New Fixed Point Results for Gamma Interpolative Contractions through Gamma Distance Mappings

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Abstract: - In this manuscript, we introduced several noteworthy contractions. These contractions emerge from the integration of two concepts: γ-distance and interpolative contraction. To demonstrate the importance of our findings, we present an application along with illustrative examples.

Key-Words: - fixed point, gamma distance, γ-interpolative contractions, γ-M- contractions.

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1 Introduction

Fixed point theory is of significant relevance in mathematics, as it guarantees the existence of solutions across a variety of problems in multiple fields. A fixed point of a function is characterized as a point that the function maps onto itself. Theorems associated with fixed points, such as Banach's fixed-point theorem, [1], are crucial as they establish the existence of these points under certain conditions. Specifically, Banach's theorem asserts that any contraction mapping that operates within a complete metric space must have a unique fixed point. Introduced by Stefan Banach in 1922, this theorem has inspired a multitude of mathematicians to explore various extensions and generalizations in different mathematical domains, as indicated by the references in [2], [3], [4], [5], [6], [7], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24],[25], [26], [27], [28].

It is important to note that fixed point theory can be applied to fractional differential equations, as evidenced by the studies referenced in [29], [30], and [31].

2 Preliminary

In the work of [8], the authors present the notion of Gamma distance and demonstrate various fixed point theorems utilizing this concept.

Definition 2.1. *[8] Let $\mathcal{D}$ be a metric on $\mathcal{L}$. A function $\gamma : [0,\infty) \times \mathcal{L} \times \mathcal{L} \to [0,\infty)$ is said to be γ-distance over $(\mathcal{L},\mathcal{D})$ if γ satisfying:*

(γ_1) *For all $w \geq 0$, $\gamma(w, v_1, v_2) \geq w\mathcal{D}(v_1, v_2)$,*

(γ_2) *for each sequences $(l_n), (v_n)$ in $\mathcal{L}$ and (w_n) in $(0,\infty)$, we have $\mathcal{D}(l_n, v_n) \to \xi > 0$ and $w_n \to w > 0$ imply $\liminf_{n\to\infty} \gamma(w_n, l_n, v_n) > \xi w$.*

The subsequent examples are attributed to the work of [8].

Example 2.2. *[8] Let $\mathcal{L} = \mathbb{R}$ and let $\mathcal{D} : \mathcal{L} \times \mathcal{L} \to [0,+\infty)$ be the usual metric on $\mathcal{L}$; i.e. $\mathcal{D}(l_1, l_2) = |l_1 - l_2|$. Then $\gamma : [0,+\infty) \times \mathcal{L} \times \mathcal{L} \to [0,+\infty)$ which is defined by*

$$\gamma(w, l_1, l_2) = \alpha\, w\, (\, |l_1| + |l_2|\,),$$

where $\alpha > 1$ is γ-distance on $(\mathcal{L},\mathcal{D})$.

In subsequent examples, the functions $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5$ are established as mappings from the domain $[0,+\infty) \times \mathcal{L} \times \mathcal{L}$ to the co-domain $[0,+\infty)$.

Example 2.3. *[8] In $(\mathcal{L},\mathcal{D})$, let $\phi:[0,\infty)\to[0,\infty)$ be a continuous function with $\phi(a)>0$ for $a>0$. Then, $\gamma_1,\gamma_2,\gamma_3,\gamma_4$ $\gamma_5$and γ_6 are γ-distance over $(\mathcal{L},\mathcal{D})$.*

1. $\gamma_1(w,l_1,l_2)=w\mathcal{D}(l_1,l_2)+\phi(\mathcal{D}(l_1,l_2)$,
2. $\gamma_2(w,l_1,l_2)=w\,\mathcal{D}(l_1,l_2)+\phi(w)$,
3. $\gamma_3(w,l_1,l_2)=e^w\,\mathcal{D}(l_1,l_2)$,
4. $\gamma_4(w,l_1,l_2)=\dfrac{(w+\varepsilon)^2+[\mathcal{D}(l_1,l_2)]^2}{2},\varepsilon>0$,
5. $\gamma_5(w,l_1,l_2)=w\,[1+\mathcal{D}(l_1,l_2)]^w\,\mathcal{D}(l_1,l_2)$,
6. $\gamma_6(w,l_1,l_2)=\dfrac{w^2+1}{2}\,\mathcal{D}(l_1,l_2)$,

The subsequent part of this study requires the inclusion of the subsequent result.

