Maximum Likelihood Estimation for the Parameters of New Mixture Maxwell–Boltzmann Distribution and Its Applications

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Abstract: This article introduces a new mixture extension of the Maxwell–Boltzmann distribution (MBD) to enhance data modeling flexibility. Maximum Likelihood Estimation (MLE) is employed to estimate parameters, and key statistical properties are analyzed. In addition, simulation studies are conducted to compare the bias, mean square error, and variance of the estimates over a wide range of sample sizes from n = 5 to 10,000. The new distribution is applied to a real-world heart attack survival dataset to demonstrate its potential in practical applications. These findings highlight the potential of the proposed distribution in survival studies and modeling right-skewed data.

Key-Words: Generalized Maxwell-Boltzmann Distribution, Maximum Likelihood Estimation, Moment Generating Function, Incomplete Moments, Mixture Distributions, Length-Biased Distributions, Sustainable Development Goal (SDG) 3

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1 Introduction

Survival analysis is a field within statistics concerned with analyzing the probability of an event of interest occurring over time, such as disease occurrence, recovery, or mortality. It is referred to by different names across disciplines: event history analysis (EHA) in sociology, reliability analysis in engineering, and duration analysis in economics. Broadly, statistical methodologies for survival analysis can be classified into three categories: (1) nonparametric statistics, exemplified by the Kaplan-Meier estimator and the log-rank test; (2) semiparametric statistics, exemplified by the Cox proportional hazards model; and (3) parametric statistics, which focus on constructing probability models of survival time distributions. Parametric survival analysis assumes that survival functions follow specific parametric distributions, such as the exponential or Weibull distributions, where the hazard rate remains constant or varies over time, respectively. Once the distribution is assumed, parameter estimation, survival function computation, confidence interval calculation, and relative risk assessment can be conducted effectively, provided that the underlying assumptions hold.

A variety of parametric survival distributions have been employed in survival modeling, including the exponential, [1], [2], [3], [4], log-normal, [5], [6], [7], gamma, [8], [9], generalized gamma, [10], [11], log-logistic, [12], [13], inverse Gaussian, [14], [15], [16], [17], complementary gamma zero-truncated Poisson distribution, [18], [19], and mixture Pareto distributions, [20]. These distributions can describe diverse survival data when the best-fit distribution is selected.

Due to the importance of selecting the appropriate survival distribution for the data, this article focuses on creating a new distribution based on the well-known MBD. Present an alternative survival model with a time-varying hazard rate that is estimated using the Maximum Likelihood Estimation (MLE) and show its applicability using real-world data.

Length-biased distributions (LBDs) are modeled to display bias behavior depending on how long a person lives or how long a particular phenomenon, such as survival data. Recently, research works have investigated modifications and improvements of the MBD. For example, Jallal et al. presented a mixture of MBD with a weighted component, enhancing its flexibility in modeling diverse datasets [21]. In addition, Castillo et al. introduced a new mixture model that combines MBD with a generalized gamma distribution, offering improved adaptability [22]. Furthermore, the study of Phaphan and Abdullahi focused on the theoretical properties of the Maxwell-Boltzmann distribution with length bias (LBMBD) [23]. They explain how to use the "optimize" and "maxLik" functions in R to apply the MLE of the distribution parameters. This article develops a new parametric survival distribution with time-dependent hazard rates by expanding the Maxwell-Boltzmann distribution. This proposed distribution is expected to be especially valuable in medical and clinical research that contributes to the Sustainable Development Goal (SDG) 3, which aims to improve health and well-being for people at all stages of life.

2 Length-biased Maxwell–Boltzmann distribution (LBMBD)

The MBD is well known in statistical fields, mechanical engineering, and applied sciences [16]. The cumulative distribution function (CDF) of MBD is given by

$$G_{MB}(x) = \frac{2}{\sqrt{\pi}} \times \gamma\left(\frac{3}{2}, \frac{x^2}{2\lambda^2}\right). \tag{1}$$

Next, the probability density function (PDF) of MBD is given by

$$g_{MB}(x) = \sqrt{\frac{2}{\pi}}\frac{x^2 \mathrm{e}^{-\frac{x^2}{2\lambda^2}}}{\lambda^3}, \tag{2}$$

where $x \geq 0$, the parameter λ is positive, and γ denotes the incomplete lower Gamma function. The expected value of the MBD is given by

$$E(X) = 2\lambda\sqrt{\frac{2}{\pi}}. \tag{3}$$

The length-biased Maxwell–Boltzmann distribution's (LBMBD) PDF and CDF are represented by [23]:

$$g_{LBMB}(x) = \frac{xf(x)}{E(X)} = \frac{x^3 \mathrm{e}^{-\frac{x^2}{2\lambda^2}}}{2\lambda^4}, \tag{4}$$

and

$$G_{LBMB}(x) = \int_0^x \frac{x^3 \mathrm{e}^{-\frac{x^2}{2\lambda^2}}}{2\lambda^4} dx. \tag{5}$$

Hence, equation (2) can be expressed as

$$G_{LBMB}(x;\lambda) = \gamma\left(2, \frac{x^2}{2\lambda^2}\right). \tag{6}$$

3 Proposed a new mixture Maxwell–Boltzmann distribution

3.1 Probability Density Function

The MBD exhibits a fundamental role in statistical mechanics and other fields. However, in many real-world scenarios, a single distribution may not be sufficient to accurately model complex data structures. Therefore, a study of mixture distributions has been developed, which provides a flexible approach by combining multiple distributions to better fit the data. In this subsection, we propose a new mixture MBD, which extends the classical MBD to accommodate right-skewed data. This new distribution will be an alternative skewed distribution, making it suitable for various scientific and engineering applications.

