On Asymptotic Expansions for Generalised and Modified Neumann Functions

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Abstract: It is shown that the generalised Bessel and confluent hypergeometric differential equations can be transformed into each other and that their solutions can be related. The confluent hypergeometric function of the first kind corresponds to the generalised Bessel function; the confluent hypergeometric function of the second kind corresponds to a modified generalised Neumann function, that is a linear combination of generalised Bessel and Neumann functions. The generalised Hankel functions of the first kind appear in the asymptotic expansion of the modified generalised Neumann function. The generalised Hankel function of the second kind is needed in the asymptotic expansions for generalised Bessel and (unmodified) generalised Neumann functions. As an application is considered the wave propagation in a cylindrical duct with (a) uniform axial flow and (b) swirl with constant angular velocity. Whereas the original Bessel differential equation applies to case (a), it is shown that the inclusion of (b) swirl leads to the generalised Bessel differential equation. The plots of generalised compared with original Bessel functions show the differences between acoustic and acoustic-vortical waves, including radial and temporal instability of the latter.

Key-Words: Generalised Bessel function, Generalised Neumann and modified functions, Generalised Hankel functions of two kinds, Confluent hypergeometric functions of two kinds, Asymptotic expansions, Asymptotic coefficients, Generalised Bessel differential equation, Confluent hypergeometric differential equation, Bessel functions, Waves in cylindrical nozzles with axial and swirling flow, Differences between acoustic and acoustic-vortical waves, Radial and temporal instability of swirling flows

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1 Introduction

The Bessel differential equation was first considered in connexion with the oscillations of a heavy chain, [1], and vibrations of a circular membrane, [2], and has, since [3], a vast number of applications supported by an extensive theory, [4]. A substantial number of applications arise from the separation of variables in the Laplace operator in cylindrical or spherical coordinates, [5], [6], that leads respectively to the cylindrical or spherical Bessel functions. The cases of specific interest include two types of cylindrical waves: (i) sound waves as compressible perturbations of a uniform flow, [7], [8]; (ii) vortical waves as incompressible perturbations of a uniform flow with superimposed rigid body rotation, [9], [10]. Whereas (i) and (ii) separately lead to the original Bessel differential equation, their coupling leads to a generalisation, [11]. Thus the consideration of coupled acoustic-vortical waves as rotational compressible perturbations of a uniform mean flow with rigid body rotation lead to the generalised Bessel equation, [12], that differs from the original in having an extra term involving a second parameter, namely the degree μ, in addition to the order ν. The generalised Bessel differential equation of order ν and degree μ may have other applications, and deserves separate study as it leads to generalisations of the Bessel, Neumann and Hankel functions with (a) some similar and also (b) other quite distinct properties.

The generalised Bessel differential equation has singularities only at the origin and infinity, like the original Bessel differential equation; the (a) similarities arise because in both cases the singularity is regular at the origin, and the (b) differences arise because the singularity at infinity is irregular with different degrees. The solutions of the generalised Bessel differential equation around the regular singularity at the origin, [12], have: (i) indices

that are exponents of the leading power depending only on the order ν; (ii) recurrence relations for the coefficients of the power series expansion depending also on the degree μ. From (i), it follows the familiar situation that generalised Bessel functions specify the general integral for non-integer degree, and generalised Neumann functions are needed for integer degree. From (ii), it follows that the series expansion for the generalised Bessel and Neumann functions differ from the original series in having finite products multiplying each term; these finite products can be expressed as ratios of Gamma functions, whose arguments become singular for zero degree.

The asymptotic solutions of the generalised Bessel differential equation, [13], are quite different from those of the original Bessel differential equation because the extra term is dominant for non-zero degree, implying that the irregular singularity at infinity is of degree two rather than one. The original Hankel functions scale asymptotically as an exponential of the variable, and are oscillating for real variable and monotonic for imaginary variable. The leading term of the asymptotic solutions of the generalised Bessel differential equation is monotonic both for real and imaginary variables: (i) there is one solution without exponential factor and with power leading term, designated generalised Hankel function of the first kind; (ii) all other solutions involve the generalised Hankel function of the second kind that scales as an exponential of the square of the variable. The solution of the generalised Bessel differential equation can be obtained as power series around the regular singularity at the origin using the Frobenius-Fuchs method, [14], [15], [16], and as asymptotic expansions around the irregular singularity at infinity using normal integrals, [17], [18]. The generalised Hankel functions of the first and second kinds do not tend to the original Hankel functions as the degree tends to zero; thus the solution of the generalised Bessel differential equation is discontinuous at infinity in the parameter degree μ corresponding to a Hopf bifurcation, [13], [19], [20].

The generalised Bessel differential equation (section 2) can be transformed to the confluent hypergeometric equation (subsection 2.1) thus relating the generalised Bessel function to the confluent hypergeometric function of the first kind (subsection 2.2). This leads for the generalised Bessel function to: (i) power series (subsection 2.2) in agreement with the Frobenius-Fuchs method [14], [15]; (ii) asymptotic expansion (subsection 2.3) as a linear combination of the generalised Hankel functions of two kinds previously found, [13], by the method of normal integrals, [17]. The confluent hypergeometric function of the second kind is related (section 3) to the modified generalised Neumann function, that is a linear combination of generalised Bessel and Neumann functions (subsection 3.1). The modified generalised Neumann function corresponds to the generalised Hankel function of the first kind since it has the same asymptotic expansion within a constant multiplying factor (subsection 3.2). The unmodified generalised Neumann function is a linear combination of the generalised Bessel function (section 2) and modified generalised Neumann function (section 3) and thus has an asymptotic expansion as a linear combination of Hankel functions of two kinds (subsection 3.3).

The original Bessel functions appear in a variety of physical situations and engineering problems, such as cylindrical and spherical acoustic, [21], [22], and electromagnetic, [23], [24], waves and vibrations of elastic cylinders and shafts, [25], [26]. As an application are considered waves in a cylindrical nozzle, in two cases: (a) purely acoustic waves in a cylindrical duct with uniform axial flow, [27], [28]; (b) coupled acoustic-vortical waves in the presence of rigid body swirl, [9], [11]. It is shown (subsection 4.1) that the extension from (a) to (b) requires replacing the original by the generalised Bessel differential equation, with both mathematical and physical implications: (i) near the axis, where the swirl velocity of the mean flow is small, the regular singularity at the origin of the differential equation (section 2) implies that acoustic-vortical waves (Figure 1) are modification of purely acoustic waves (subsection 4.2); (ii) far from the axis, where the swirl velocity of the mean flow is large (Figure 2), the higher degree of the irregular singularity at infinity of the differential equation (section 3) implies significant contrasts, between purely acoustic waves radially decaying and stable in time, and acoustic-vortical waves that are a radial and temporal instability of the swirling mean flow (subsection 4.3).

The conclusion (section 5) highlights the main mathematical and physical aspects, that are closely related, using two diagrams and three other figures. The first diagram in Figure 7 (Appendix) illustrates the relations among: (i–v) the generalised Bessel, unmodified and modified generalised Neumann, and the Hankel functions of two kinds; (vi–vii) the confluent hypergeometric functions of two kinds. A second diagram in Figure 8 (Appendix) relates the seven coefficients that connect (i) to (v), and depend on the order ν and degree μ in the generalised Bessel differential equation, and do not involve the independent variable z. The Figure 3 (Appendix), Figure 4 (Appendix) and Figure 5 (Appendix) compare, for the fundamental mode $m = 0$ and first two harmonics $m = 1, 2$, the radial waveforms of cylindrical waves, in the cases of: (i) purely acoustic

waves, that are represented by the original Bessel functions, and are stable in time and exhibit algebraic asymptotic decay like the inverse square root of distance from axis, to conserve the wave energy flux; (ii) coupled acoustic-vortical waves, that are unstable as exponentials of time and exponentials of the square of distance from the axis, with the temporal and spatial instabilities increasing with angular velocity of the swirling mean flow, that feeds energy into the waves. Thus the interaction (Figure 6, Appendix) of sinusoidal sound waves with vorticity due to rigid body rotation can: (i) modify oscillations for weak swirl; (ii) suppress oscillations for leading to monotonic divergence. The latter (ii) will become dominant at large radius when the mean flow that has azimuthal swirl velocity becomes large compared with the uniform axial velocity.

2 Asymptotic Expansion for the Generalised Bessel Function

The generalised Bessel differential equation can be transformed to a confluent hypergeometric equation (subsection 2.1), via changes of independent and dependent variables. This leads to a relation between the generalised Bessel functions and confluent hypergeometric function of the first kind (subsection 2.2) that can be used to obtain (subsection 2.3) the asymptotic expansion for the generalised Bessel function, [12], and its relation with generalised Hankel functions, [13].

2.1 Transformation between the Generalised Bessel and the Confluent Hypergeometric Differential Equations

The confluent hypergeometric differential equation has been known for a long time, [29], [30], in contrast with the more recent introduction of the generalised Bessel differential equation, [12], [13]. The latter is defined as a first step toward the transformation between the two.

Definition 1. The generalised Bessel differential equation with complex order ν and degree μ is defined by

$$z^2Q'' + z\left(1 - \frac{\mu}{2}z^2\right)Q' + \left(z^2 - \nu^2\right)Q = 0, \quad (1)$$

where $'$ denotes the derivative with respect to the variable z.

Remark 1. The original Bessel differential equation corresponds to zero degree with $\mu = 0$.

Theorem 1. *The generalised Bessel differential equation* (1) *is transformed to confluent hypergeometric differential equation*

$$wS'' + (b - w)S' - aS = 0, \quad (2a)$$

with parameters

$$b = 1 \pm \nu, \quad (2b)$$

$$a = \pm\frac{\nu}{2} - \frac{1}{\mu}, \quad (2c)$$

via the changes of independent and dependent variables:

$$w = \frac{\mu z^2}{4}, \quad (3a)$$

$$S(w) = z^{\mp\nu}Q\left(z^2\right). \quad (3b)$$

Proof. The generalised Bessel differential equation (1) has cubic coefficients, and the change of independent variable,

$$s = z^2, \quad (4a)$$

$$Q(z) = R(s), \quad (4b)$$

leads to an ordinary differential equation with quadratic coefficients:

$$4s^2R'' + 2s\left(2 - \frac{\mu s}{2}\right)R' + \left(s - \nu^2\right)R = 0. \quad (4c)$$

Another change this time of the dependent variable, with arbitrary constant parameter α,

$$R(s) = s^\alpha T(s), \quad (5a)$$

leads to the ordinary differential equation:

$$4s^2T'' + 2s\left(2 + 4\alpha - \frac{\mu s}{2}\right)T' + \left[4\alpha^2 - \nu^2 + (1 - \mu\alpha)s\right]T = 0. \quad (5b)$$

The choice

$$\alpha = \pm\frac{\nu}{2} \quad (6a)$$

of constant in (5a) allows division of (5b) by s leading to a differential equation

$$sT'' + \left(1 \pm \nu - \frac{\mu s}{4}\right)T' + \left[\frac{1}{4}\left(1 \mp \frac{\mu\nu}{2}\right)\right]T = 0. \quad (6b)$$

The ensemble of transformations (4a), (4b), (5a) and (6a) reduces the solution of the generalised Bessel differential equation (1) to the solution of an ordinary differential equation with linear coefficients (6b):

$$Q(z) = R(s) = s^{\pm\nu/2}T(s) = z^{\pm\nu}T\left(z^2\right). \quad (7)$$

The power in (7) has for exponents the indices near the regular singularity at the origin and the square of

the variable z^2 appears in the power series for the generalised Bessel function. The change of variable

$$w = \frac{\mu s}{4}, \tag{8a}$$

$$S(w) = T(s), \tag{8b}$$

transforms the differential equation (6b) to

$$wS'' + (1 \pm \nu - w) S' + \left(\frac{1}{\mu} \mp \frac{\nu}{2} \right) S = 0. \tag{8c}$$

The differential equation (8c) is of the confluent hypergeometric type (2a) with parameters (2b) and (2c). □

2.2 Relation between the Generalised Bessel Function and Confluent Hypergeometric Function of First Kind

From the relations (2a) to (2c), it may be expected the next theorem.

