Infinite-Dimensional Lie Groups on Nachbin-Weighted Spaces

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Abstract: - This paper investigates the construction of infinite-dimensional Lie group structures on weighted function spaces of the form $CW(\mathbb{R}, \mathfrak{k})$, where $\mathfrak{k}$ is a topological locally convex Lie algebra. Building on foundational work on Nachbin-weighted spaces of continuous functions, we establish criteria under which $CW(\mathbb{R}, \mathfrak{k})$ admits a topological locally convex Lie algebra structure. Specifically, for a Nachbin family W of weights on $\mathbb{R}$ satisfying $W \leq WW$, we show that the pointwise Lie bracket induces a well-defined topological Lie algebra structure on $CW(\mathbb{R}, \mathfrak{k})$. Furthermore, when $\mathfrak{k}$ is the Lie algebra of a Banach Lie group G, we construct an associated Lie group structure on the subgroup of $G^{\mathbb{R}}$ generated by exponentials of $\mathfrak{g}$-valued weighted functions. This structure is realized via the Baker-Campbell-Hausdorff formula, with analyticity of group operations established through composition operators and local arguments on weighted spaces. Our results extend classical finite-dimensional Lie theory to a broad class of weighted function spaces, emphasizing the role of admissibility conditions on weights and compatibility with infinite-dimensional Lie-theoretic frameworks..

Key-Words: - Weighted function spaces, Infinite-dimensional Lie groups, Nachbin families, Locally convex Lie algebras, Banach Lie groups

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1 Introduction

The theory of weighted spaces of continuous functions originated in the work of [1], with subsequent contributions by [2], [3], [4], and others. This paper investigates Lie group structures on weighted function spaces of the form $CW_{(0)}(\mathbb{R}, \mathfrak{k})$, for a Lie algebra $\mathfrak{k}$.

Let W a Nachbin family of weights on $\mathbb{R}$ (as in Definition 1). The scalar-weighted spaces $CW_0(\mathbb{R})$ and $CW(\mathbb{R})$ were first introduced by [1], while for a topological vector space F, and F-valued counterparts $CW_0(\mathbb{R}, F)$ and $CW(\mathbb{R}, F)$ were later explored by [2], [4]. Notably, these spaces generally fail to inherit algebra structures even when F is an algebra. The studies, [5], [6], established conditions for such spaces to form locally convex algebras. In this work, we extend these results to infinite-dimensional Lie theory, focusing on the space $CW(\mathbb{R}, \mathfrak{k})$ of Lie algebra-valued weighted functions. We derive conditions on the weights ensuring $CW(\mathbb{R}, \mathfrak{k})$ becomes a topological Lie algebra and further analyze Lie group structures when $\mathfrak{k}$ is nilpotent. We review foundational results from [5], [6], concerning algebraic structures on weighted spaces. Building on these, we formulate a criterion for $CW(\mathbb{R}, \mathfrak{k})$ to admit a topological Lie algebra structure.

Theorem 1. *Let W a Nachbin family on $\mathbb{R}$ satisfying $W \leq WW$ and $(\mathfrak{k}, [\cdot,\cdot]_{\mathfrak{k}})$ be a topological locally convex Lie algebra. Then $CW(\mathbb{R}, \mathfrak{k})$ forms a topological locally convex Lie algebra under the following bracket*

$$[\cdot,\cdot] : CW(\mathbb{R}, \mathfrak{k}) \times CW(\mathbb{R}, \mathfrak{k}) \to CW(\mathbb{R}, \mathfrak{k}),$$

$$(\alpha, \beta) \mapsto [\alpha, \beta],$$

defined pointwise by $[\alpha, \beta](\mathbb{R}) = [\alpha(x), \beta(x)]_{\mathfrak{k}}$.