Lemma 2.4. *Let $(\mathcal{L},\mathcal{D})$ be a metric space and (l_n) is a sequence where*

$$\lim_{n\to\infty}\mathcal{D}(l_n,l_{n+1})=0. \qquad (1)$$

If (l_n) is not Cauchy, then an $\epsilon>0$ exists, and two sub-sequences (l_{n_k}) and (l_{m_k}) of (l_n) such that $\lim_{k\to\infty}\mathcal{D}(l_{n_k},l_{m_k})$ and $\lim_{k\to\infty}\mathcal{D}(l_{n_k-1},l_{m_k-1})$ equal ϵ.

Proof. Since (l_n) is not Cauchy. Therefore, there is $\epsilon>0$ and two sub-sequences (l_{n_k}) and (l_{m_k}) of (l_n) such that (m_k) is selected as the minimum index such that

$$\mathcal{D}(l_{n_k},l_{m_k})\geq\epsilon,\ m_k>n_k>k. \qquad (2)$$

So,

$$\mathcal{D}(l_{n_k},l_{m_k-1})<\epsilon. \qquad (3)$$

Using the triangle inequality and Equations (2),(3), we get

$$\begin{aligned}\epsilon\leq\mathcal{D}(l_{n_k},l_{m_k}) &\leq\mathcal{D}(l_{n_k-1},l_{m_k})+\mathcal{D}(l_{m_k},l_{m_k-1})\\ &<\epsilon+\mathcal{D}(l_{m_k},l_{m_k-1}).\end{aligned}$$

By considering condition 1 and evaluating the limit as $k\to\infty$, we get

$$\lim_{k\to\infty}\mathcal{D}(l_{n_k},l_{m_k})=\epsilon. \qquad (4)$$

Again, using the triangle inequality, we get $\mathcal{D}(l_{n_k-1},l_{m_k-1})-\mathcal{D}(l_{n_k},l_{m_k})$ $\leq\mathcal{D}(l_{n_k-1},l_{n_k})+\mathcal{D}(l_{m_k},l_{m_k-1})$, and $\mathcal{D}(l_{n_k},l_{m_k})-\mathcal{D}(l_{n_k-1},l_{m_k-1})$ $\leq\mathcal{D}(l_{n_k},l_{n_k-1})+\mathcal{D}(l_{m_k-1},l_{m_k})$. Therefore, $|\mathcal{D}(l_{n_k-1},l_{m_k-1})-\mathcal{D}(l_{n_k},l_{m_k})|$ $\leq\mathcal{D}(l_{n_k-1},l_{n_k})+\mathcal{D}(l_{m_k},l_{m_k-1})$.

So, Taking the limit as $k\to\infty$ and considering condition 1 and Equation (4), we get

$$\lim_{k\to\infty}\mathcal{D}(l_{n_k-1},l_{m_k-1})=\epsilon. \qquad (5)$$

□

Henceforth, we mean by $(\mathcal{L},\mathcal{D})$ a metric space and γ a γ-distance over $(\mathcal{L},\mathcal{D})$.

3 Fixed Point Results for Weak γ-interpolative Contractions

Definition 3.1. *Suppose there is γ over $(\mathcal{L},\mathcal{D})$. A self mapping $f:\mathcal{L}\to\mathcal{L}$ is said to be weak γ-interpolative contractions if there are $a\in[1,2)$ and such that*

$$\gamma(\mathcal{D}(fl_1,fl_2),fl_1,fl_2)\leq[\mathcal{D}(l_1,fl_1)]^a[\mathcal{D}(l_2,fl_2)]^{2-a}. \qquad (6)$$

Remark 3.2. *If $f:\mathcal{L}\to\mathcal{L}$ is a weak γ-interpolative contractions, then for each $l_1,l_2\in\mathcal{L}$, we have*

$$[\mathcal{D}(fl_1,fl_2)]^2\leq[\mathcal{D}(l_1,fl_1)]^a[\mathcal{D}(l_2,fl_2)]^{2-a}.$$

Lemma 3.3. *Suppose $f:\mathcal{L}\to\mathcal{L}$ is a weak γ-interpolative contractions. If $l_1,l_2\in\mathcal{F}_f$, then $l_1=l_2$.*