$$f(x) = \frac{1}{\lambda+1}g_{MB}(x) + \frac{\lambda}{\lambda+1}g_{LBMB}(x) \tag{7}$$

Inserting equations (2) and (4) into equation (7), we get

$$f(x) = \frac{x^2 \mathrm{e}^{-\frac{x^2}{2\lambda^2}}}{(\lambda+1)\lambda^3}\left[\sqrt{\frac{2}{\pi}} + \frac{x}{2}\right]. \tag{8}$$

3.2 Verification of the new mixture Maxwell–Boltzmann distribution

Before applying a newly proposed distribution, it is essential to verify its validity through mathematical and statistical property. The new mixture Maxwell–Boltzmann distribution must satisfy fundamental statistical criteria which is as follow:

$$\int\limits_0^\infty \frac{x^2 \mathrm{e}^{-\frac{x^2}{2\lambda^2}}}{(\lambda+1)\lambda^3}\left[\sqrt{\frac{2}{\pi}} + \frac{x}{2}\right] dx = 1. \tag{9}$$

Let

$$k = \frac{x^2}{2\lambda^2} \Rightarrow x = \lambda\sqrt{2k}, \tag{10}$$

and

$$dx = \frac{\lambda^2 dk}{x}. \tag{11}$$

We substitute equations (10) and (11) into (9)

$$\frac{1}{(\lambda+1)}\left[\frac{2}{\sqrt{\pi}}\int_0^\infty k^{\left(\frac{1}{2}+1\right)-1}\mathrm{e}^{-k}dk + \lambda\int_0^\infty k\mathrm{e}^{-k}dk\right] = 1, \tag{12}$$

then

$$\frac{1}{(\lambda+1)}\left[\frac{2}{\sqrt{\pi}}\frac{1}{2}\sqrt{\pi} + \lambda\Gamma(2)\right] = 1. \tag{13}$$

This clearly shown that, the probability density function defined in (9) is valid PDF.

3.3 Cumulative Distribution Function

The cumulative distribution function (CDF) of the new mixture Maxwell–Boltzmann distribution is as follows:

$$\begin{aligned} F(x) &= \int_0^x f(t)dt \\ &= \int_0^x \left(\frac{1}{\lambda+1}g(t) + \frac{\lambda}{\lambda+1}g_{LBMB}(t)\right)dt \\ &= \frac{1}{\lambda+1}G(x) + \frac{\lambda}{\lambda+1}G_{LBMB}(x). \end{aligned} \tag{14}$$

Therefore, the CDF can be expressed as

$$F(x) = \frac{1}{\lambda+1}\left[\frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2}, \frac{x^2}{2\lambda^2}\right) + \lambda\gamma\left(2, \frac{x^2}{2\lambda^2}\right)\right]. \tag{15}$$

3.4 Hazard Function

The hazard function can be obtained by using the following relation:

$$h(x) = \frac{f(x)}{1 - F(x)}. \tag{16}$$

From equations (8) and (15), we can write the hazard function of the proposed distribution as follows:

$$h(x) = \frac{\frac{x^2\mathrm{e}^{-\frac{x^2}{2\lambda^2}}}{(\lambda+1)\lambda^3}\left[\sqrt{\frac{2}{\pi}} + \frac{x}{2}\right]}{1 - \frac{1}{\lambda+1}\left[\frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2}, \frac{x^2}{2\lambda^2}\right) + \lambda\gamma\left(2, \frac{x^2}{2\lambda^2}\right)\right]}. \tag{17}$$

4 Properties of the new mixture Maxwell–Boltzmann distribution

4.1 The r^{th} ordinary moment of the new mixture Maxwell–Boltzmann distribution

The r^{th} ordinary moment of the new mixture Maxwell–Boltzmann distribution is defined by

$$\mu_r' = E(X^r) = \int_0^\infty x^r f(x)dx. \tag{18}$$

Substituting equation (8) into (18) and integrate with respect to x, lead to the r^{th} ordinary moment of the new mixture Maxwell–Boltzmann distribution is obtained as follows:

$$\mu_r' = \int_0^\infty \frac{x^{r+2}\mathrm{e}^{-\frac{x^2}{2\lambda^2}}}{(\lambda+1)\lambda^3}\left[\sqrt{\frac{2}{\pi}} + \frac{x}{2}\right]dx, \tag{19}$$

or

$$\mu_r' = \frac{1}{\lambda+1}E(X^r) + \frac{\lambda}{\lambda+1}E_{LBMB}(X^r), \tag{20}$$

where

$$E(X^r) = 2^{\frac{r+1}{2}}\lambda^r\sqrt{\frac{2}{\pi}};\ r = 1, 2, 3 \ldots.$$

And, we have

$$E_{LBMB}(X^r) = (2\lambda^2)^{\frac{r}{2}}\Gamma\left(\frac{r}{2}+2\right);\ r = 1, 2, 3 \ldots$$

$$\mu_r' = \frac{1}{\lambda+1}2^{\frac{r+1}{2}}\lambda^r\sqrt{\frac{2}{\pi}} + \frac{\lambda}{\lambda+1}(2\lambda^2)^{\frac{r}{2}}\Gamma\left(\frac{r}{2}+2\right), \tag{21}$$

when $r = 1$, we obtained mean

$$\mu_1' = \frac{\sqrt{8}\lambda}{\sqrt{\pi}(\lambda+1)} + \frac{3\sqrt{2\pi}\lambda^2}{4(\lambda+1)}, \tag{22}$$

when $r = 2$

$$\mu_2' = \frac{(2\lambda)^2}{\sqrt{\pi}(\lambda+1)} + \frac{4\lambda^3}{\lambda+1}. \tag{23}$$