Theorem 2. *The generalised Bessel function of order ν and degree μ is related to the confluent hypergeometric function of the first kind by*

$$J_\nu^\mu(z) = \frac{(z/2)^\nu}{\Gamma(1+\nu)} F\left(\frac{\nu}{2} - \frac{1}{\mu}; 1+\nu; \frac{1}{4}\mu z^2 \right). \tag{9}$$

Proof. Substituting the confluent hypergeometric function of the first kind,

$$S(w) = F(a; b; w), \tag{10a}$$

in (7) leads to

$$Q(z) = z^{\pm\nu} T\left(z^2\right) = z^{\pm\nu} F\left(a; b; \frac{1}{4}\mu z^2 \right). \tag{10b}$$

Since the generalised Bessel function and confluent hypergeometric function of first kind are the solutions without logarithmic singularity of the respective differential equations, they must be proportional through a factor not involving the independent variable. Thus the generalised Bessel function of order ν and degree μ should be related to the confluent hypergeometric function of the first kind by

$$J_{\pm\nu}^\mu(z) = A_1(\mu, \pm\nu) z^{\pm\nu} \times F\left(\pm\frac{\nu}{2} - \frac{1}{\mu}; 1 \pm \nu; \frac{1}{4}\mu z^2 \right), \tag{11}$$

where the multiplying factor can depend on the parameters (μ, ν), but cannot depend on the variable z. The coefficient function can be determined by comparing the series expansions of the generalised Bessel function and of the confluent hypergeometric function of the first kind. The latter has series expansion, [30], [31],

$$w = \frac{1}{4}\mu z^2, \quad F(a; b; w) = \frac{\Gamma(b)}{\Gamma(a)} \sum_{j=0}^{\infty} \frac{\Gamma(a+j)}{\Gamma(b+j)} \frac{w^j}{j!}, \tag{12}$$

using the variable (3a) and parameters (2b) and (2c). The substitution of (12) in (11) leads to

$$J_{\pm\nu}^\mu(z) = A_1(\mu, \pm\nu) z^{\pm\nu} \frac{\Gamma(1 \pm \nu)}{\Gamma(\pm\nu/2 - 1/\mu)} \times \sum_{j=0}^{\infty} \frac{\Gamma(j \pm \nu/2 - 1/\mu)}{j! \Gamma(1 \pm \nu + j)} \left(\frac{1}{4}\mu z^2 \right)^j. \tag{13}$$

The comparison of (13) with the generalised Bessel series, [12],

$$J_{\pm\nu}^\mu(z) = \frac{(z/2)^{\pm\nu}}{\Gamma(\pm\nu/2 - 1/\mu)} \times \sum_{j=0}^{\infty} \frac{\left(\mu z^2/4\right)^j}{j!} \frac{\Gamma(j \pm \nu/2 - 1/\mu)}{\Gamma(1 \pm \nu + j)}, \tag{14}$$

leads to

$$A_1(\mu, \pm\nu) 2^{\pm\nu} \Gamma(1 \pm \nu) = 1. \tag{15}$$

The substitution of (15) in (11) proves (9). □

Remark 2. The confluent hypergeometric equation (2a) has another solution,

$$S(w) = w^{1-b} F(1 + a - b; 2 - b; w), \tag{16}$$

that corresponds by (2b), (2c) and (3a) to the solution (7) of the generalised Bessel equation,

$$Q(z) = \left(\frac{\mu z}{4} \right)^{\mp\nu} F\left(1 + a - b; 2 - b; \frac{1}{4}\mu z^2 \right) = \left(\frac{\mu z}{4} \right)^{\mp\nu} F\left(\frac{\mp\nu}{2} - \frac{1}{\mu}; 1 \mp \nu; \frac{1}{4}\mu z^2 \right), \tag{17}$$

that coincides with (9) to within a multiplying factor independent of z. Thus both solutions (10a) and (16) of the confluent hypergeometric equation (2a) lead to the generalised Bessel functions (9)≡(17) of orders $\pm\nu$ as solutions of the generalised Bessel equation (1).

Remark 3. The solutions (14) are linearly independent (theorem 9 in [12]) if the order ν is not an integer; the solutions (10a) and (16) of the confluent hypergeometric equation (2a) are linearly independent if b is not zero or an integer in agreement with $b = 1 + \nu$ in (2b). The case of ν as a negative

integer leads to the generalised Neumann function (section 3) that relates for b zero or a negative integer to the confluent hypergeometric function of second kind.

The theorem 2 has the inverse.

Theorem 3. *The confluent hypergeometric function of the first kind with parameters (a, b) and variable w is related by*

$$F(a;b;w) = \Gamma(b)\left(\frac{w}{\mu}\right)^{(1-b)/2} \times J_{b-1}^{2/(b-2a-1)}\left(2\sqrt{\frac{w}{\mu}}\right) \tag{18}$$

to the generalised Bessel function of order

$$\nu = b - 1, \tag{19a}$$

degree

$$\mu = \frac{2}{b-2a-1}, \tag{19b}$$

and variable

$$z = 2\sqrt{\frac{w}{\mu}}. \tag{19c}$$

Proof. From (2b)≡(19a), (2c)≡(19b) and (3a)≡(19c) with upper signs follows coincidence of (9)≡(18). □

The generalised Bessel function is broadly equivalent to the confluent hypergeometric function of the first kind. This relation can be used in both directions. For example the original Bessel function is the particular case of zero degree of the generalised Bessel function leading by (9) to the next relation with the confluent hypergeometric function of the first kind:

$$J_\nu(z) = \lim_{\mu\to 0} J_\nu^\mu(z) = \frac{(z/2)^\nu}{\Gamma(1+\nu)} \lim_{\mu\to 0} F\left(\frac{\nu}{2} - \frac{1}{\mu}; 1+\nu; \frac{1}{4}\mu z^2\right). \tag{20}$$

Using instead of the degree μ the parameter

$$\lambda = -\frac{1}{\mu} \tag{21a}$$

leads to

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(1+\nu)} \times \lim_{\lambda\to\pm\infty} F\left(\frac{\nu}{2} + \lambda; 1+\nu; -\frac{(z/2)^2}{\lambda}\right), \tag{21b}$$

that is the usual form of the original Bessel function as a particular case of the confluent hypergeometric function of the first kind (p. 506 of [32]). Conversely the confluent hypergeometric function of the first kind is used next to obtain the asymptotic expansion for the generalised Bessel function.

2.3 Asymptotic Relation between Generalised Bessel and Hankel Functions

The confluent hypergeometric functions of the first kind has the asymptotic expansion, [29], [30]

$$\begin{aligned} F(a;b;w) \sim{}& \frac{\Gamma(b)}{\Gamma(b-a)}(-w)^{-a} \\ &\times \left\{1 + \sum_{j=1}^{M-1} \frac{(-w)^{-j}}{j!} \right. \\ &\times \prod_{l=0}^{j-1} (l+a)(l+1+a-b) \\ &\left. + O\left(w^{-M}\right)\right\} \\ &+ \frac{\Gamma(b)}{\Gamma(a)} e^w w^{a-b} \left\{1 + \sum_{j=1}^{N-1} \frac{w^{-j}}{j!} \right. \\ &\times \prod_{l=0}^{j-1} (l-a)(l-1+b-a) \\ &\left. + O\left(w^{-N}\right)\right\}. \end{aligned} \tag{22}$$

Substitution of (22) in (9) specifies the asymptotic expansion for the generalised Bessel function:

$$\begin{aligned} J_\nu^\mu(z) \sim{}& \frac{(z/2)^\nu}{\Gamma(1+\nu/2+1/\mu)} \left(-\frac{\mu z^2}{4}\right)^{-\nu/2+1/\mu} \\ &\times \left\{1 + \sum_{j=1}^{M-1} \frac{\left(-\mu z^2/4\right)^{-j}}{j!} \prod_{l=0}^{j-1} \left[\left(l - \frac{1}{\mu}\right)^2 - \frac{\nu^2}{4}\right] \right. \\ &\left. + O\left(z^{-2M}\right)\right\} + \frac{(z/2)^\nu}{\Gamma(\nu/2-1/\mu)} \exp\left(\frac{1}{4}\mu z^2\right) \\ &\times \left(\frac{\mu z^2}{4}\right)^{-1-1/\mu-\nu/2} \left\{1 + \sum_{j=1}^{N-1} \frac{\left(\mu z^2/4\right)^{-j}}{j!} \right. \\ &\left. \times \prod_{l=0}^{j-1} \left[\left(l + \frac{1}{\mu}\right)^2 - \frac{\nu^2}{4}\right] + O\left(z^{-2N}\right)\right\}. \end{aligned} \tag{23}$$

This can be restated as follows.

Definition 2. The generalised Hankel functions of the first and second kinds are defined by the next asymptotic expansions:

$$H_{\mu,\nu}^{(1)}(z) \sim z^{2/\mu}\left\{1+\sum_{j=1}^{N-1}\frac{\left(-\mu z^2/4\right)^{-j}}{j!}\times\prod_{l=0}^{j-1}\left[\left(l-\frac{1}{\mu}\right)^2-\frac{\nu^2}{4}\right]+O\left(z^{-2N}\right)\right\}, \tag{24a}$$

$$H_{\mu,\nu}^{(2)}(z) \sim \exp\left(\frac{\mu z^2}{4}\right) z^{-2-2/\mu}\left\{1+\sum_{j=1}^{N-1}\frac{\left(\mu z^2/4\right)^{-j}}{j!}\prod_{l=0}^{j-1}\left[\left(l+\frac{1}{\mu}\right)^2-\frac{\nu^2}{4}\right]+O\left(z^{-2N}\right)\right\}. \tag{24b}$$

Remark 4. The definitions (24a) and (24b) agree with respectively theorems 1 and 2 in [13], concerning asymptotic solutions of the generalised Bessel differential equation. Thus any linear combination of (24a) and (24b) is an asymptotic solution of the generalised Bessel differential equation (1).