Proof. The proof follows from Remark 3.2. □

Lemma 3.4. *Suppose f is weak γ-interpolative contractions, and $l_0\in\mathcal{L}$. Then for the Picard sequence (l_n) generated by f at l_0, if $l_n\neq l_{n+1}$ for each $n\in\mathbb{N}$, then*

$$\lim_{n\to\infty}\mathcal{D}(l_n,l_{n+1})=0.$$

Proof. Since $l_n\neq l_{n+1}$, by Condition 6

$$\gamma(\mathcal{D}(l_n,l_{n+1}),l_n,l_{n+1})\leq[\mathcal{D}(l_{n-1},l_n)]^a[\mathcal{D}(l_n,l_{n+1})]^{2-a}.$$

From Remark 3.2, we have

$$[\mathcal{D}(l_n,l_{n+1})]^a\leq[\mathcal{D}(l_{n-1},l_n)]^a.$$

Hence $(\mathcal{D}(l_n,l_{n+1}):\ n\in\mathbb{N})$ is a non-increasing sequence in $(0,\infty)$. So, there is $\eta\geq0$ such that $\lim_{n\to\infty}\mathcal{D}(L_n,l_{n+1})=\eta$. Suppose to the contrary; that is $\eta>0$. Let $w_n=\mathcal{D}(v_{n-1},v_n)$. Then, $t_n\to\eta$. Therefore, by γ_2 of Definition 2.1, we have

$$\eta^2<\liminf_{n\to\infty}\gamma(\mathcal{D}(v_{n-1},v_n),v_n,v_{n+1})\leq\eta^2,$$

which is a contradiction. Hence the result. □

Theorem 3.5. *Suppose that $(\mathcal{L},\mathcal{D})$ is complete and there is γ-distance γ on $(\mathcal{L},\mathcal{D})$. Assume that $f:\mathcal{L}\to\mathcal{L}$ is weak γ-interpolative contractions. Then $\mathcal{F}_f$ consists of only one element.*

Proof. Let $l_0 \in \mathcal{L}$ be an arbitrary element, and consider the Picard sequence (l_n) generated by the function f at the point l_0. If there exists an integer $k \in \mathbb{N}$ such that $l_k = l_{k+1}$, it follows that $l_k \in \mathcal{F}_f$. Therefore, we will assume that for every $n \in \mathbb{N}$, $l_n \neq l_{n+1}$. We assert that the sequence (l_n) is a Cauchy sequence in the metric space $(\mathcal{L}, \mathcal{D})$. To argue otherwise, we assume that (l_n) is not Cauchy. Consequently, there exists a positive number $\epsilon > 0$ and two subsequences (l_{n_k}) and (l_{m_k}) of (l_n), where (m_k) is selected as the smallest index for which

$$\mathcal{D}(l_{n_k}, l_{m_k}) \geq \epsilon, \ m_k > n_k > k. \tag{7}$$

This implies that

$$\mathcal{D}(l_{n_k}, l_{m_k-1}) < \epsilon. \tag{8}$$

Using the triangle inequality and Equations (7),(8), we get

$$\begin{aligned} \epsilon \leq \mathcal{D}(l_{n_k}, l_{m_k}) &\leq \mathcal{D}(l_{n_k-1}, l_{m_k}) + \mathcal{D}(l_{m_k}, l_{m_k-1}) \\ &< \epsilon + \mathcal{D}(l_{m_k}, l_{m_k-1}). \end{aligned}$$

Taking the limit as $k \to \infty$ and considering Lemma 3.4, we get

$$\lim_{k\to\infty} \mathcal{D}(l_{n_k}, l_{m_k}) = \epsilon. \tag{9}$$

Again, using the triangle inequality, we get
$\mathcal{D}(l_{n_k-1}, l_{m_k-1}) - \mathcal{D}(l_{n_k}, l_{m_k})$
$\leq \mathcal{D}(l_{n_k-1}, l_{n_k}) + \mathcal{D}(l_{m_k}, l_{m_k-1})$,
and
$\mathcal{D}(l_{n_k}, l_{m_k}) - \mathcal{D}(l_{n_k-1}, l_{m_k-1})$
$\leq \mathcal{D}(l_{n_k}, l_{n_k-1}) + \mathcal{D}(l_{m_k-1}, l_{m_k})$.
Therefore, $|\mathcal{D}(l_{n_k-1}, l_{m_k-1}) - \mathcal{D}(l_{n_k}, l_{m_k})|$
$\leq \mathcal{D}(l_{n_k-1}, l_{n_k}) + \mathcal{D}(l_{m_k}, l_{m_k-1})$.
So, Taking the limit as $k \to \infty$ and considering Lemma 3.4 and Equation (9), we get

