.0916	0.0917	0.0916
	Lower	9.1135	9.4127	9.4127	9.4127
	Upper	10.3229	10.6177	10.6177	10.6177
	AL	1.2093	1.2050	1.2050	1.2050
	CP	1.0000	1.0000	1.0000	1.0000
200	Est.	10.0050	10.0050	9.9955	10.0074
	Bias	0.0050	0.0050	-0.0045	0.0074
	MSE	0.0484	0.0483	0.0482	0.0484
	Lower	9.7466	9.5871	9.5871	9.5871
	Upper	10.6025	10.4398	10.4398	10.4398
	AL	0.8558	0.8527	0.8527	0.8527
	CP	1.0000	1.0000	1.0000	1.0000

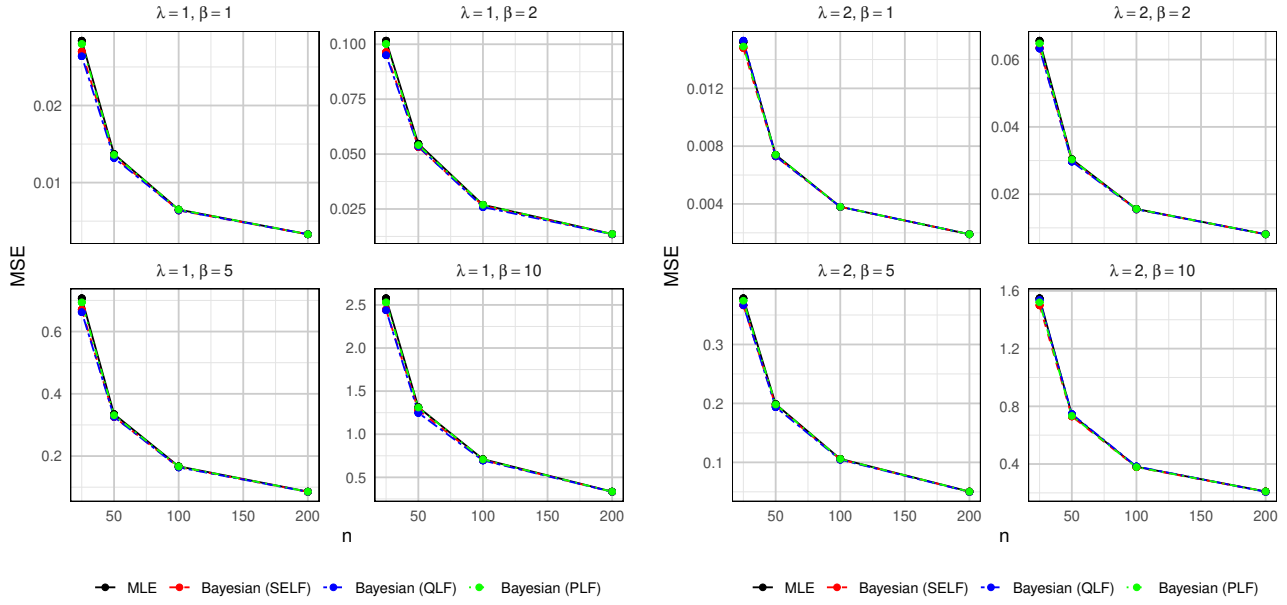


(a) $\lambda = 1$ (b) $\lambda = 2$



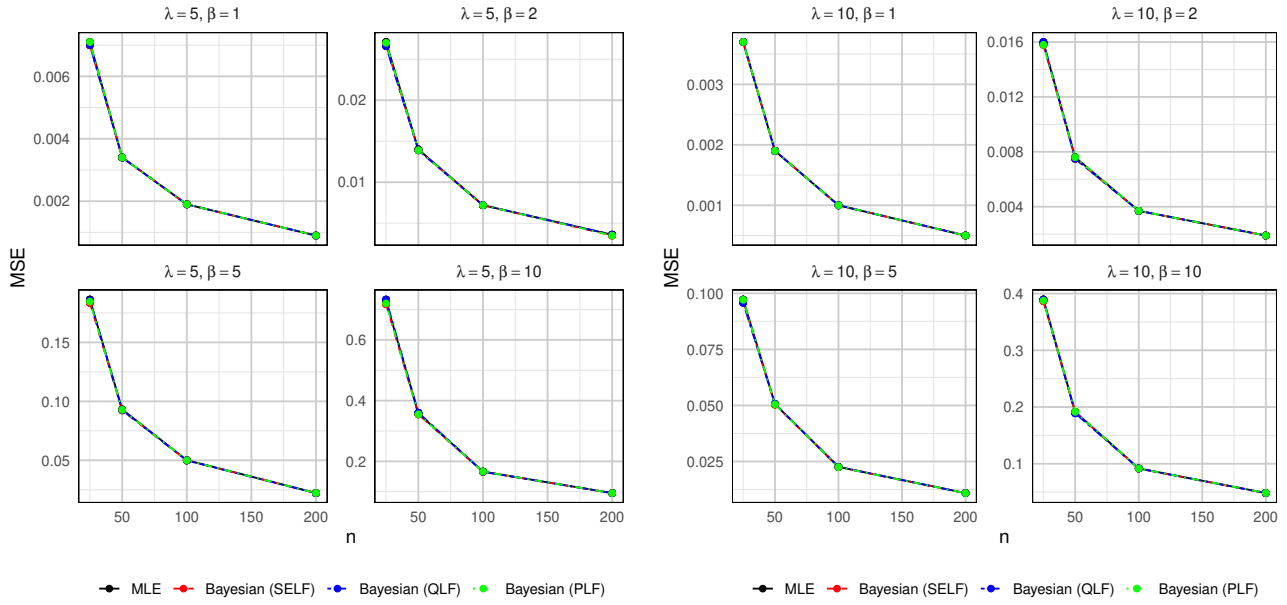
(c) $\lambda = 5$ (d) $\lambda = 10$

Fig. 1: Line plots of estimated values $\hat{\beta}$ across four values of $\beta = 1, 2, 5, 10$ for each λ, evaluated at sample sizes $n = 25, 50, 100, 200$. Source: created by the authors.