The combination of (24a) and (24b) corresponding to the generalised Bessel functions (23) is given by the next theorem.

Theorem 4. *The generalised Bessel function of order ν and degree μ has the asymptotic expansion*

$$J_\nu^\mu(z) \sim \frac{2^{-2/\mu}}{\Gamma\left(1+\nu/2+1/\mu\right)}(-\mu)^{-\nu/2+1/\mu}\times H_{\mu,\nu}^{(1)}(z)+\frac{2^{2+2/\mu}}{\Gamma\left(\nu/2-1/\mu\right)}\mu^{-1-\nu/2-1/\mu}H_{\mu,\nu}^{(2)}(z), \tag{25}$$

as a linear combination of the generalised Hankel functions of the first (24a) *and second* (24b) *kinds.*

Corollary 1. *The asymptotic expansion* (23) *of the generalised Bessel function of arbitrary order ν and non-zero degree $\mu \neq 0$ as a linear combination of generalised Hankel functions of the first* (24a) *and second* (24b) *kinds,*

$$J_\nu^\mu(z) \sim D_+(\mu,\nu)\,H_{\mu,\nu}^{(1)}(z)+D_-(\mu,\nu)\,H_{\mu,\nu}^{(2)}(z), \tag{26}$$

has coefficients

$$D_+(\mu,\nu) = \frac{2^{-2/\mu}}{\Gamma\left(1+\nu/2+1/\mu\right)}\mu^{-\nu/2+1/\mu}\times e^{-\mathrm{i}\pi\nu/2+\mathrm{i}\pi/\mu}, \tag{27a}$$

$$D_-(\mu,\nu) = \frac{2^{2+2/\mu}}{\Gamma\left(\nu/2-1/\mu\right)}\mu^{-1-\nu/2-1/\mu}, \tag{27b}$$

that satisfy the relation

$$D_+(\mu,\nu)\,D_-(\mu,-\nu)-D_+(\mu,-\nu)\,D_-(\mu,\nu) = \frac{2}{\mu\pi\mathrm{i}}\left(e^{-\mathrm{i}\pi\nu}-e^{\mathrm{i}\pi\nu}\right) = -\frac{4}{\mu\pi}\sin(\pi\nu). \tag{28}$$

Proof. From (27a) and (27b) follow

$$\begin{aligned} D_+(\mu,\nu)\,D_-(\mu,-\nu) &= \frac{4}{\mu}\frac{e^{-\mathrm{i}\pi\nu/2+\mathrm{i}\pi/\mu}}{\Gamma\left(1+\nu/2+1/\mu\right)\Gamma\left(-\nu/2-1/\mu\right)} \\ &= -\frac{4}{\mu\pi}\sin\left(\frac{\pi\nu}{2}+\frac{\pi}{\mu}\right)e^{-\mathrm{i}\pi\nu/2+\mathrm{i}\pi/\mu} \\ &= -\frac{2}{\mu\pi\mathrm{i}}\left(e^{\mathrm{i}2\pi/\mu}-e^{-\mathrm{i}\pi\nu}\right), \end{aligned} \tag{29a}$$

$$D_+(\mu,-\nu)\,D_-(\mu,\nu) = -\frac{2}{\mu\pi\mathrm{i}}\left(e^{\mathrm{i}2\pi/\mu}-e^{\mathrm{i}\pi\nu}\right). \tag{29b}$$

The difference of (29a) and (29b) proves (28). □

Remark 5. The relations (25), (26), (27a), (27b) and (28) agree with the theorems 5 and 6 in [13], that was proved by a different method of comparison of the Wronskians of generalised Bessel and Hankel functions.

3 Asymptotic Expansion for the Modified and Generalised Neumann Function

The confluent hypergeometric function of the first kind (subsection 2.1) corresponds to the generalised Bessel function (subsection 2.2), and the confluent hypergeometric function of the second kind corresponds to (i) a linear combination of the generalised Bessel and Neumann functions (subsection 3.2) designated the modified generalised Neumann function (subsection 3.1); the modified generalised Neumann function corresponds to the generalised Hankel function of the first kind, since it has the same asymptotic expansion to within a constant factor (subsection 3.2). The unmodified generalised Neumann function is a linear combination of (i) generalised Bessel and (ii)

modified generalised Neumann functions, and thus its asymptotic expansion (subsection 3.1) involves both the generalised Hankel functions of two kinds.

3.1 Relation with the Confluent Hypergeometric Function of the Second Kind

The general integral of the generalised Bessel equation (1) when the order ν is an integer involves (theorem 19 in [12]) the generalised Neumann function as a second particular integral linearly independent from the generalised Bessel function. Likewise the general integral of the confluent hypergeometric equation (2a) when the parameter (2b) is zero or a negative integer involves the confluent hypergeometric function of the second kind that is linearly independent from that of the first kind. The generalised Bessel function (9) is related to the confluent hypergeometric function of the first kind both for integer and non-integer order ν and degree μ. In the case of negative integer order $\nu = -n$ or parameter $b = 1 - n$ equal to zero or a negative integer, the confluent hypergeometric function of the second kind with the same parameters must be a linear combination of the generalised Bessel and Neumann functions, that is designated the modified Neumann function.

This suggests the following.

Definition 3. The modified generalised Neumann function with arbitrary order ν and integer degree n is defined by

$$\tilde{Y}_n^\mu(z) = -\frac{\Gamma(-n/2-1/\mu)}{\pi}\left(-\frac{z}{2}\right)^n \times G\left(\frac{n}{2}-\frac{1}{\mu};1+n;\frac{\mu z^2}{4}\right), \tag{30}$$

in terms of the confluent hypergeometric function of the second kind.

This leads to the next theorem.

Theorem 5. *The modified generalised Neumann function of arbitrary degree μ and integer degree n is related by*

$$\tilde{Y}_n^\mu(z) = Y_n^\mu(z) + \frac{1}{\pi}\log(\mu)\, J_n^\mu(z) \tag{31}$$

to the generalised Bessel J_ν^μ and Neumann Y_ν^μ functions.

Proof. The modified generalised Neumann function of arbitrary degree μ and integer order n is related to the confluent hypergeometric function of the second kind by

$$\tilde{Y}_n^\mu(z) = A_3(\mu,n)\, z^n G\left(\frac{n}{2}-\frac{1}{\mu};1+n;\frac{1}{4}\mu z^2\right), \tag{32}$$

where the coefficient can depend on the parameters (μ, n), but cannot depend on the variable z and thus (32) is a solution of the generalised Bessel differential equation (1). The confluent hypergeometric function of the second kind has the series expansion, [29], [30]

$$\begin{aligned} G(a;1+n;w) = {} & \frac{(-1)^{n+1}}{\Gamma(a)\,\Gamma(a-n)} \\ & \times \left\{ \log w \sum_{j=0}^{\infty} \frac{w^j}{j!}\frac{\Gamma(a+j)}{(j+n)!} \right. \\ & - (-w)^{-n} \sum_{j=0}^{n-1} (-w)^j \frac{(n-j-1)!}{j!} \\ & \times \Gamma(j+a-n) - \sum_{j=0}^{\infty} \frac{w^j}{j!}\frac{\Gamma(a+j)}{(j+n)!} \\ & \times [\psi(1+j)+\psi(1+j+n) \\ & \left. - \psi(j+a)] \right\}, \end{aligned} \tag{33}$$

that has a logarithmic singularity as the generalised Neumann function (theorem 13 in [12]).

Substituting (33) in (32) leads to the expression for the modified generalised Neumann function:

$$\begin{aligned} \tilde{Y}_n^\mu(z) = {} & \frac{z^n(-1)^{n+1}A_3(\mu,n)}{\Gamma(n/2-1/\mu)\,\Gamma(-n/2-1/\mu)} \\ & \times \left\{ \log\left(\frac{1}{4}\mu z^2\right) \sum_{j=0}^{\infty} \frac{\left(\mu z^2/4\right)^j}{j!\,(j+n)!} \right. \\ & \times \Gamma\left(j+\frac{n}{2}-\frac{1}{\mu}\right) - \left(-\frac{1}{4}\mu z^2\right)^{-n} \\ & \times \sum_{j=0}^{n-1} \left(-\frac{\mu z^2}{4}\right)^j \frac{(n-j-1)!}{j!} \\ & \times \Gamma\left(j-\frac{n}{2}-\frac{1}{\mu}\right) - \sum_{j=0}^{\infty} \frac{\left(\mu z^2/4\right)^j}{j!\,(j+n)!} \\ & \times \Gamma\left(j+\frac{n}{2}-\frac{1}{\mu}\right) \left[\psi(1+j)\right. \\ & \left. \left. + \psi(1+j+n) - \psi\left(j+\frac{n}{2}-\frac{1}{\mu}\right)\right] \right\}. \end{aligned} \tag{34}$$

The first, second and third terms on the right-hand side of (34) involve respectively the generalised Bessel function J_n^μ and the preliminary X_n^μ and complementary Z_n^μ functions (theorem 13 in [12]):

$$\begin{aligned}\tilde{Y}_n^\mu(z) = \frac{2^n(-1)^{n+1}A_3(\mu,n)}{\Gamma(-n/2-1/\mu)} \\ \times\left\{\log\left(\frac{1}{4}\mu z^2\right)J_n^\mu(z)\right. \\ \left.+\pi\left[X_n^\mu(z)+Z_n^\mu(z)\right]\right\}.\end{aligned} \tag{35}$$

Choosing the coefficient to satisfy

$$A_3(\mu,n) = \frac{(-1)^{n+1}}{\pi}2^{-n}\Gamma\left(-\frac{n}{2}-\frac{1}{\mu}\right) \tag{36}$$

simplifies the modified generalised Neumann function to

$$\tilde{Y}_n^\mu(z) = \frac{1}{\pi}\log\left(\frac{1}{4}\mu z^2\right)J_n^\mu(z)+X_n^\mu(z)+Z_n^\mu(z). \tag{37}$$

This is similar to the generalised Neumann function (theorem 13 in [12]) except for the logarithmic factor:

$$Y_n^\mu(z) = \frac{2}{\pi}\log\left(\frac{z}{2}\right)J_n^\mu(z)+X_n^\mu(z)+Z_n^\mu(z). \tag{38}$$

Using

$$\log\left(\frac{1}{4}\mu z^2\right) = 2\log\left(\frac{z}{2}\right)+\log\mu \tag{39a}$$

in (37) yields

$$\begin{aligned}\tilde{Y}_n^\mu(z) = \left[\frac{2}{\pi}\log\left(\frac{z}{2}\right)+\frac{1}{\pi}\log(\mu)\right]J_n^\mu(z) \\ +X_n^\mu(z)+Z_n^\mu(z).\end{aligned} \tag{39b}$$

Subtracting (38) from (39b) gives

$$\tilde{Y}_n^\mu(z)-Y_n^\mu(z) = \frac{1}{\pi}\log(\mu)\,J_n^\mu(z), \tag{40}$$

that coincides with (31) and confirms that the modified generalised Neumann function is specified by (31) as a linear combination of generalised Bessel (14) and Neumann (38) functions. □

3.2 Relation with Generalised Bessel and Neumann Functions

The generalised Bessel function of arbitrary order ν and non-zero degree $\mu \neq 0$ has an asymptotic expansion (26) as a linear combination with coefficients (27a)–(27b) of generalised Hankel functions of the first (24a) and second (24b) kinds. The asymptotic expansion of the former does not and that of latter does involve exponential factors. It is shown next that the modified generalised Neumann function scales on the generalised Hankel function of the first kind (24a) and thus its asymptotic expansion does not involve an exponential factor. From (31) it follows that this is not the case for the generalised Neumann function whose asymptotic expansion involves an exponential factor arising from the generalised Hankel of the second kind (24b). The modified generalised Neumann function is considered first together with its asymptotic expansion.