The variance of new mixture Maxwell–Boltzmann distribution is

$$\begin{aligned} Var(X) =& \frac{(2\lambda)^2}{\sqrt{\pi}(\lambda+1)} + \frac{(2\lambda)^2}{\lambda+1} \\ &- \left[\frac{(2\lambda)^2}{\sqrt{\pi}(\lambda+1)} + \frac{4\lambda^3}{\lambda+1}\right]^2. \end{aligned} \tag{24}$$

5 The Moment Generating Function of the new mixture Maxwell–Boltzmann distribution

The moment generating function of the new mixture Maxwell–Boltzmann distribution can be written as

$$E(e^{Xt}) = M_X(t) = \sum_{r=0}^{\infty} \frac{t^r E(X^r)}{r!}. \tag{25}$$

By inserting equation (21) into (25) lead to the moment generating function of the new mixture of Maxwell–Boltzmann distribution is obtained as follows:

$$\begin{aligned} E(e^{Xt}) =& M_X(t) \\ =& \sum_{r=0}^{\infty} \frac{t^r}{r!} \Bigg\{ \frac{1}{\lambda+1} 2^{\frac{r+1}{2}} \lambda^r \sqrt{\frac{2}{\pi}} \\ &+ \frac{\lambda}{\lambda+1} (2\lambda^2)^{\frac{r}{2}} \Gamma\left(\frac{r}{2}+2\right) \Bigg\}. \end{aligned} \tag{26}$$

6 Maximum Likelihood Estimation for λ

The log-likelihood function for the given probability density function (PDF) is:

$$L(\lambda) = \prod_{i=1}^{n} f(x_i). \tag{27}$$

Taking the natural logarithm,

$$\log L(\lambda) = \sum_{i=1}^{n} \log f(x_i). \tag{28}$$

Substituting the PDF:

$$f(x) = \frac{x^2 e^{-x^2/2\lambda^2}}{(\lambda+1)\lambda^3} \left[\sqrt{\frac{2}{\pi}} + \frac{x}{2} \right], \tag{29}$$

we obtain:

$$\begin{aligned} \log L(\lambda) =& \sum_{i=1}^{n} \left[\log x_i^2 - \frac{x_i^2}{2\lambda^2} - \log(\lambda+1) \right] \\ &+ \sum_{i=1}^{n} \left[-3\log\lambda + \log\left(\sqrt{\frac{2}{\pi}} + \frac{x_i}{2} \right) \right]. \end{aligned} \tag{30}$$

To find the MLE for λ, we take the derivative:

$$\frac{d}{d\lambda} \log L(\lambda) = \sum_{i=1}^{n} \left[\frac{x_i^2}{\lambda^3} - \frac{1}{\lambda+1} - \frac{3}{\lambda} \right]. \tag{31}$$

Setting it equal to zero:

$$\sum_{i=1}^{n} \frac{x_i^2}{\lambda^3} = \sum_{i=1}^{n} \left[\frac{1}{\lambda+1} + \frac{3}{\lambda} \right]. \tag{32}$$

Define:

$$S_2 = \sum_{i=1}^{n} x_i^2. \tag{33}$$

Then,

$$\frac{S_2}{\lambda^3} = \frac{n}{\lambda+1} + \frac{3n}{\lambda}. \tag{34}$$

Multiplying by $\lambda^3(\lambda+1)$:

$$S_2(\lambda+1) = n\lambda^3 + 3n\lambda^2(\lambda+1). \tag{35}$$

Expanding:

$$S_2\lambda + S_2 = 4n\lambda^3 + 3n\lambda^2. \tag{36}$$

Rearranging into a cubic equation:

$$4n\lambda^3 + 3n\lambda^2 - S_2\lambda - S_2 = 0. \tag{37}$$

A unique positive real root exists by Descartes' Rule of Signs. Therefore, when n is large, the term $4n\lambda^3$ dominates over $3n\lambda^2$, $-S_2\lambda$, and $-S_2$, meaning we can approximate:

$$4n\lambda^3 \approx S_2. \tag{38}$$

$$\lambda^3 \approx \frac{S_2}{4n}. \tag{39}$$

Solving for λ we take the cube root of both sides, then $\hat{\lambda}$ is

$$\hat{\lambda} \approx \sqrt[3]{\frac{S_2}{4n}}, \tag{40}$$

where

$$S_2 = \sum_{i=1}^{n} x_i^2. \tag{41}$$

7 Simulation Result

7.1 Graphical Representations of the PDF and Hazard Function

In this subsection, we present simulations of the PDF and hazard function of the new mixture Maxwell–Boltzmann distribution. Simulation graphs are shown for varying values of the parameter $\lambda = 0.7, 1.4, 2.1, 2.8, 3.5$ to compare their effects on the shapes of the PDF, and hazard function, displayed in Figure 1 and Figure 2, respectively.