$$\lim_{k\to\infty} \mathcal{D}(l_{n_k-1}, l_{m_k-1}) = \epsilon. \tag{10}$$

Now, set $w_k = \mathcal{D}(l_{n_k-1}, l_{m_k-1})$, $a_k = l_{n_k}$ and $b_k = l_{m_k}$. Then, $\lim_{k\to\infty} w_k = \lim_{k\to\infty} d(a_k, b_k) = \epsilon > 0$. So, by Lemma 3.4, (γ_2) and condition 6, we get

$$\epsilon^2 < \liminf_{k\to\infty} \gamma(w_k, a_k, b_k) \leq 0,$$

a contradiction. Hence, (l_n) is Cauchy, so there is $l \in \mathcal{L}$ such that (l_n) converges to l. Now, by Remark 3.2, we have

$$[\mathcal{D}(fl, l_{n+1})]^2 \leq [\mathcal{D}(l, fl)]^a [\mathcal{D}(l_n, l_{n+1})]^{2-a}.$$

So, if we pass $n \to +\infty$, we get $l_n \to fl$, and so, then by uniqueness of the limit, we get $l = fl$.

The uniqueness follows from Lemma 3.3. □

By the same argument in Theorem 3.5, we get the following result.

Theorem 3.6. *Suppose that $(\mathcal{L}, \mathcal{D})$ is complete, $f : \mathcal{L} \to \mathcal{L}$, and there is γ-distance γ on $(\mathcal{L}, \mathcal{D})$, such that f satisfies the following condition for all $l_1, l_2, \in \mathcal{L}$.*

$$\gamma(\mathcal{D}(l_1, l_2,), fl_1, fl_2) \leq [\mathcal{D}(l_1, fl_1)]^a [\mathcal{D}(l_2, fl_2)]^{2-a}, \tag{11}$$

where $a \in [1, 2)$. Then $\mathcal{F}_f$ consists of only one element.

Example 3.7. *Suppose $\mathcal{L} = \{0, 1, ..., 10\}$, define mapping $f : \mathcal{L} \to \mathcal{L}$ by :*

$$fl = \begin{cases} 0, & l \in \{0, 1, 2\}; \\ 1, & l \in \{3, 4, \ldots, 7\}; \\ 2, & l \in \{8, 9, 10\}. \end{cases}$$

Then $\mathcal{F}_f$ consists of only one element.
To prove this, we need to show that $\forall\ l_1, l_2 \in \mathcal{L}$, we have

$$\gamma(\mathcal{D}(fl_1, fl_2), fl_1, fl_2) \leq [\mathcal{D}(l_1, fl_1)]^a [\mathcal{D}(l_2, fl_2)]^{2-a}.$$

Now, let
$\gamma(\mathcal{D}(fl_1, fl_2), fl_1, fl_2)$
$= \left(\dfrac{\mathcal{D}^2(fl_1, fl_2) + 1}{2}\right) \mathcal{D}(fl_1, fl_2)$.
Where $\mathcal{D}(l_1, l_2) = |l_1 - l_2|$ with $a = 1.05$, so we have: $\left(\dfrac{\mathcal{D}^2(fl_1, fl_2) + 1}{2}\right) \mathcal{D}(fl_1, fl_2)$
$\leq \mathcal{D}^{1.05}(l_1, fl_1)\mathcal{D}^{0.95}(l_2, fl_2)$.
For all $l \in \mathcal{L}$ with $l \neq fl$ we have the following Cases:
Case (1): If $l_1, l_2 \in \{0, 1, 2\}$ or $l_1, l_2 \in \{3, 4, \ldots, 7\}$ or $l_1, l_2 \in \{8, 9, 10\}$, then

$$\gamma(\mathcal{D}(fl_1, fl_2), fl_1, fl_2) = 0 \leq [\mathcal{D}(l_1, fl_1)]^{1.05} [\mathcal{D}(l_2, fl_2)]^{0.95}.$$