Theorem 6. *The modified generalised Neumann function of arbitrary degree μ and non-zero order $n \neq 0$ is given by*

$$\begin{aligned}\tilde{Y}_n^\mu(z) = \frac{1}{\pi}\frac{(z/2)^n}{\Gamma(n/2-1/\mu)} \\ \times\left\{\log\left(\frac{1}{4}\mu z^2\right)\sum_{j=0}^{\infty}\frac{\left(\mu z^2/4\right)^j}{j!\,(j+n)!}\right. \\ \times\Gamma\left(j+\frac{n}{2}-\frac{1}{\mu}\right)-\left(-\frac{1}{4}\mu z^2\right)^{-n} \\ \times\sum_{j=0}^{n-1}\left(-\frac{\mu z^2}{4}\right)^j\frac{(n-j-1)!}{j!} \\ \times\Gamma\left(j-\frac{n}{2}-\frac{1}{\mu}\right)-\sum_{j=0}^{\infty}\frac{\left(\mu z^2/4\right)^j}{j!\,(j+n)!} \\ \times\Gamma\left(j+\frac{n}{2}-\frac{1}{\mu}\right)\left[\psi(1+j)\right. \\ \left.\left.+\,\psi(1+j+n)-\psi\left(j+\frac{n}{2}-\frac{1}{\mu}\right)\right]\right\}.\end{aligned} \tag{41}$$

Proof. The substitution of (36) in (34) leads to (41). □

Theorem 7. *The modified generalised Neumann function of non-zero degree and integer order has the*

asymptotic expansion

$$\tilde{Y}_n^{\mu}(z) \sim -\frac{(-z/2)^n}{\pi}\left(\frac{\mu z^2}{4}\right)^{-n/2+1/\mu} \times \Gamma\left(-\frac{n}{2}-\frac{1}{\mu}\right)\left\{1+\sum_{j=1}^{N-1}\frac{\left(-\mu z^2/4\right)^{-j}}{j!} \times \prod_{l=0}^{j-1}\left[\left(l-\frac{1}{\mu}\right)^2-\frac{n^2}{4}\right]+O\left(z^{-2N}\right)\right\} \tag{42}$$

that coincides with the generalised Hankel function of the first kind (24a)*,*

$$\tilde{Y}_n^{\mu}(z) \sim \tilde{E}(\mu,n)\, H_{\mu,n}^{(1)}(z), \tag{43a}$$

to within a constant multiplying factor:

$$\tilde{E}(\mu,n) = -\frac{(-1)^n}{\pi}2^{-2/\mu}\mu^{-n/2+1/\mu} \times \Gamma\left(-\frac{n}{2}-\frac{1}{\mu}\right). \tag{43b}$$

Proof. The confluent hypergeometric function of the second kind, [29], [30], has the asymptotic expansion

$$G(a;b;w) \sim w^{-a}\left\{1+\sum_{j=1}^{N-1}\frac{(-w)^{-j}}{j!} \times \prod_{l=0}^{j-1}(l+a)(l+a-b+1) + O\left(w^{-N}\right)\right\}. \tag{44}$$

Substituting (44) in the relation between the modified generalised Neumann function and the confluent hypergeometric function of the second kind (30) proves (42) and hence (43a)–(43b). □

The asymptotic expansions of the generalised Bessel – (26), (27a) and (27b) – and modified Neumann – (43a) and (43b) – functions lead through (31) to the asymptotic expansion for the unmodified generalised Neumann function (section 3.3).

3.3 Asymptotic Expansion for the Generalised Neumann Function

The asymptotic expansions for (subsection 2.3) the generalised Bessel function and for (subsection 3.2) the generalised modified Neumann function can be used in the relation with the generalised Neumann function to obtain the asymptotic expansion for the latter.

The substitution of (26) and (43a) in (31) leads to

$$Y_n^{\mu}(z) \sim \left[\tilde{E}(\mu,n)-\frac{1}{\pi}\log(\mu)\, D_+(\mu,n)\right] \times H_{\mu,n}^{(1)}(z) - \frac{1}{\pi}\log(\mu)\, D_-(\mu,n)\, H_{\mu,n}^{(2)}(z), \tag{45}$$

that can be restated as the next theorem.

Theorem 8. *The generalised Neumann function with non-zero degree $\mu \neq 0$ and integer order $\nu = n$ has the asymptotic expansion*

$$Y_n^{\mu}(z) \sim E_+(\mu,n)\, H_{\mu,n}^{(1)}(z) + E_-(\mu,n)\, H_{\mu,n}^{(2)}(z), \tag{46}$$

as a linear combination of generalised Hankel functions of the first (24a) *and second* (24b) *kinds with coefficients*

$$E_+(\mu,n) = \tilde{E}(\mu,n) - \frac{1}{\pi}\log(\mu)\, D_+(\mu,n), \tag{47a}$$

$$E_-(\mu,n) = -\frac{1}{\pi}\log(\mu)\, D_-(\mu,n), \tag{47b}$$

where (47a) *involves* (43b)*.*

Proof. The substitution of (47a), (47b) and (43b) in (46) leads back to (45). □

Definition 4. The cross-commutator of the coefficients – (27a),(27b) and (47a),(47b) – of the asymptotic expansion of respectively the generalised Bessel (26) and Neumann (46) functions is defined by

$$E(\mu,n) \equiv D_+(\mu,n)\, E_-(\mu,n) - D_-(\mu,n)\, E_+(\mu,n). \tag{48}$$

Corollary 2. *The commutator* (48) *satisfies*

$$E(\mu,n) = -\tilde{E}(\mu,n)\, D_-(\mu,n) \tag{49}$$

and is given by

$$E(\mu,n) = 4\frac{(-1)^n}{\pi}\mu^{-1-n}\frac{\Gamma(-n/2-1/\mu)}{\Gamma(n/2-1/\mu)}. \tag{50}$$

Proof. Using (47a) and (47b) in (48) proves (49); the substitution of (27b) and (43b) leads to (50). □

Remark 6. The result (48) agrees with the theorem 7 in [13], that was proved using a different method of Wronskians.

Definition 5. The modified cross-commutator is defined exchanging the sign $n \to -n$ in one of the first factors in (48):

$$\hat{E}(\mu,n) \equiv D_+(\mu,-n)\,E_-(\mu,n) - D_-(\mu,-n)\,E_+(\mu,n). \quad (51)$$

Corollary 3. *The modified cross-commutator* (51) *is given by*

$$\hat{E}(\mu,n) = \frac{(-1)^n}{\pi}\frac{4}{\mu}. \quad (52)$$

Proof. Substitution of (47a) and (47b) in (51) gives

$$\hat{E}(\mu,n) = -\frac{1}{\pi}\log(\mu)\,D(\mu,n) - D_-(\mu,-n)\,\tilde{E}(\mu,n), \quad (53)$$

where: (i) the coefficient of the logarithmic term is

$$\frac{4\sin(\pi\nu)}{\mu\pi} = D_+(\mu,-\nu)\,D_-(\mu,\nu) - D_+(\mu,\nu)\,D_-(\mu,-\nu) \equiv D(\mu,\nu), \quad (54)$$

and vanishes for n as an integer $n \in \mathbb{Z}$,

$$D(\mu,n) = \frac{4}{\mu\pi}\sin(n\pi) = 0; \quad (55)$$

(ii) the second term on the right-hand side of (51) is a constant evaluated from (27b) and (43b):

$$-D_-(\mu,-n)\,\tilde{E}(\mu,n) = \frac{(-1)^n}{\pi}\frac{4}{\mu}. \quad (56)$$

□

Remark 7. Using (55) simplifies the modified cross-commutator (53) to:

$$\hat{E}(\mu,n) = -D_-(\mu,-n)\,\tilde{E}(\mu,n). \quad (57)$$

Corollary 4. *The ratio of the two cross-commutators* (48) *and* (51) *is*

$$\frac{\hat{E}(\mu,n)}{E(\mu,n)} = \frac{D_-(\mu,-n)}{D_-(\mu,n)} = \mu^n\frac{\Gamma(n/2-1/\mu)}{\Gamma(-n/2-1/\mu)}. \quad (58)$$

Proof. The ratio of (57) and (49) proves the first relation of (58) and the second follows from (27b). □

4 Application to Acoustic-vortical Waves in a Swirling Flow

The original Bessel functions, [3], [4], have many applications, such as cylindrical and spherical acoustic, [7], [8], [20], [22], [27], [28], elastic, [25], [26], and electromagnetic, [23], [24], waves. The generalised Bessel functions were introduced, [12], [13], for acoustic-vortical waves, [11], in a swirling flow, and the comparison (section 4) with purely acoustic waves in a uniform flow (subsection 4.1) shows that the related mathematical and physical properties are: (i) moderately modified near the rotation axis, where the tangential velocity of the mean flow is small (subsection 4.2); (ii) quite different far from the rotational axis, where the strong swirl causes radial and temporal instabilities (subsection 4.3).

The inlets of compressors and exhaust nozzles of turbines, and turbomachinery in ducts adds swirl to the axial flow, leading to acoustic-vortical waves, that relate to the stability of the mean flow.