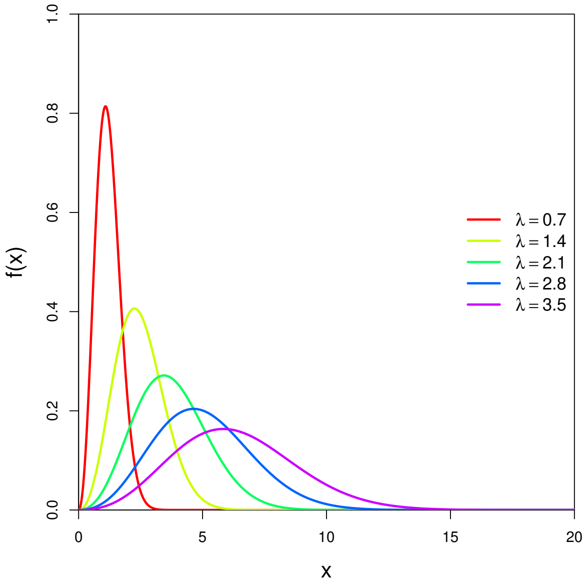

Fig. 1: The PDF of the new mixture Maxwell–Boltzmann distribution plotted for different values of the parameter $\lambda = 0.7, 1.4, 2.1, 2.8$, and 3.5.

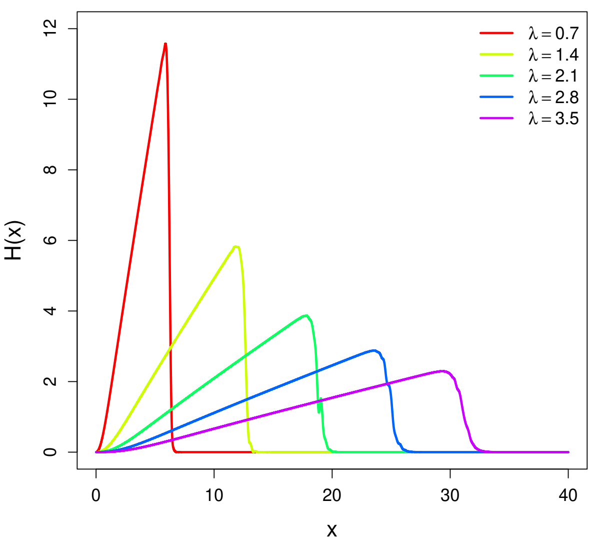

Fig. 2: The hazard function of the new mixture Maxwell–Boltzmann distribution plotted for different values of the parameter $\lambda = 0.7, 1.4, 2.1, 2.8$, and 3.5.

7.2 Performance of Numerical Estimators and Closed-Form Solution

This subsection employs an iterative approach using the R programming language. The "optimise" function, which integrates the golden section search with successive parabolic interpolation, is utilized alongside the "maxLik" function version (1.5-2.1), which applies the BFGS optimization algorithm. These methods are compared with the closed-form expression (40).

Random numbers for the new mixture Maxwell–Boltzmann distribution were generated using the acceptance–rejection method via the Maxwell–Boltzmann distribution, implemented in the shotGroups library. The R package shotGroups (version 0.8.2) provides functions for analyzing data based on the Maxwell–Boltzmann distribution, including the PDF, CDF, and random number generation. The random number generation procedure of the proposed distribution is presented in Algorithm 1. The experimental was conducted with sample sizes $n = 5, 10, 20, 30, 50, 100, 500, 1,000, 10,000$ and parameter values $\lambda = 0.7, 1.4, 2.1, 2.8$. Each scenario was repeated 1,000 times to ensure statistical reliability.

As observed in Table 1, Table 2, Table 3, together with Figure 4, Figure 5 and Figure 6 presented in the Appendix, the mean of the parameter estimates across all methods closely approximates the true parameter values for both small and large sample sizes. However, for $n < 5$, the BFGS method exhibits higher bias and MSE compared to other methods. Furthermore, the MLE obtained via the closed-form expression have a lower simulated variance than the other methods.

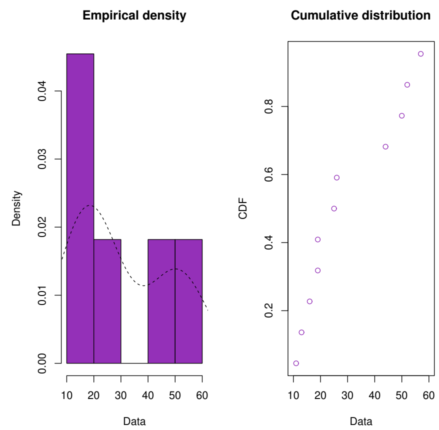

Fig. 3: Histogram and empirical density (left panel), and Cumulative distribution of the observed survival time of the patient (in days) (right panel).

Algorithm 1: Random Number Generation for New Mixture MBD

Input: Number of samples n, parameter λ
Output: Matrix X with n samples
Define the New Mixture MBD PDF:
$f(X, \lambda) = \frac{1}{\lambda+1} g_{MB}(X, \lambda) + \frac{\lambda}{\lambda+1} g_{LBMB}(X, \lambda)$;
Define the ratio function:
$f_M(X, \lambda) = \frac{f(X, \lambda)}{h(X, \lambda)}$, where h is the PDF of Maxwell–Boltzmann distribution;
Compute $M = \max_{X \in [0,10]} f_M(X, \lambda)$;
Initialize empty list $X = []$;
while *length(X)* $< n$ **do**
  Let $N = \lceil n \cdot M \rceil$;
  Generate $y_1, \ldots, y_N$ from Maxwell–Boltzmann(λ);
  Generate $u_1, \ldots, u_N$ from Uniform$(0, M)$;
  foreach *pair* (y_i, u_i) *in* (y, u) **do**
    if $u_i < f_M(y_i, \lambda)$ **then**
      Append y_i to X;
    end
  end
end
Trim X to the first n samples;
Convert X to a matrix with one column;
return X;

8 Applications to Data Set

The heart attack dataset is an real data of a survival time dataset, as described in [20]. All patients in the dataset experienced a heart attack at some point and were deceased by the end of the survival period. The dataset includes the following survival times: 11, 19, 16, 57, 26, 13, 50, 19, 25, 52, and 44. As shown in Figure 3, this dataset exhibits a right-skewed distribution. Therefore, three right-skewed distributions were selected for a goodness-of-fit comparison: the Maxwell–Boltzmann distribution, the length-biased Maxwell–Boltzmann distribution, and the new mixture Maxwell–Boltzmann distribution.