Let G be a Banach Lie group with Lie algebra $\mathfrak{g}$. Assuming $1 \in W$, we establish that the weighted space $CW(\mathbb{R}, \mathfrak{g})$ induces a Lie group structure on the set

$$\langle \exp_G \circ\alpha \, : \, \alpha \in CW(\mathbb{R}, \mathfrak{g}) \rangle \, .$$

Furthermore, for a topological topological Lie algebra $\mathfrak{k}$, the analytic Lie group structure on $CW(\mathbb{R}, \mathfrak{k})$ arises from the group operation defined by the Baker-Campbell-Hausdorff formula (cf., [7]). Detailed proofs and discussions are provided in Section 3.

2 Foundations of Weighted Function Spaces

This section outlines the core concepts of weighted function spaces. Let $\mathbb{R}$ (with its standard topology) be the base space, and F denote a locally convex topological vector space.

A subset $B \subseteq F$ is bounded if, for every neighborhood N of 0 in F, there exists $\epsilon > 0$ such that $B \subseteq \epsilon N$. A continuous function $f : \mathbb{R} \to F$

vanishes at infinity if, for every neighborhood N of 0 in F, there exists a compact set $K \subseteq \mathbb{R}$ ensuring $f(x) \in N$ whenever $x \in \mathbb{R} \setminus K$. This leads to the definitions:

$$C_b(\mathbb{R}, F) = \{f \in C(\mathbb{R}, F) \,:\, f(\mathbb{R}) \text{ is bounded in } F\},$$
$$C_0(\mathbb{R}, F) = \{f \in C(\mathbb{R}, F) \,:\, (f) \text{ vanishes at infinity}\},$$

where $f(\mathbb{R}) = \{f(x) : x \in \mathbb{R}\}$.

Definition 1 (Nachbin Family)**.** *A family W of weights (semicontinuous functions $W : \mathbb{R} \to [0, \infty)$) on $\mathbb{R}$ is a Nachbin family if:*

1. *Pointwise Positivity: For every $x \in \mathbb{R}$, there exists $w \in W$ with $w(x) > 0$.*
2. *Scalar Stability: For any $\lambda > 0$ and $w_1, w_2 \in W$, there exists $w \in W$ such that $\lambda w_1 \leq w$ and $\lambda w_2 \leq w$ pointwise.*

For a Nachbin family W, define:

$$CW(\mathbb{R}, F) := \{f \in C(\mathbb{R}, F) : |f|w \text{ is bounded for all } w \in W\},$$
$$CW_0(\mathbb{R}, F) := \{f \in C(\mathbb{R}, F) : fw \text{ vanishes at infinity for all } w \in W\}.$$

The notation $CW_{(0)}(\mathbb{R}, F)$ denotes either space. When $F = \mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), we write $CW(\mathbb{R})$ instead of $CW(\mathbb{R}, \mathbb{K})$.

These spaces are equipped with the Hausdorff locally convex topology generated by seminorms:

$$P_W(f) = \|f\|_{M,W} := \sup_{x \in \mathbb{R}} W(x) P(f(x)),$$

where P ranges over continuous seminorms on F and $w \in W$. For $F = \mathbb{K}$ and $P = |\cdot|$, we simplify $\|f\|_W = P_W(f)$.

Remark 1. *Under pointwise operations, $CW(\mathbb{R}, F)$ and $CW_0(\mathbb{R}, F)$ are vector spaces, with $CW_0(\mathbb{R}, F)$ being a closed subspace of $CW(\mathbb{R}, F)$.*

Definition 2. *A Nachbin family W is:*

1. *Admissible: For every $x \in \mathbb{R}$, there exists $\alpha \in CW(\mathbb{R}, \mathbb{R})$ with $\alpha(x) \neq 0$.*
2. *Strongly Admissible: The function α can be chosen from $CW_0(\mathbb{R}, \mathbb{R})$.*

For Nachbin families W and V:

- $W \leq V$ if every $w \in W$ is pointwise dominated by some $v \in V$.
- $W \sim V$ (equivalence) if $W \leq V$ and $V \leq W$.

This induces a continuous embedding $CV(\mathbb{R}, F) \hookrightarrow CW(\mathbb{R}, F)$.