4.1 Waves in a Cylindrical Nozzle with Axial and Swirling Flow

The pressure perturbation of linear non-dissipative free sound waves in a compressible fluid with uniform velocity $\mathbf{V}_0$ satisfies the convected wave equation, [8], [22]

$$\left\{\nabla^2 - \frac{1}{c^2}\left(\frac{\partial}{\partial t} + \mathbf{V}_0\cdot\boldsymbol{\nabla}\right)^2\right\} p(\mathbf{x},t) = 0, \quad (59)$$

where c is the sound speed. In cylindrical coordinates (r,ϕ,z) with the z axis along the uniform flow velocity U,

$$\mathbf{V}_0 = \mathbf{e}_z U, \quad (60)$$

the convected wave equation (59) becomes

$$\left\{\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial z}\right)^2\right\} \times p(r,\phi,z,t) = 0. \quad (61)$$

Since the coefficients depend only on the distance from the axis, exist waves with exponential dependence on time, axial and azimuthal coordinates,

$$p(r,\phi,z,t) = \sum_{m=-\infty}^{+\infty} e^{\mathrm{i}m\phi}\int_{-\infty}^{+\infty}\mathrm{d}k \times \int_{-\infty}^{+\infty}\mathrm{d}\omega\exp[\mathrm{i}(kz-\omega t)]\,P(r;k,m,\omega), \quad (62)$$

where: (i) the Fourier series involve the integer azimuthal wavenumber m; (ii) the double Fourier integral involves the frequency ω and axial wavenumber k. Substituting (62) in (61), it follows that the pressure perturbation spectrum $P(r;m,k,\omega)$, for a wave with frequency ω, and azimuthal m and axial k wavenumbers, has a radial dependence specified by

$$r^2\frac{\mathrm{d}^2P}{\mathrm{d}r^2}+r\frac{\mathrm{d}P}{\mathrm{d}r}+\left(\kappa^2r^2-m^2\right)P=0, \tag{63}$$

that is a linear second-order ordinary differential equation.

In (63) appears the frequency Doppler shifted by the mean flow,

$$\overline{\omega}=\omega-kU, \tag{64}$$

and the radial wavenumber,

$$\kappa=\sqrt{\frac{(\omega-kU)^2}{c^2}-k^2}=\sqrt{\frac{\overline{\omega}^2}{c^2}-k^2}. \tag{65}$$

The change of variables

$$\xi\equiv\kappa r, \tag{66a}$$

$$P(r;k,m,\omega)\equiv Q(\xi), \tag{66b}$$

transforms (63) to a Bessel differential equation of order m:

$$\xi^2\frac{\mathrm{d}^2Q}{\mathrm{d}\xi^2}+\xi\frac{\mathrm{d}Q}{\mathrm{d}\xi}+\left(\xi^2-m^2\right)Q=0. \tag{67}$$

Thus the acoustic pressure spectrum is given for finite radius by a linear combination of Bessel and Neumann functions,

$$0\leq r<\infty:\quad P(r;m,k,\omega)=A_+J_m(\kappa r)+A_-Y_m(\kappa r), \tag{68}$$

and asymptotically by a linear combination of Hankel functions of two kinds,

$$0<r:\quad P(r;m,k,\omega)=B_+H_m^{(1)}(\kappa r)+B_-H_m^{(2)}(\kappa r). \tag{69}$$

If a swirling flow (Figure 1) with constant angular velocity Ω is superimposed on the uniform flow (60), the total mean flow velocity becomes

$$\mathbf{V}_0=\mathbf{e}_zU+\mathbf{e}_\phi\Omega r, \tag{70}$$

and the pressure perturbation spectrum (62) satisfies, [11], the differential equation

$$r^2\frac{\mathrm{d}^2P}{\mathrm{d}r^2}+r\left[1-\left(\frac{\Omega r}{c_0}\right)^2\right]\frac{\mathrm{d}P}{\mathrm{d}r}+\left(\kappa^2r^2-m^2\right)P=0, \tag{71}$$

where: (i) the sound speed depends on the radius r,

$$[c_0(r)]^2=c_{00}^2-\frac{\gamma-1}{2}\Omega^2r^2, \tag{72}$$

with c_{00} the sound speed on axis and γ the adiabatic exponent, and is assumed that the swirl Mach number M is small, but not negligible,

$$M^4\ll1:\quad M=\frac{\Omega r}{c_{00}}; \tag{73}$$

(ii) with the same approximation the radial wavenumber is given by

$$\kappa=\sqrt{\frac{\overline{\omega}^2}{c_{00}^2}-k^2+\left(\frac{2k\Omega}{\overline{\omega}}\right)^2-2\left(\frac{\Omega}{c_{00}}\right)^2\left(3+m\frac{\Omega}{\overline{\omega}}\right)}. \tag{74}$$

In the absence of swirl $\Omega=0$, the sound speed (72) is constant $c_0(r)=c_{00}$, the radial wavenumber (74) simplifies to (65), and the differential equation (71) reduces to (63), leading to the original Bessel differential equation (67). In the presence of swirl $\Omega\neq0$ the same change of variable (66a)–(66b) applied to acoustic-vortical waves (71) leads to the generalised Bessel differential equation

$$\xi^2\frac{\mathrm{d}^2Q}{\mathrm{d}\xi^2}+\xi\left(1-\frac{\mu}{2}\xi^2\right)\frac{\mathrm{d}Q}{\mathrm{d}\xi}+\left(\xi^2-m^2\right)Q=0, \tag{75}$$

with degree specified by the swirl

$$\mu=2\left(\frac{\Omega}{c_{00}\kappa}\right)^2. \tag{76}$$

Thus the difference between purely acoustic waves (63) in a uniform flow (60) and acoustic-vortical waves (71) in a swirling flow (70) relates respectively to the original (67) and generalised (75) Bessel differential equations, whose solutions are considered for small (subsection 4.2) and large (subsection 4.3) radius.

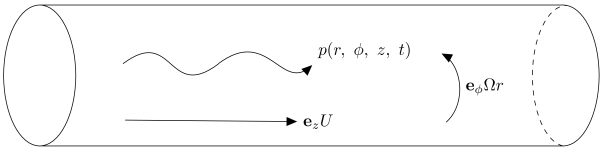

Figure 1: The linear compressible vortical perturbation of a uniform flow with swirl in a cylindrical duct corresponds to acoustic-vortical waves in a nozzle.

4.2 Vortical Modification of Sound Waves for Small Radius

For small radius the swirl in (70) is a small addition to the uniform mean flow (60), and the properties of acoustic-vortical waves appear as a modification of sound waves. The solution of the generalised Bessel differential equation (1)≡(75) is a linear combination of generalised Bessel and Neumann functions with: (i) variable (66a) distance from axis r made dimensionless multiplying by the radial wavenumber (74); (ii) order m equal to the azimuthal wavenumber in (62); (iii) degree equal to the dimensionless swirl parameter (76),

$$r_1 \leq r \leq r_2 : \\ P(r;m,k,\omega) = A_+ J_m^\mu(\kappa r) + A_- Y_m^\mu(\kappa r), \quad (77)$$

and holds (Figure 2) for an annular duct of inner and outer radii respectively r_1 and r_2. For a cylindrical duct $r_1 = 0$, the generalised Neumann function (37), that has a logarithmic singularity on axis, must be omitted $A_- = 0$, and the pressure perturbation spectrum is specified by the generalised Bessel function alone,

$$0 \leq r \leq r_2 : \quad P(r;m,k,\omega) = A_+ J_m^\mu(\kappa r). \quad (78)$$

Note that only one arbitrary constant A_+ remains.

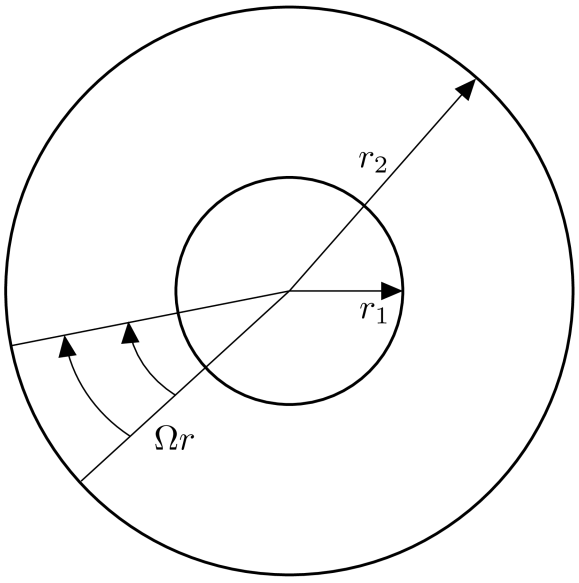

Figure 2: The nozzle can be an annular duct with inner and outer radii, respectively r_1 and r_2, with axial and swirling flow in between, and can reduce to: (i) the interior of a cylinder in Figure 1 if the inner radius is zero $r_1 = 0$; (ii) the exterior of a cylinder of radius r_1 if the outer radius is infinite $r_2 = \infty$.

Using (14) with (66a) and (76) the pressure perturbation spectrum (78) of acoustic-vortical waves in a cylindrical nozzle with uniform axial flow and rigid body swirl (70) is given by

$$P(r;m,k,\omega) = A_+ \frac{(\kappa r/2)^m}{\Gamma(m/2-1/\mu)} \\ \times \sum_{j=0}^{\infty} \left(\frac{\Omega^2 r^2}{2c_{00}^2}\right)^j \frac{\Gamma(j+m/2-1/\mu)}{j!\Gamma(1+j+m)}. \quad (79)$$

Using the descending property of the Gamma function,

$$\frac{\Gamma(j+m/2-1/\mu)}{\Gamma(m/2-1/\mu)} = \prod_{l=0}^{j-1}(l+m/2-1/\mu) \\ = (-\mu)^{-j} \prod_{l=0}^{j-1}[1-\mu(l+m/2)], \quad (80)$$

together with

$$\left(\frac{\Omega^2 r^2}{2c_{00}^2}\right)^j = \left(\frac{\mu\kappa^2}{4} r^2\right)^j, \quad (81)$$

in (79) leads to the pressure spectrum:

$$P(r;m,k,\omega) = A_+ \left(\frac{\kappa r}{2}\right)^m \left\{ \frac{1}{m!} \right. \\ + \sum_{j=1}^{\infty} \frac{(-1)^j}{j!} \frac{(\kappa r/2)^{2j}}{(j+m)!} \\ \left. \times \prod_{l=0}^{j-1} \left[1 - (2l+m) \left(\frac{\Omega}{c_{00}\kappa}\right)^2 \right] \right\}. \quad (82)$$

In the absence of swirl $\Omega = 0$, for purely acoustic waves, the product in the last factor of (82) is omitted and: (i) for propagating waves, with real $\kappa = |\kappa|$ radial wavenumber (65), the series is oscillatory; (ii) for imaginary $\kappa = \mathrm{i}|\kappa|$ radial wavenumber the series is non-oscillatory leading to evanescent or divergent waves.