The parameters of the new mixture Maxwell–Boltzmann distribution were estimated using the closed-form expression (40), while the other distributions were fitted using Maximum Likelihood Estimation (MLE) through a combination of the golden section search and successive parabolic interpolation methods from the optimise function. The Akaike Information Criterion (AIC) was used as the evaluation metric, with the best model being determined by the lowest AIC value. According to the results in Table 4, the new mixture Maxwell–Boltzmann distribution yielded the lowest AIC, indicating that it provides the best fit among the candidate distributions.

Table 1. This table shows the mean estimated values, bias, MSE, and the variance of the MLE.

n	λ	$\hat{\lambda}$	Bias($\hat{\lambda}$)	MSE($\hat{\lambda}$)	var($\hat{\lambda}$)
5	0.7	0.740	0.040	0.009	0.008
	1.4	1.195	-0.205	0.060	0.019
	2.1	1.574	-0.526	0.308	0.031
	2.8	1.913	-0.887	0.832	0.045
10	0.7	0.744	0.044	0.006	0.004
	1.4	1.194	-0.206	0.051	0.009
	2.1	1.577	-0.523	0.289	0.015
	2.8	1.929	-0.871	0.781	0.022
20	0.7	0.746	0.046	0.004	0.002
	1.4	1.201	-0.200	0.045	0.005
	2.1	1.590	-0.510	0.268	0.008
	2.8	1.932	-0.868	0.765	0.012
30	0.7	0.744	0.044	0.003	0.001
	1.4	1.202	-0.198	0.042	0.003
	2.1	1.590	-0.510	0.266	0.005
	2.8	1.937	-0.864	0.753	0.007
50	0.7	0.747	0.047	0.003	0.001
	1.4	1.203	-0.197	0.041	0.002
	2.1	1.592	-0.508	0.262	0.003
	2.8	1.938	-0.862	0.747	0.004
100	0.7	0.748	0.048	0.003	0.000
	1.4	1.206	-0.194	0.039	0.001
	2.1	1.593	-0.507	0.258	0.002
	2.8	1.939	-0.861	0.744	0.002
500	0.7	0.747	0.047	0.002	0.000
	1.4	1.207	-0.193	0.038	0.000
	2.1	1.594	-0.506	0.257	0.000
	2.8	1.940	-0.860	0.740	0.001
1,000	0.7	0.748	0.048	0.002	0.000
	1.4	1.206	-0.194	0.038	0.000
	2.1	1.594	-0.506	0.257	0.000
	2.8	1.940	-0.860	0.740	0.000
10,000	0.7	0.748	0.048	0.002	0.000
	1.4	1.206	-0.194	0.038	0.000
	2.1	1.595	-0.505	0.256	0.000
	2.8	1.940	-0.860	0.739	0.000

9 Conclusion

This article introduced a new mixture extension of the Maxwell–Boltzmann distribution (MBD) and demonstrated its effectiveness in fitting to the right-skewed data. Using the maximum likelihood estimate (MLE), we estimate its parameters and analyze key statistical properties. In the real-world application part, the AIC results show that the new mixture MBD gives the best fit. This confirms

Table 2. The mean estimates, bias, MSE, and simulated variance of the MLE via BFGS from the *maxLik* function.

n	λ	$\tilde{\lambda}$	Bias($\tilde{\lambda}$)	MSE($\tilde{\lambda}$)	var($\tilde{\lambda}$)
5	0.7	0.083	-0.617	1.102	0.014
	1.4	1.303	-0.097	0.248	0.052
	2.1	1.225	-0.875	3.289	0.113
	2.8	18.694	15.894	3,521.796	0.196
10	0.7	0.696	-0.004	0.007	0.007
	1.4	1.383	-0.018	0.026	0.026
	2.1	2.070	-0.030	0.055	0.054
	2.8	3.279	0.479	251.742	0.098
20	0.7	0.699	-0.001	0.003	0.003
	1.4	1.392	-0.008	0.014	0.014
	2.1	2.094	-0.006	0.029	0.029
	2.8	2.839	0.039	3.342	0.051
30	0.7	0.696	-0.004	0.002	0.002
	1.4	1.394	-0.006	0.009	0.009
	2.1	2.092	-0.008	0.019	0.019
	2.8	2.790	-0.010	0.032	0.032
50	0.7	0.700	0.000	0.001	0.001
	1.4	1.394	-0.006	0.005	0.005
	2.1	2.096	-0.004	0.012	0.012
	2.8	2.793	-0.007	0.020	0.019
100	0.7	0.700	0.000	0.001	0.001
	1.4	1.400	0.000	0.003	0.003
	2.1	2.098	-0.002	0.006	0.006
	2.8	2.794	-0.006	0.011	0.011
500	0.7	0.699	-0.001	0.000	0.000
	1.4	1.400	0.000	0.001	0.001
	2.1	2.098	-0.002	0.001	0.001
	2.8	2.796	-0.004	0.002	0.002
1,000	0.7	0.700	0.000	0.000	0.000
	1.4	1.399	-0.001	0.000	0.000
	2.1	2.098	-0.002	0.001	0.001
	2.8	2.796	-0.004	0.001	0.001
10,000	0.7	0.700	0.000	0.000	0.000
	1.4	1.400	0.000	0.000	0.000
	2.1	2.100	0.000	0.000	0.000
	2.8	2.797	-0.004	0.000	0.000