Case (2): If $l_1 \in \{0, 1, 2\}$ & $l_2 \in \{3, 4, \ldots, 7\}$, then

$$\begin{aligned} \gamma(\mathcal{D}(fl_1, fl_2), fl_1, fl_2) &= \frac{1^2 + 1}{2}(1) \\ &\leq [1]^{1.05}[2]^{0.95} \\ &\leq [\mathcal{D}(l_1, fl_1)]^{1.05}[\mathcal{D}(l_2, fl_2)]^{0.95}. \end{aligned}$$

Case (3): If $l_1 \in \{0, 1, 2\}$ & $l_2 \in \{8, 9, 10\}$, then

$$\begin{aligned} \gamma(\mathcal{D}(fl_1, fl_2), fl_1, fl_2) &= \frac{2^2 + 1}{2}(2) \\ &\leq [1]^{1.05}[6]^{0.95} \\ &\leq [\mathcal{D}(l_1, fl_1)]^{1.05}[\mathcal{D}(l_2, fl_2)]^{0.95}. \end{aligned}$$

Case (4): If $l_1 \in \{3, 4, \ldots, 7\}$ & $l_2 \in \{8, 9, 10\}$, then

$$\begin{aligned}\gamma(\mathcal{D}(fl_1, fl_2), fl_1, fl_2) &= \frac{1^2+1}{1}(2)\\ &\le [2]^{1.05}[6]^{0.95}\\ &\le [\mathcal{D}(l_1, fl_1)]^{1.05}[\mathcal{D}(l_2, fl_2)]^{0.95}.\end{aligned}$$

The remaining cases follow the same pattern. Hence, all conditions of γ-interpolative contractions are met by f. Theorem 3.5 confirms that $\mathcal{F}_f$ cosists of only one element.

4 Fixed Point Results for γ-M-contractions

Definition 4.1. *Suppose there is γ over $(\mathcal{L}, \mathcal{D})$. A self mapping $f : \mathcal{L} \to \mathcal{L}$ is said to be γ-M-contraction if there are $a \in [1, 2)$ and such that*

$$\gamma(\mathcal{D}(fl_1, fl_2), fl_1, fl_2) \le \max\{(\mathcal{D}(l_1, l_2)^2,\ [\mathcal{D}(l_1, fl_1)]^a[\mathcal{D}(l_2, fl_2)]^{2-a}\}. \quad (12)$$

Remark 4.2. *If $f : \mathcal{L} \to \mathcal{L}$ is γ-M-contraction, then for each $l_1, l_2 \in \mathcal{L}$, we have* $[\mathcal{D}(fl_1, fl_2)]^2 \le \max\{(\mathcal{D}(l_1, l_2)^2,\ [\mathcal{D}(l_1, fl_1)]^a[\mathcal{D}(l_2, fl_2)]^{2-a}\}$.

Lemma 4.3. *Suppose $f : \mathcal{L} \to \mathcal{L}$ is γ-M-contraction. If $l_1, l_2 \in \mathcal{F}_f$, then $l_1 = l_2$.*

Proof. The proof follows from Remark 4.2. □

Lemma 4.4. *Suppose f is γ-M-contraction, and $l_0 \in \mathcal{L}$. Then for the Picard sequence (l_n) generated by f at l_0, if $l_n \neq l_{n+1}$ for each $n \in \mathbb{N}$, then*

$$\lim_{n\to\infty} \mathcal{D}(l_n, l_{n+1}) = 0.$$

Proof. Since $l_n \neq l_{n+1}$, by Condition 12
$\gamma(\mathcal{D}(l_n, l_{n+1}), l_n, l_{n+1}) \le \max\{[\mathcal{D}(l_{n-1}, l_n)]^2, [\mathcal{D}(l_{n-1}, l_n)]^a[\mathcal{D}(l_n, l_{n+1})]^{2-a}\}$.
From Remark 4.2, we have
$[\mathcal{D}(l_n, l_{n+1})]^2 \le \max\left\{[\mathcal{D}(l_{n-1}, l_n)]^2,\ [\mathcal{D}(l_{n-1}, l_n)]^a[\mathcal{D}(l_n, l_{n+1})]^{2-a}\right\}$.
Hence, either $[\mathcal{D}(l_n, l_{n+1})]^2 \le [\mathcal{D}(l_{n-1}, l_n)]^2$ or $[\mathcal{D}(l_n, l_{n+1})]^a \le [\mathcal{D}(l_{n-1}, l_n)]^a$. Therefore in each case, we have $(\mathcal{D}(l_n, l_{n+1}) : n \in \mathbb{N})$ is a non-increasing sequence in $(0, \infty)$. So, there is $\eta \ge 0$ such that $\lim_{n\to\infty} \mathcal{D}(l_n, l_{n+1}) = \eta$. Suppose to the contrary; that is $\eta > 0$. Let $w_n = \mathcal{D}(l_{n-1}, l_n)$. Then, $w_n \to \eta$. Therefore, by γ_2 of Definition 2.1, we have