In the presence of swirl $\Omega \neq 0$, if it is small in the sense of small swirl parameter $\mu \ll 1$ in (76), the term in square brackets is positive if (l,m) are not too large, and the series remains (i) oscillatory for real κ and (ii) monotonic for imaginary κ. However, for stronger swirl, the term in square brackets is negative and the conclusions (i) and (ii) are reversed. In particular for strong swirl,

$$\Omega > c_{00}\kappa, \quad (83)$$

corresponding to $\mu > 1$, all terms in square brackets are negative, thus the series is monotonic for propagating waves, indicating instability. This is shown plotting the generalised Bessel function of first kind over the range

$$0 \leq \kappa r \leq 10 \tag{84a}$$

for four values of the swirl parameter,

$$\mu = 0, 0.2, 0.5, 1.0, \tag{84b}$$

for the fundamental mode and first two harmonics,

$$m = 0, 1, 2, \tag{84c}$$

in Figure 3 (Appendix), Figure 4 (Appendix) and Figure 5 (Appendix) respectively.

The Figure 3 (Appendix), for the fundamental mode $m = 0$, shows that for purely acoustic waves (top left), represented by the original Bessel function of order zero, there are oscillations, with decaying amplitude due to cylindrical spreading, in the absence of swirl $\mu = 0$. For increasing swirl $\mu \neq 0$, the acoustic-vortical waves represented by the generalised Bessel function of degree μ and zero order, the Figure 3 (Appendix) shows that: (top right) for weak swirl $\mu = 0.2$ the oscillations remain for small radius and then diverge for larger radius; (bottom left) stronger swirl $\mu = 0.5$ suppresses the oscillations and leads to monotonic divergence; (bottom right) increasing further the swirl to $\mu = 1.0$ leads to faster divergence with opposite sign. The Figure 4 (Appendix) for the first harmonic $m = 1$ shows: (top left) decaying oscillations for acoustic waves; (top right) increasing oscillations for acoustic-vortical waves with weak swirl $\mu = 0.2$; (bottom left) divergence for stronger swirl $\mu = 0.5$, faster than for the fundamental ($m = 0$ in Figure 3, bottom left, Appendix); (bottom right) even faster divergence for stronger swirl. The Figure 5 (Appendix) for the second harmonic $m = 2$ confirms the effects of swirl shown by Figure 3 (Appendix) and Figure 4 (Appendix): (i) weak swirl can amplify oscillations leading to divergence at larger distances; (ii) stronger swirl suppresses oscillations and leads to monotonic divergence. These qualitative results can be explained qualitatively in Figure 6 (Appendix) as the superposition of the velocities associated with: (a) oscillations with finite amplitude for acoustic waves; (b) swirl with rigid body rotation with azimuthal velocity proportional to the radius. The result is: (i) for small radius the oscillations may be amplified or reduced by the swirl; (ii) for large radius the azimuthal velocity leads to overwhelming divergence. The transition from (i) to (ii) occurs at smaller radius for larger swirl; for strong swirl the oscillations may be suppressed, leading to monotonic divergence.

4.3 Asymptotic Instability in Radial Direction and Time

The Figure 3 (Appendix), Figure 4 (Appendix) and Figure 5 (Appendix) show the following contrast: (i) linear acoustic waves are oscillatory stable perturbations of a uniform compressible mean flow; (ii) adding swirl to the mean flow, in this case with constant angular velocity, leads to acoustic-vortical waves, which are unstable with growing oscillations or monotonic divergence. The spatial stability in the radial direction corresponds to asymptotic solutions for large distance from axis of the (a) original and (b) generalised Bessel differential equations, respectively for (a) acoustic and (b) acoustic-vortical waves. The pressure perturbation of acoustic waves is given (69) by a linear combination of Hankel functions of two kinds, with asymptotic approximations, [32],

$$H_m^{(1,2)}(\kappa r) \sim \sqrt{\frac{2}{\pi\kappa r}} \exp\left[\pm \mathrm{i}\left(\kappa r - \frac{m\pi}{2} - \frac{\pi}{4}\right)\right] + O\left(r^{-3/2}\right), \tag{85}$$

corresponding to: (i) waves propagating radially outward (or inward) for the plus (or minus) sign; (iii) for cut-on (or cut-off) modes with real (or imaginary) radial wavenumber (65) to oscillating wave (86a) [or monotonic decay or growth (86b)], that is,

$$\exp(\mathrm{i}\kappa r) = \exp(\pm \mathrm{i}|\kappa| r) \tag{86a}$$

if $\kappa = \pm|\kappa|$ for cut-on modes, or

$$\exp(\mathrm{i}\kappa r) = \exp(\mp|\kappa| r) \tag{86b}$$

if $\kappa = \pm \mathrm{i}|\kappa|$ for cut-off modes; (iii) the amplitude decays with the inverse square root of the distance from axis, to conserve the energy flux across a cylinder of large radius,

$$\lim_{r\to\infty}\left\{r|P(r;m,k,\omega)|^2\right\} = \text{const.}, \tag{87}$$

because sound waves do not exchange energy with the compressible uniform mean flow.

Adding swirl to the uniform stream leads to acoustic-vortical waves, that can exchange energy with the mean flow, leading to different asymptotic scaling of the pressure perturbation spectrum, that can be represented alternatively and equivalently, by: (i) a linear combination (77) of generalised Bessel and Neumann functions, extending (68) from acoustic to acoustic-vortical waves; (ii) a linear combination of generalised Hankel functions of two kinds,

$$P(r;m,\kappa,\omega) = B_+ H_{\mu,m}^{(1)}(\kappa r) + B_- H_{\mu,m}^{(2)}(\kappa r), \tag{88}$$

generalising similarly (69). The generalised Hankel function of first kind (24a) diverges asymptotically like a power of the distance from axis,

$$H_{\mu,m}^{(1)}(\kappa r) \sim (\kappa r)^{2/\mu} \left\{ 1 + \sum_{j=1}^{N-1} \frac{1}{j!} \left(-\frac{\Omega^2 r^2}{2c_{00}^2} \right)^{-j} \times \prod_{l=0}^{j-1} \left[\left(l - \frac{1}{\mu} \right)^2 - \frac{m^2}{4} \right] + O\left[(\kappa r)^{-2N} \right] \right\}, \tag{89}$$

corresponding to the "acoustic" mode. The Hankel function of second kind (24b) scales asymptotically as an exponential of the square of the radius,

$$H_{\mu,m}^{(2)}(\kappa r) \sim \exp\left(\frac{\Omega^2 r^2}{2c_{00}^2} \right) (\kappa r)^{-2-2/\mu} \left\{ 1 + \sum_{j=1}^{N-1} \frac{1}{j!} \left(-\frac{\Omega^2 r^2}{2c_{00}^2} \right)^{-j} \times \prod_{l=0}^{j-1} \left[\left(l + \frac{1}{\mu} \right)^2 - \frac{m^2}{4} \right] + O\left[(\kappa r)^{-2N} \right] \right\}, \tag{90}$$

corresponding to the "vortical mode". Thus the pressure perturbation spectrum of acoustic-vortical waves always diverges asymptotically for large distance from the axis, with predominance of the exponential "vortical" mode (90) over the algebraic "acoustic" mode (89).

The generalised Bessel differential equation (75) for acoustic-vortical waves converges to the original Bessel differential equation (67) for acoustic waves, in the limit $\mu \to 0$ of zero swirl parameter. It follows, as be checked from (82) in the limit of zero swirl $\Omega \to 0$, or zero degree $\mu \to 0$, that the generalised Bessel function tends to the original Bessel function,

$$\lim_{\mu \to 0} J_\nu^\mu(z) = J_\nu(z), \tag{91}$$

and likewise for the generalised and original Neumann functions,

$$\lim_{\mu \to 0} Y_\nu^\mu(z) = Y_\nu(z). \tag{92}$$

However the generalised Hankel functions of the first (89) and second (90) kinds do not converge to the original Hankel functions (85) for zero degree,

$$\lim_{\mu \to 0} H_{\mu,\nu}^{(1,2)}(z) \neq H_\nu^{(1,2)}(z), \tag{93}$$

and indeed in the generalised Bessel differential equation (75) the term

$$\frac{\mu}{2}\xi^2 = \frac{(\kappa r)^2}{2} 2 \left(\frac{\Omega}{c_{00}\kappa} \right)^2 = \left(\frac{\Omega r}{c_{00}} \right)^2 \tag{94}$$

becomes dominant as $r \to \infty$, as the swirl velocity $\Omega r \to \infty$ dominates the uniform axial flow (60) in the total mean flow velocity (70). The inversion of the variable,

$$\eta = \frac{1}{\xi}, \tag{95a}$$

$$W(\eta) = Q(\xi) \tag{95b}$$

transforms the generalised Bessel differential equation (75) to

$$\eta^2 \frac{\mathrm{d}^2 W}{\mathrm{d}\eta^2} + \eta \left(1 + \frac{\mu}{2\eta^2} \right) \frac{\mathrm{d}W}{\mathrm{d}\eta} + \left(\frac{1}{\eta^2} - m^2 \right) W = 0, \tag{96}$$

showing that asymptotically, as $r \to \infty$, $\xi \to \infty$ and $\eta \to 0$: (i) in the absence of swirl $\mu = 0$, for acoustic waves, the original Bessel equation has an $O\left(\eta^{-2}\right)$ coefficient of W as the only singular factor, corresponding to an irregular singularity of degree one $\vartheta = 1$ at infinity $\xi \to \infty$ or $\eta \to 0$; (ii) in the presence of swirl $\mu \neq 0$ for acoustic-vortical waves, the non-zero degree in the generalised Bessel differential equations introduces a singular factor μ/η multiplying $\mathrm{d}W/\mathrm{d}\eta$, and corresponding to an irregular singularity of higher degree two $\vartheta = 2$ at infinity. Thus the generalised Bessel differential equation has asymptotically a Hopf-type bifurcation, [13], [19], [20], for $\mu = 0$ due to the contrast between uniform and swirling mean flow.