(a) $\lambda = 1$

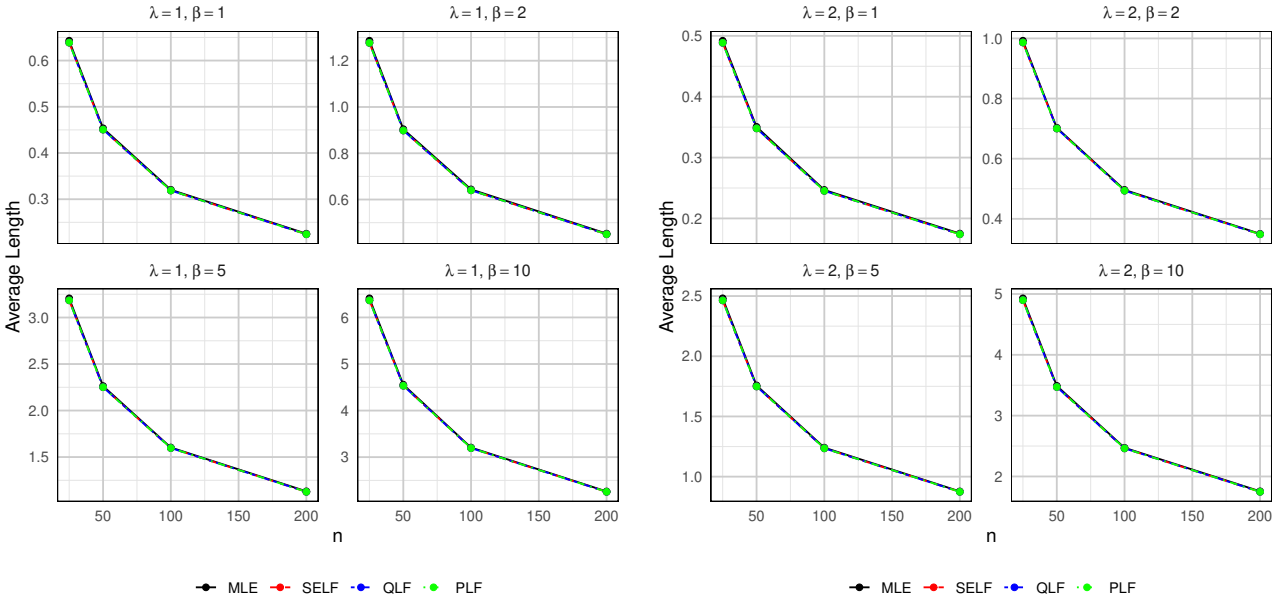
(b) $\lambda = 2$



(c) $\lambda = 5$

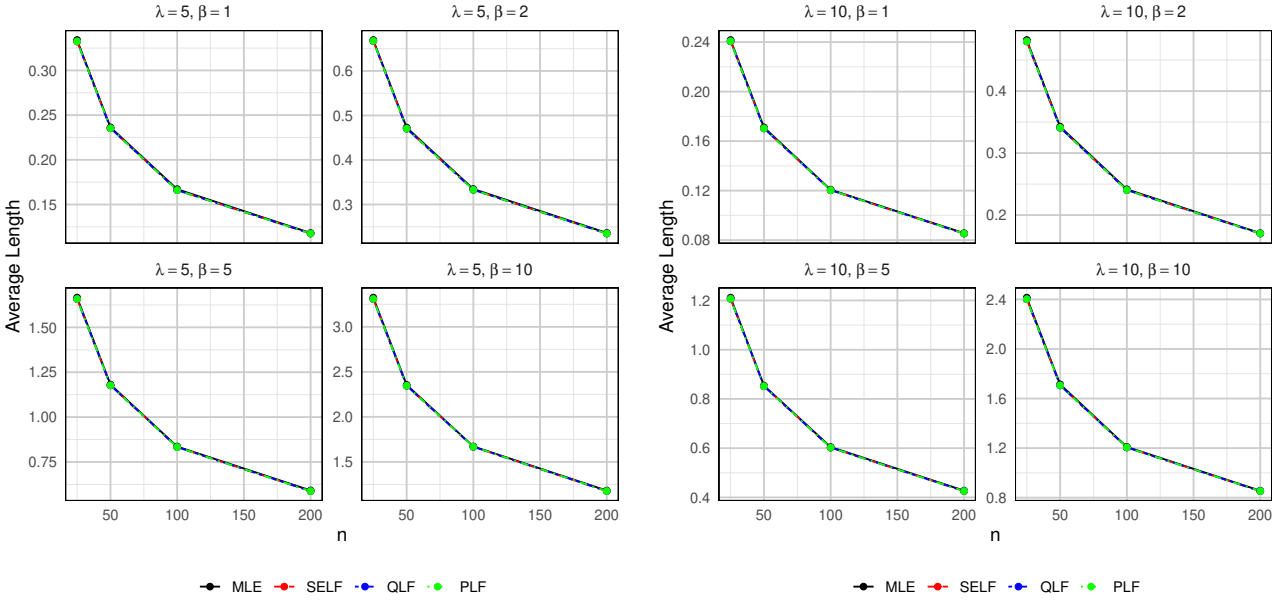
(d) $\lambda = 10$

Fig. 2: Line plots of MSE across four values of $\beta = 1, 2, 5, 10$ for each λ, evaluated at sample sizes $n = 25, 50, 100, 200$. Source: created by the authors.



(a) $\lambda = 1$

(b) $\lambda = 2$



(c) $\lambda = 5$

(d) $\lambda = 10$

Fig. 3: Line plots of AL across four values of $\beta = 1, 2, 5, 10$ for each λ, evaluated at sample sizes $n = 25, 50, 100, 200$. Source: created by the authors.