the new mixture MBD suitability for real data, particularly in survival analysis and right-skewed data modeling. These findings highlight the distribution's potential for broader applications in statistical modeling and data analysis. In addition, the plots of the PDF and hazard function show the suitability for data exhibiting right-skewness and time-dependent hazard characteristics. Therefore, this study exposes the appropriateness of the new mixture MBD for analyzing right-skewed time-to-event data in survival analysis.

Future research could explore alternative

Table 3. The mean estimates, bias, MSE, and simulated variance of the MLE via the combination of golden section search and successive parabolic interpolation method from the optimise function.

n	λ	$\acute{\lambda}$	Bias($\acute{\lambda}$)	MSE($\acute{\lambda}$)	var($\acute{\lambda}$)
5	0.7	0.693	-0.007	0.014	0.014
	1.4	1.388	-0.013	0.053	0.052
	2.1	2.070	-0.030	0.114	0.113
	2.8	2.751	-0.050	0.198	0.196
10	0.7	0.696	-0.004	0.007	0.007
	1.4	1.383	-0.018	0.026	0.026
	2.1	2.070	-0.030	0.055	0.054
	2.8	2.778	-0.022	0.098	0.098
20	0.7	0.699	-0.001	0.003	0.003
	1.4	1.392	-0.008	0.014	0.014
	2.1	2.094	-0.006	0.029	0.029
	2.8	2.782	-0.018	0.052	0.051
30	0.7	0.696	-0.004	0.002	0.002
	1.4	1.394	-0.006	0.009	0.009
	2.1	2.092	-0.008	0.019	0.019
	2.8	2.790	-0.010	0.032	0.032
50	0.7	0.700	0.000	0.001	0.001
	1.4	1.394	-0.006	0.005	0.005
	2.1	2.096	-0.004	0.012	0.012
	2.8	2.793	-0.007	0.020	0.019
100	0.7	0.700	0.000	0.001	0.001
	1.4	1.400	0.000	0.003	0.003
	2.1	2.098	-0.002	0.006	0.006
	2.8	2.794	-0.006	0.011	0.011
500	0.7	0.699	-0.001	0.000	0.000
	1.4	1.400	0.000	0.001	0.001
	2.1	2.098	-0.002	0.001	0.001
	2.8	2.796	-0.004	0.002	0.002
1,000	0.7	0.700	0.000	0.000	0.000
	1.4	1.399	-0.001	0.000	0.000
	2.1	2.098	-0.002	0.001	0.001
	2.8	2.796	-0.004	0.001	0.001
10,000	0.7	0.700	0.000	0.000	0.000
	1.4	1.400	0.000	0.000	0.000
	2.1	2.100	0.000	0.000	0.000
	2.8	2.797	-0.004	0.000	0.000

Table 4. AIC for MBD, length-biased of MBD, and new mixture MBD

Fitted Distribution	λ	AIC
Maxwell–Boltzmann	5.99	5.33
Length-biased of Maxwell–Boltzmann	5.99	4.70
New Mixture Generalized Maxwell–Boltzmann	6.65	4.52

estimation techniques and further extend the model's applicability to diverse datasets. Additionally, the proposed distribution can be applied to larger real-world datasets, as demonstrated in this simulation study, to evaluate its performance across different sample sizes.

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During the preparation of this work, the authors used ChatGPT in order to improve English writing. After using this tool/service, the authors reviewed and edited the content as needed and takes full responsibility for the content of the publication.

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Conflicts of Interest
The authors have no conflicts of interest to declare.

Appendix

Figure 4, Figure 5 and Figure 6 described in subsection 7.2 are presented in this section.