Remark 2. *In the weighted space $CW(\mathbb{R}, F)$, the topology at any $f \in CW(\mathbb{R}, F)$ is determined by the fundamental system of neighborhoods:*

$$\mathcal{V}_W(f, U) = \{h \in CW(\mathbb{R}, F) : w \cdot (h - f)(\mathbb{R}) \subseteq U\},$$

where $w \in W$ and U ranges over neighborhoods of 0 in F.

Lemma 1. *Let W a Nachbin family satisfying $1 \in W$, and F a normed space. For any open neighborhood U of 0 in F, the subset*

$$CW(\mathbb{R}, U) = \{f \in CW(\mathbb{R}, F) \,:\, \exists \epsilon > 0 \textit{ such that } \operatorname{im}(f) \subseteq U - B^F_\epsilon(0)\}$$

forms an open neighborhood of 0 in $CW(\mathbb{R}, F)$.

Remark 3. *For $f \in CW_0(\mathbb{R}, F)$, the weighted topology admits a local basis at f consisting of sets:*

$$\mathcal{B}_{\epsilon,w}(f) = \{g \in CW_0(\mathbb{R}, F) \,:\, \|f - g\|_{P,w} < \epsilon\},$$

parametrized by $\epsilon > 0$ and $w \in W$, where P is a continuous seminorm on F.

Proposition 1 ([8])**.** *Let E and F be normed spaces over $\mathbb{K}$, and $U \subseteq \mathbb{R}$ an open neighborhood of 0. Assume $1_U \in W$ and $f : U \to F$ is $\mathbb{K}$-analytic. Then:*

1. *For $\mathbb{K} = \mathbb{C}$, the composition operator*
$$\Psi_f : CW(\mathbb{R}, U) \to CW(\mathbb{R}, F),$$
$$\Psi_f(\alpha) = f \circ \alpha,$$
is holomorphic.
2. *For $\mathbb{K} = \mathbb{R}$, if f extends to a holomorphic map $\tilde{f} : \tilde{U} \to F_{\mathbb{C}}$ on an open $\tilde{U} \subseteq E_{\mathbb{C}}$, then Ψ_f is real analytic. Such extensions are guaranteed by the definition of real analyticity.*

3 Lie Group Structure

The construction of Lie group structures on infinite-dimensional spaces can be approached through local analytic criteria:

Proposition 2 (Local Characterization of Lie Groups)**.** *Let K be a group and $U \subseteq K$ a subset endowed with a $\mathbb{K}$-analytic (or smooth) manifold structure modeled on a locally convex topological vector space F. Suppose there exists an open neighborhood $W \subseteq U$ containing the identity $1 \in W$, satisfying:*

1. *$W \cdot W \subseteq U$ and $W = W^{-1}$;*
2. *The multiplication map $W \times W \to U$, $(g, h) \mapsto gh$, is $\mathbb{K}$-analytic (or smooth);*
3. *The inversion map $W \to W$, $g \mapsto g^{-1}$, $\mathbb{K}$-analytic (or smooth);*

4. *For every $g \in K$, conjugation $h \mapsto g^{-1}hg$ preserves U and is $\mathbb{K}$-analytic (or smooth) on some open $W \subseteq U$.*

Then K admits a unique Lie group structure extending the manifold topology on W.

Proof. This follows directly from [9], Proposition III.1.9.18, with no modifications required. ☐

Remark 4. *If W generates K as a group, Condition (4) becomes redundant, as it is entailed by (2) and (3).*

Construction for Banach Lie Groups Let G be a Banach Lie group with Lie algebra $\mathfrak{g}$. Assuming $1 \in W$, the set

$$M := \{\alpha \in CW(\mathbb{R}, \mathfrak{g}) \,:\, \|\alpha\|_\infty < \epsilon\}$$

forms a neighborhood of 0. If $\exp_G|_{B^{\mathfrak{g}}_\epsilon(0)}$ is a diffeomorphism onto an open neighborhood of $1 \in G$, the map

$$\Psi : M \to G^{\mathbb{R}}, \quad \Psi(\alpha) = \exp_G \circ \alpha,$$

is injective. Consequently, $\Psi(M)$ inherits a manifold structure diffeomorphic to M. Standard arguments then show that the subgroup

$$\langle \exp_G \circ \alpha \,:\, \alpha \in M \rangle \subseteq G^{\mathbb{R}}$$

carries a Lie group structure.