Since the wave speed is finite the asymptotic spatial instability of acoustic-vortical waves for large distance from axis $r \to \infty$ corresponds to a temporal instability for long time $t \to \infty$, as shown directly next. The temporal dependence of the pressure perturbation (62) on the frequency can be expressed in terms of the Doppler shifted frequency (64):

$$\exp(-\mathrm{i}\omega t) = \exp(-\mathrm{i}kUt) \exp(-\mathrm{i}\overline{\omega} t). \tag{97}$$

The spatial radial stability is evaluated with real frequency ω, allowing the radial wavenumber (74) to be real for cut-on modes (86a) and imaginary (86b) for cut-off modes, corresponding respectively to $\kappa^2 > 0$ and $\kappa^2 < 0$. In both cases κ^2 is real in (74) that can

be rewritten as a quartic polynomial of the Doppler shifted frequency:

$$0 = \overline{\omega}^4 - \overline{\omega}^2 \left[\left(k^2 + \kappa^2 \right) c_{00}^2 + 6\Omega^2 \right] - 2m\Omega^3 \overline{\omega} + 4k^2 \Omega^2 c_{00}^2. \quad (98)$$

The Doppler shifted frequency as a root of (98) may be complex, and the corresponding term in (97),

$$\exp\left(-\mathrm{i}\overline{\omega} t\right) = \exp\left[-\mathrm{i}t\Re\left(\overline{\omega}\right)\right] \exp\left[t\Im\left(\overline{\omega}\right)\right], \quad (99)$$

shows that: (i) there is temporal oscillation specified by the real part $\Re\left(\overline{\omega}\right)$; (ii) the imaginary part $\Im\left(\overline{\omega}\right)$ leads to constant amplitude, decay or instability if it is respectively zero, negative or positive. Thus acoustic-vortical waves will be temporally unstable if at least one of the four roots of (98) has positive imaginary part. Since the quartic polynomial has real coefficients, there are three possibilities: (i) four real roots, equal or distinct; (ii) one complex conjugate pair and two real roots equal or distinct; (iii) two pairs of complex conjugate roots, equal or distinct. It can be shown, [11], that the case (i) is impossible, and thus there is at least one pair of complex conjugate roots, and the root(s) with positive imaginary part implies temporal instability in both possible cases (ii) and (iii).

5 Conclusion

The confluent hypergeometric function of the first kind (12) corresponds to the generalised Bessel function (9)≡(18) and the confluent hypergeometric function of the second kind (33) corresponds to the modified generalised Neumann function (30) and to the generalised Hankel function of the first kind (43a)–(43b). The asymptotic expansions for the confluent hypergeometric functions of two kinds, [30], [32], can be used to obtain, as alternatives or complements to [12], [13], asymptotic expansions for the generalised Bessel functions – equations (26), (27a) and (27b) –, for the modified – equation (34) – and unmodified – equations (46), (47a), (47b) and (43b) – generalised Neumann functions, relating them to the asymptotic expansions for the generalised Hankel functions of first (24a) and second (24b) kinds, that are the basic expansions. Thus (Figure 7, Appendix) the basic asymptotic expansions are: (i) the generalised Hankel function of the first kind (24a), that specify the modified generalised Neumann function (42) to within a constant factor (43a)–(43b); (ii) the generalised Hankel function of the second kind (24b) whose asymptotic expansion involves an exponential factor unlike (i). The asymptotic expansion (23) for the generalised Bessel function (14) is a linear combination (25)≡(26) of (i) and (ii) with coefficients (27a)–(27b). The asymptotic expansion (45) for the generalised Neumann function (31) is a linear combination (46) of (i) and (ii) with coefficients (47a), (47b) and (43b). The coefficients in the preceding asymptotic expansions satisfy the relations (48) to (56) as indicated in the Figure 8 (Appendix).

Some of these relations can also be obtained from the direct solution of the generalised Bessel differential equations using: (i) the Frobenius-Fuchs, [14], [15], [16], method to obtain, [10], series expansions around the regular singularity at the origin, specifying the generalised Bessel functions, and with a logarithmic singularity for the generalised Neumann function; (ii) the normal integrals, [17], [18], to obtain, [13], asymptotic expansions around the irregular singularity at infinity leading to the generalised Hankel functions of the first and second kinds. The relation between the generalised Bessel, [12], and confluent hypergeometric, [30], [32], differential equations provides an alternative approach relating: (i) the generalised Bessel function to the confluent hypergeometric function of the first kind (section 2); (ii) the confluent hypergeometric function of the second kind to the modified generalised Neumann function and generalised Hankel function of first kind (section 3); (iii) the unmodified generalised Neumann function relates to the generalised Hankel functions of two kinds, and hence the confluent hypergeometric functions of two kinds to the modified generalised Neumann function (section 3). The two approaches are complementary since one may provide a simpler proof of some results whereas the other may provide more details of some relations.

The first diagram in Figure 7 (Appendix) illustrates the relations among: (i–ii) the confluent hypergeometric functions of two kinds at the bottom; (iii–vii) above the generalised Bessel functions, the unmodified and modified generalised Neumann functions, and the generalised Hankel functions of two kinds. The second diagram in Figure 8 (Appendix) concerns the seven coefficients that relate (i) to (v), and depend on the order ν and degree μ in the generalised Bessel differential equation, and do not involve the independent variable z.

The generalised Bessel differential equation (1)≡(75) was introduced, [12], in the context, [11], of acoustic-vortical waves (71) whose pressure perturbation spectrum (62) is specified alternatively and equivalently by a linear combination: (i) of (77) generalised Bessel and Neumann functions in an annular nozzle (Figure 2), with only the former appearing in the case of a cylindrical duct (78)≡(82); (ii) of (88) generalised Hankel functions of the first (89) and second (90) kinds. For purely acoustic waves (61) in a uniform stream

the sound speed c is constant and appears together with the Doppler shifted frequency (64) in the radial wavenumber (65). Adding (Figure 1) to the uniform stream (60) rigid body swirl (70) changes the radial wavenumber to (74) and introduces the swirl parameter (76). The pressure perturbation spectrum is specified: (i) as function of distance from axis of rotation made dimensionless (66a) multiplying by the axial wavenumber; (ii) the azimuthal wavenumber m in (62) specifies the order of the original Bessel, Neumann and Hankel functions for acoustic waves (68)≡(69); (iii) for acoustic-vortical waves (77)≡(88) the generalised Bessel, Neumann and Hankel functions have degree specified by the swirl parameter (76). The waveforms for the fundamental azimuthal mode $m = 0$, and first $m = 1$ and second $m = 2$ harmonics, respectively in Figure 3 (Appendix), Figure 4 (Appendix) and Figure 5 (Appendix), show that: (a) acoustic waves with $\mu = 0$ are stable oscillations decaying radially due to cylindrical geometric spreading (85) that conserves the asymptotic energy flux (87), and thus do not exchange energy with the mean flow; (b) acoustic-vortical waves with $\mu \neq 0$ are radially (and hence also temporally unstable) with oscillatory or monotonic amplitude growth due to interaction with the swirling mean flow. This contrast between (a) purely acoustic and (b) acoustic-vortical waves, shown quantitatively in Figure 3 (Appendix), Figure 4 (Appendix) and Figure 5 (Appendix) for the fundamental and first two harmonics, is explained qualitatively in Figure 6 (Appendix), with the acoustic oscillations with finite amplitude overwhelmed by the rigid body rotation at larger radii, leading to faster instability for stronger swirl.

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APPENDIX

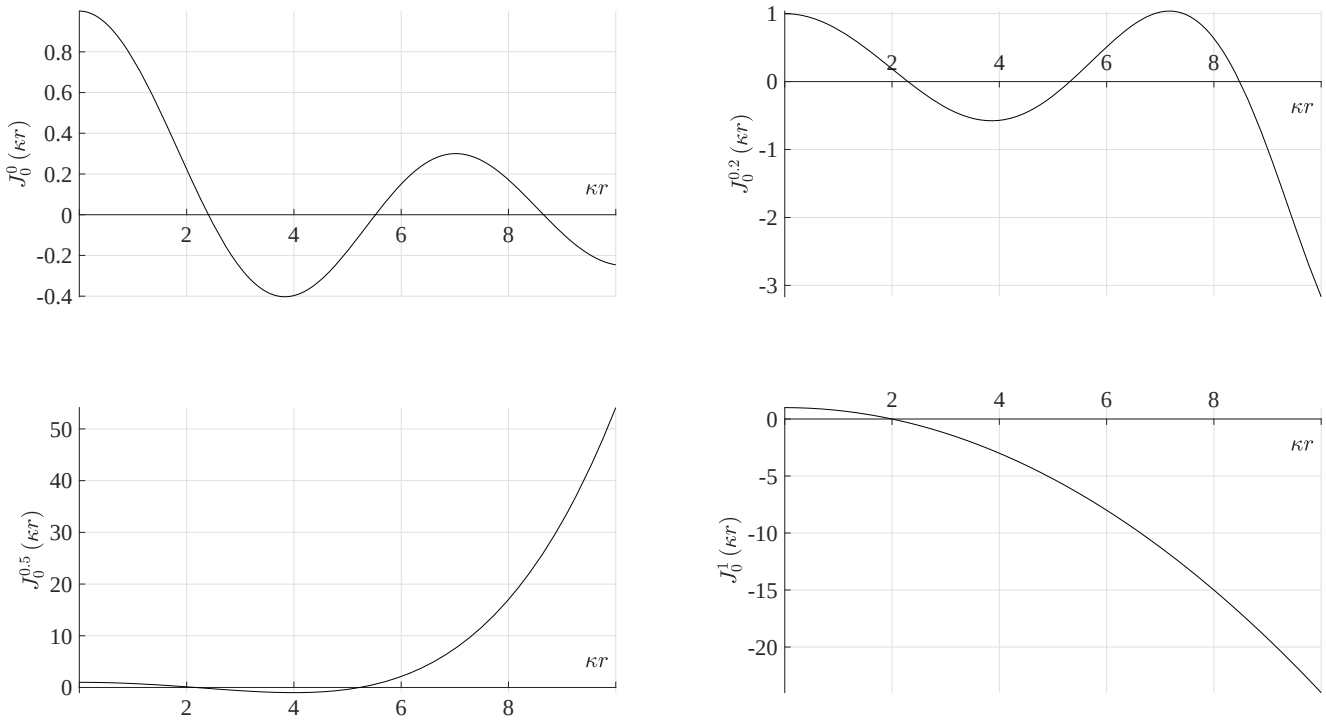

Figure 3: The pressure perturbation spectrum of acoustic-vortical waves in a cylindrical nozzle with axial flow and rigid body swirl are represented by generalised Bessel functions $J_m^\mu(\chi)$ of (84a) the distance from axis multiplied (66a) by the radial wavenumber – (74) and (65) – for four values (84b) of the swirl parameter (76) for the fundamental azimuthal mode $m = 0$.