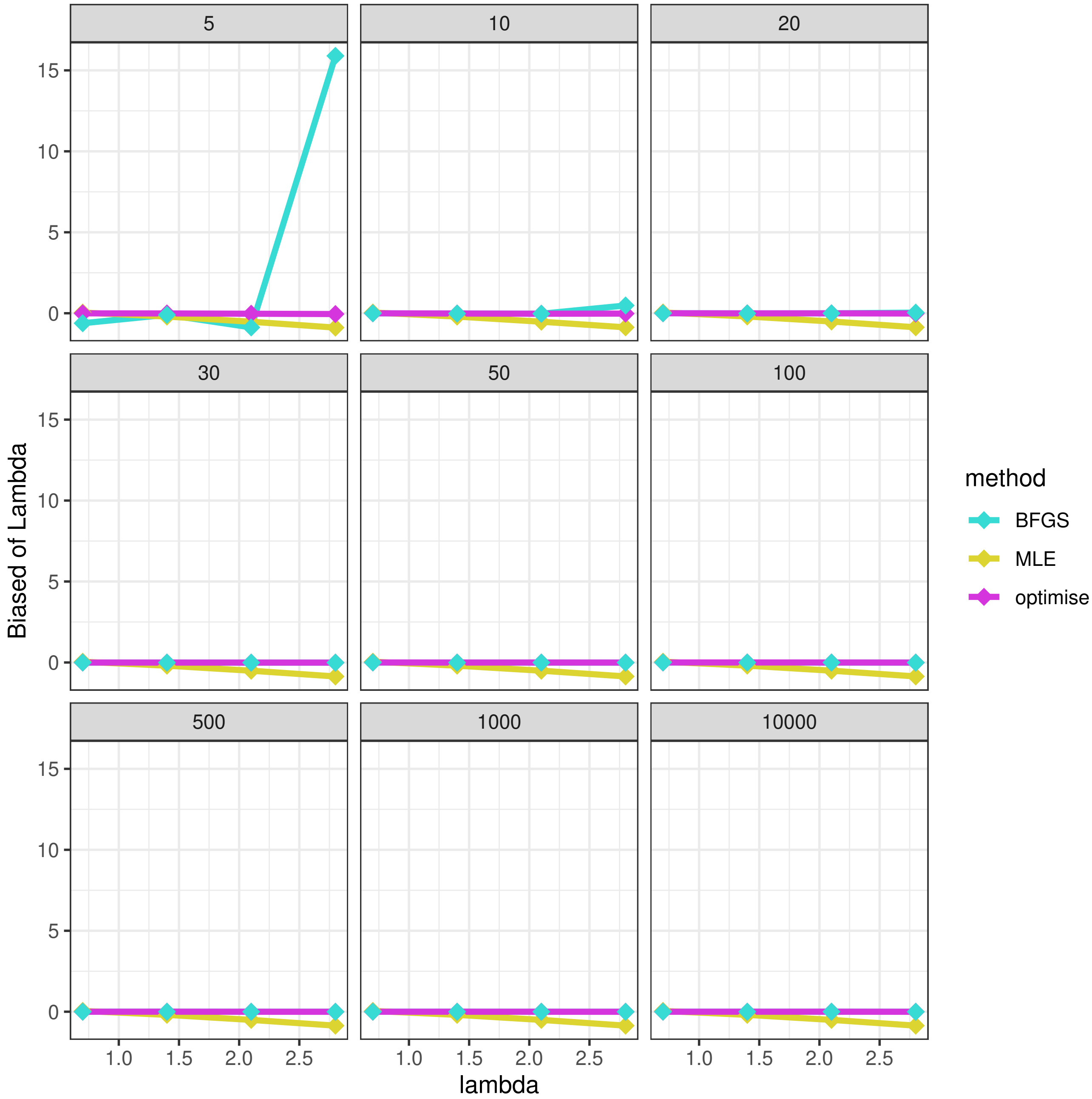

Fig. 4: Line chart of bias of λ for $n = 5, 10, 20, 30, 50, 100, 500, 1,000, 10,000$. The x-axis denotes the values of λ varying from $\lambda = 0.7, 1.4, 2.1, 2.8$ and the y-axis denotes the bias of λ.