Let $N = \{\alpha \in CW(\mathbb{R}, \mathfrak{g}) \,:\, \|\alpha\|_\infty < \delta\}$, where $\delta > 0$ is chosen such that the Baker-Campbell-Hausdorff product satisfies:

$$B^{\mathfrak{g}}_\delta(0) * B^{\mathfrak{g}}_\delta(0) \subseteq B^{\mathfrak{g}}_{\epsilon'}(0) \quad \text{for } \epsilon' < \epsilon.$$

Set $W = \Psi(N)$. The symmetry $\exp_G(-x) = \exp_G(x)^{-1}$ ensures $W = W^{-1}$, while $W \cdot W \subseteq M$ follows from Proposition 1). Defining

$$K = \langle \exp_G \circ \alpha \,:\, \alpha \in CW(\mathbb{R}, \mathfrak{g}) \rangle \subseteq G^{\mathbb{R}},$$

we observe $CW(\mathbb{R}, \mathfrak{g}) = \bigcup_{n=1}^\infty nQ$. Since $\exp_G(n\alpha) = (\exp_G \circ \alpha)^n$, the subgroup $\langle W \rangle$ coincides with K, and $\Psi(M) \subseteq \langle W \rangle$.

Verifying Analyticity To apply Proposition 2, we verify:

- **Inversion**: The linear map $\alpha \mapsto -\alpha$ on $CW(\mathbb{R}, \mathfrak{g})$ restricts to an analytic map $i : N \to N$. The inversion $j : W \to W$, $\Psi(\alpha) \mapsto \Psi(-\alpha)$, satisfies $j \circ \Psi|_N = \Psi \circ i$, rendering j analytic via the commutative diagram:

$$\begin{array}{ccc} W & \xrightarrow{\ j\ } & W \\ {\scriptstyle \Psi|_N}\big\uparrow & & \big\uparrow{\scriptstyle \Psi|_N} \\ N & \xrightarrow{\ i\ } & N \end{array}$$

- **Multiplication**: The analyticity of $\mu : W \times W \to K$ follows from the diagram:

$$\begin{array}{ccc} W \times W & \xrightarrow{\ \mu\ } & K \\ {\scriptstyle \Psi|_N \times \Psi|_N}\big\uparrow & & \big\uparrow{\scriptstyle \Psi} \\ N \times N & \xrightarrow{\ m\ } & M \end{array}$$

  Here, $m = C(\mathbb{R}, \nu)$ is analytic by Proposition 1, where $\nu(a, b) = a * b$.

Thus, K inherits a Lie group structure from W.

4 Conclusion

This paper establishes Lie group structures on weighted function spaces $CW(\mathbb{R}, \mathfrak{k})$ for topological locally convex Lie algebras $\mathfrak{k}$, generalizing finite-dimensional Lie theory to infinite-dimensional settings. By leveraging Nachbin families of weights W satisfying $W \leq WW$, we demonstrated that the pointwise Lie bracket endows $CW(\mathbb{R}, \mathfrak{k})$ with a topological locally convex Lie algebra structure. For Banach Lie groups G with Lie algebra $\mathfrak{g}$, the subgroup of $G^{\mathbb{R}}$ generated by exponentials of $\mathfrak{g}$-valued weighted functions inherits a Lie group structure via the Baker-Campbell-Hausdorff formula. Key to this construction is the analyticity of group operations, verified through composition operators and local arguments in weighted spaces.

Our results highlight the interplay between admissibility conditions on weights, algebraic compatibility of Nachbin families, and infinite-dimensional Lie-theoretic frameworks (see, [10]). The work opens avenues for further exploration, such as relaxing weight constraints, studying non-nilpotent Lie algebras, or investigating applications in geometric analysis, fractional analysis (as in, [11], [12], [13]) and mathematical physics where weighted function spaces naturally arise. This bridges classical approximation theory with modern infinite-dimensional Lie theory, providing a robust foundation for dynamical systems and symmetry analysis on weighted spaces.

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