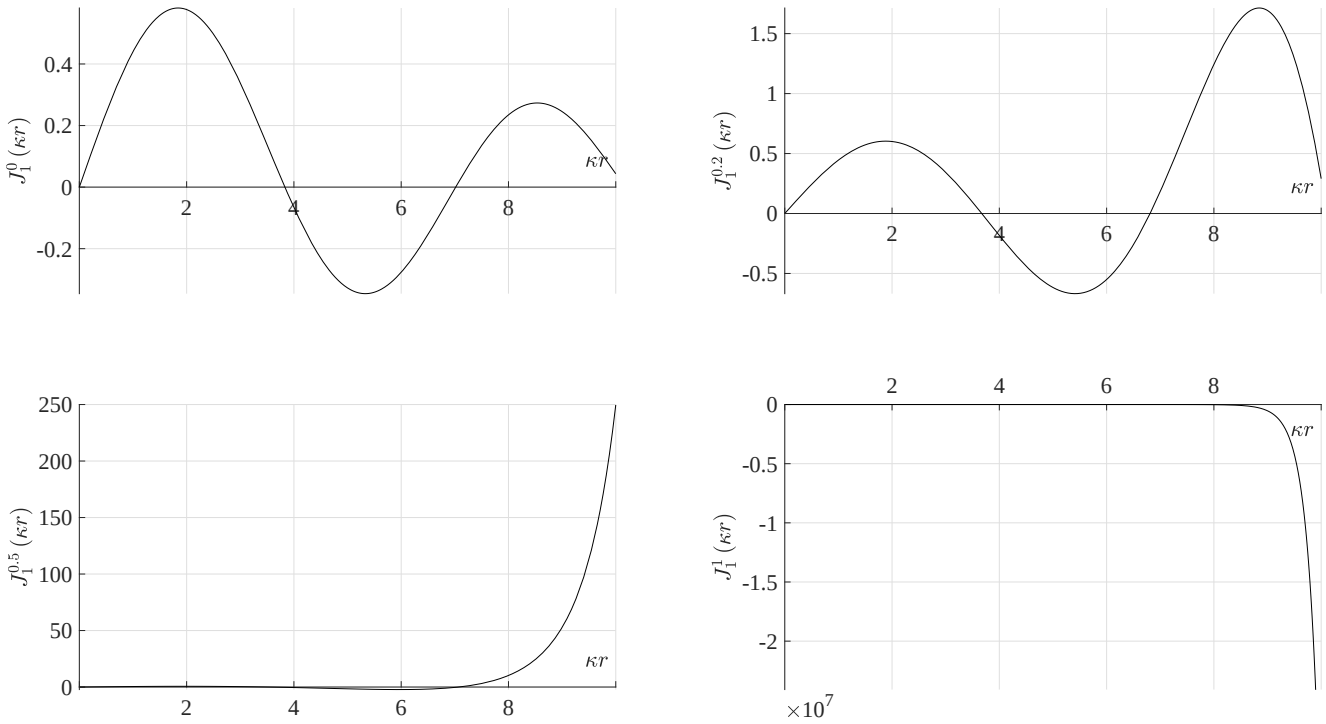

Figure 4: As figure 3 (Appendix) for the first azimuthal harmonic $m = 1$.

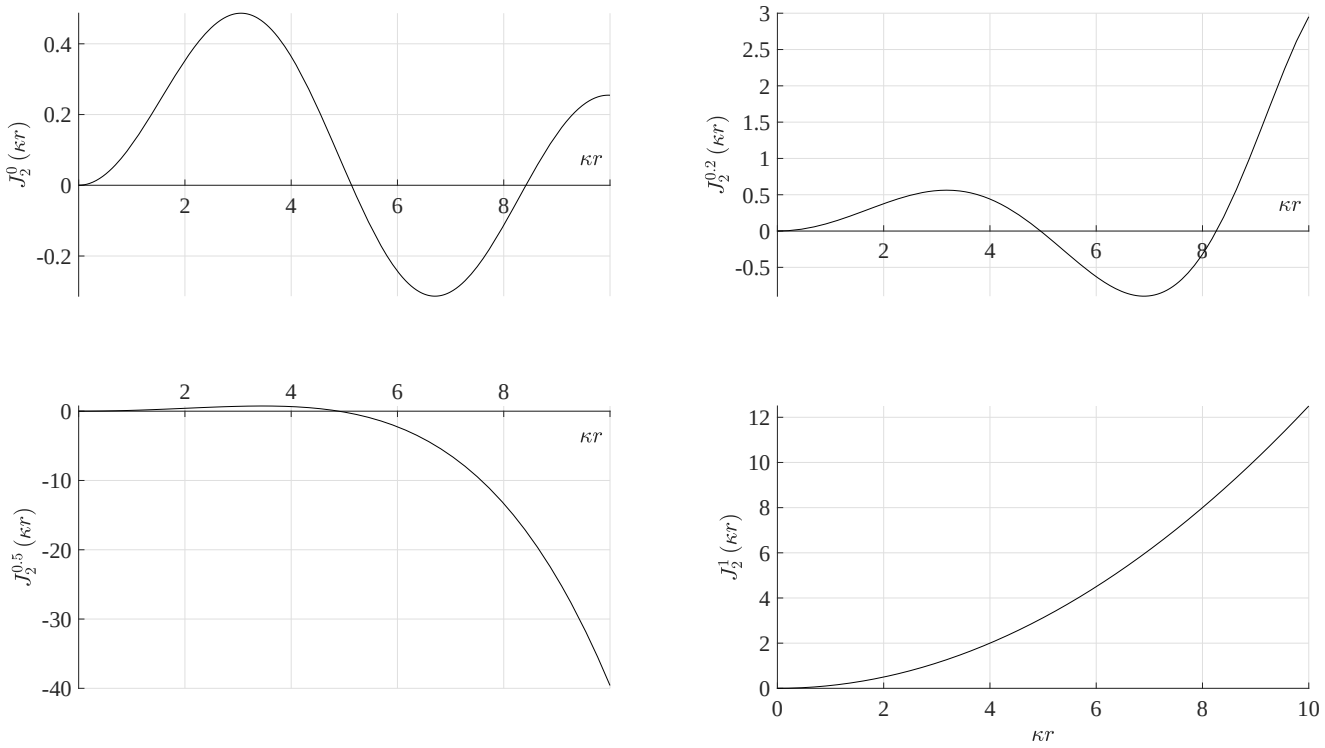

Figure 5: As figure 3 (Appendix) for the second azimuthal harmonic $m = 2$.

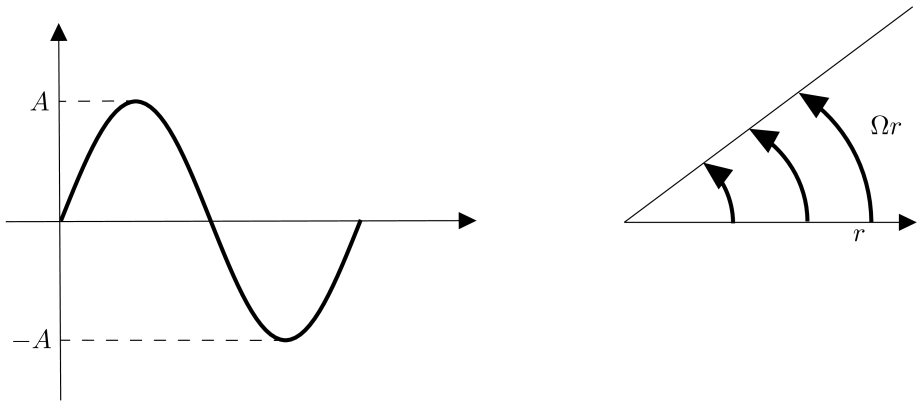

Figure 6: The superposition of (left) sinusoidal oscillations with (right) rigid body rotation: (i) can amplify or reduce oscillations for small radius; (ii) for large radius leads to monotonic instability.

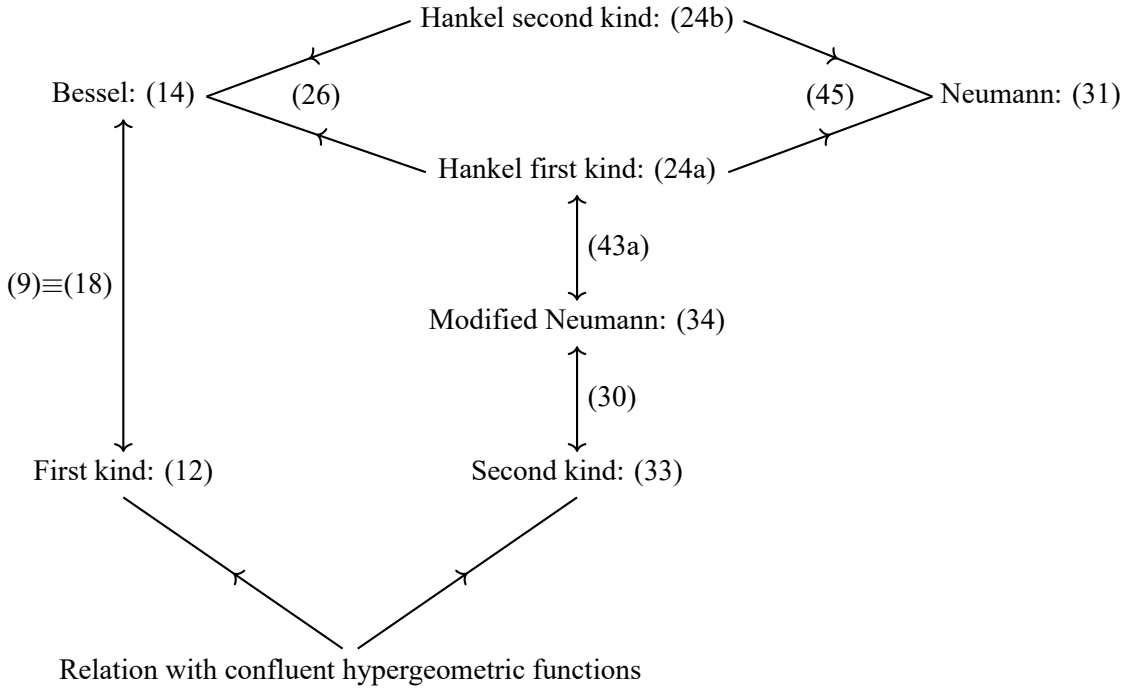

Figure 7: Relation between the generalised Bessel functions and the confluent hypergeometric functions.

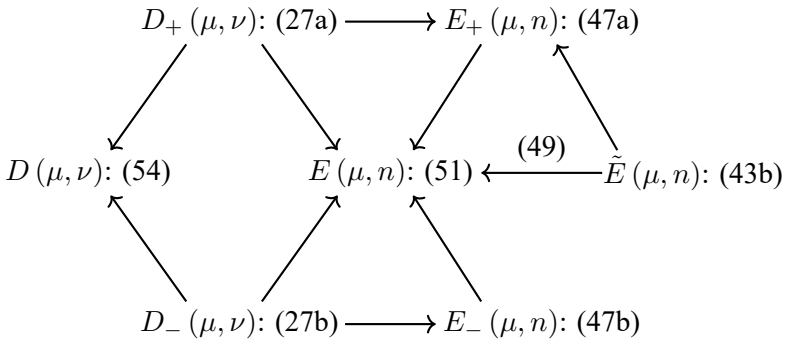

Figure 8: Relations among asymptotic coefficients.

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Conflicts of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.