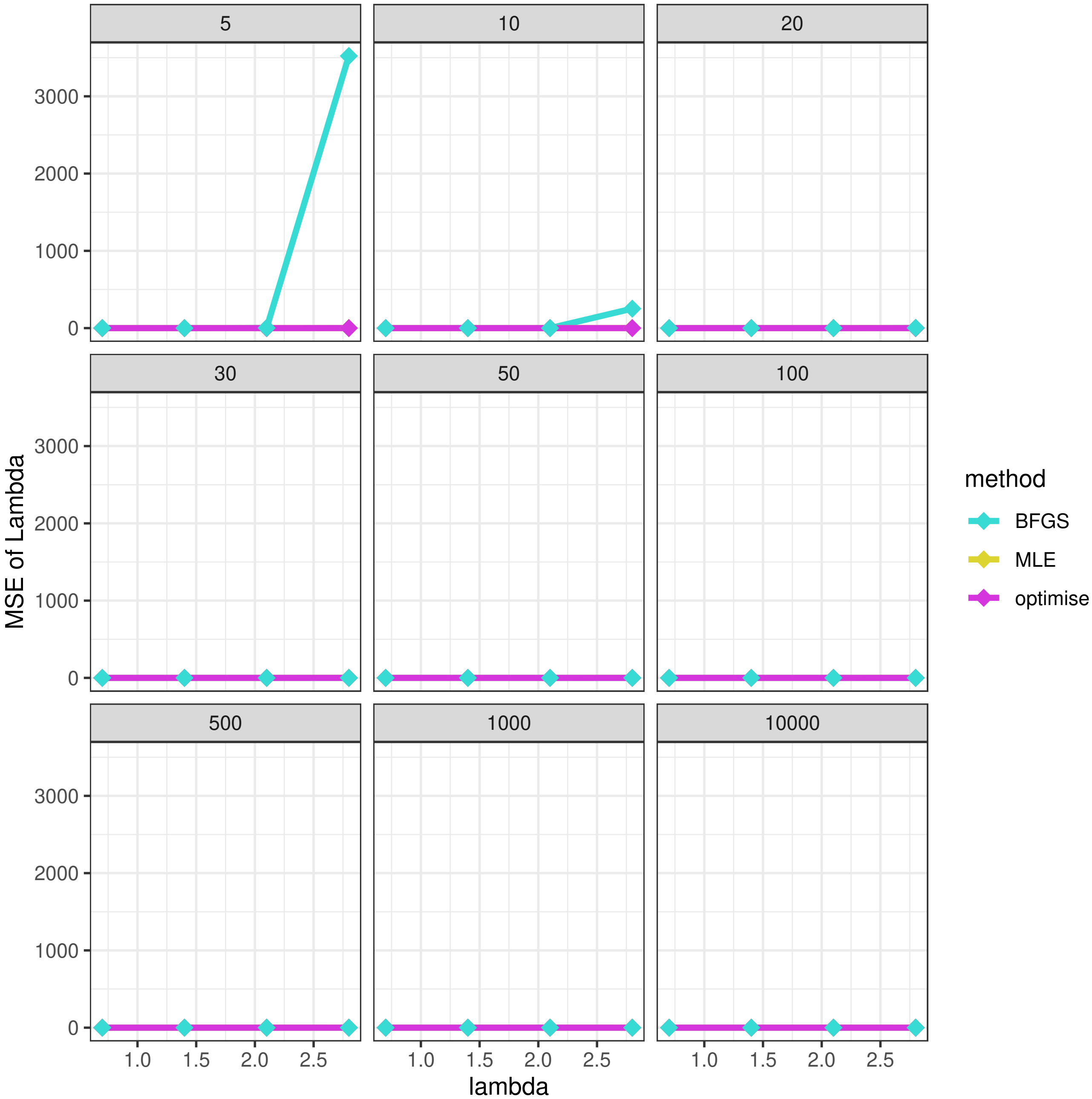

Fig. 5: Line chart of MSE of λ for $n = 5, 10, 20, 30, 50, 100, 500, 1,000, 10,000$. The x-axis denotes the values of λ varying from $\lambda = 0.7, 1.4, 2.1, 2.8$ and the y-axis denotes the bias of λ.

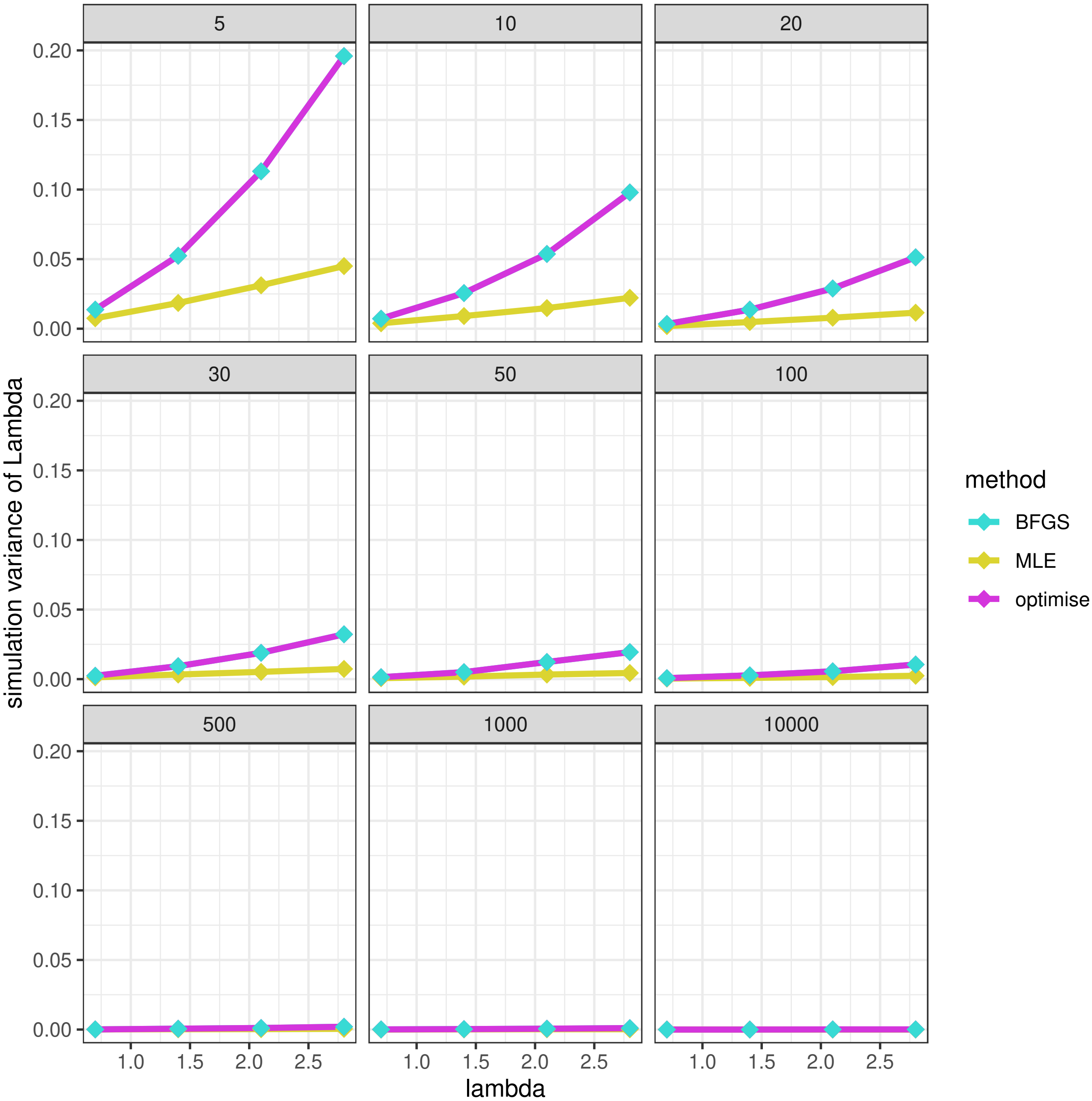

Fig. 6: Line chart of simulation variance of λ for $n = 5, 10, 20, 30, 50, 100, 500, 1,000, 10,000$. The x-axis denotes the values of λ varying from $\lambda = 0.7, 1.4, 2.1, 2.8$ and the y-axis denotes the bias of λ.