$$\eta^2 < \liminf_{n\to\infty} \gamma(\mathcal{D}(l_{n-1}, l_n), l_n, l_{n+1}) \le \eta^2,$$

which is a contradiction. Hence the result. □

Theorem 4.5. *Suppose that $(\mathcal{L}, \mathcal{D})$ is complete and there is γ-distance γ on $(\mathcal{L}, \mathcal{D})$. Assume that $f : \mathcal{L} \to \mathcal{L}$ is γ-M-contraction. Then $\mathcal{F}_f$ consists of only one element.*

Proof. Let $l_0 \in \mathcal{L}$ be an arbitrary element, and consider the Picard sequence (l_n) generated by the function f at the point l_0. If there exists an integer $k \in \mathbb{N}$ such that $l_k = l_{k+1}$, it follows that $l_k \in \mathcal{F}_f$. Therefore, we will assume that for every $n \in \mathbb{N}$, $l_n \neq l_{n+1}$. We assert that the sequence (l_n) is a Cauchy sequence in the metric space $(\mathcal{L}, \mathcal{D})$. To argue otherwise, we assume that (l_n) is not Cauchy. Consequently, there exists a positive number $\epsilon > 0$ and two subsequences (l_{n_k}) and (l_{m_k}) of (l_n), where (m_k) is selected as the smallest index for which

$$\mathcal{D}(l_{n_k}, l_{m_k}) \ge \epsilon,\ m_k > n_k > k. \quad (13)$$

This implies that

$$\mathcal{D}(l_{n_k}, l_{m_k-1}) < \epsilon. \quad (14)$$

Using the triangle inequality and Equations (13),(14), we get

$$\begin{aligned}\epsilon \le \mathcal{D}(l_{n_k}, l_{m_k}) &\le \mathcal{D}(l_{n_k-1}, l_{m_k}) + \mathcal{D}(l_{m_k}, l_{m_k-1})\\ &< \epsilon + \mathcal{D}(l_{m_k}, l_{m_k-1}).\end{aligned}$$

Taking the limit as $k \to \infty$ and considering Lemma 4.4, we get

$$\lim_{k\to\infty} \mathcal{D}(l_{n_k}, l_{m_k}) = \epsilon. \quad (15)$$

Again, using the triangle inequality, we get
$\mathcal{D}(l_{n_k-1}, l_{m_k-1}) - \mathcal{D}(l_{n_k}, l_{m_k}) \le \mathcal{D}(l_{n_k-1}, l_{n_k}) + \mathcal{D}(l_{m_k}, l_{m_k-1})$,
and
$\mathcal{D}(l_{n_k}, l_{m_k}) - \mathcal{D}(l_{n_k-1}, l_{m_k-1}) \le \mathcal{D}(l_{n_k}, l_{n_k-1}) + \mathcal{D}(l_{m_k-1}, l_{m_k})$.
Therefore,
$|\mathcal{D}(l_{n_k-1}, l_{m_k-1}) - \mathcal{D}(l_{n_k}, l_{m_k})| \le \mathcal{D}(l_{n_k-1}, l_{n_k}) + \mathcal{D}(l_{m_k}, l_{m_k-1})$.
So, by taking the limit as $k \to \infty$ and taking into account Lemma 4.4 and Equation (9), we get

$$\lim_{k\to\infty} \mathcal{D}(l_{n_k-1}, l_{m_k-1}) = \epsilon. \quad (16)$$

Now, set $w_k = \mathcal{D}(l_{n_k-1}, l_{m_k-1})$, $a_k = l_{n_k}$ and $b_k = l_{m_k}$. Then, $\lim_{k\to\infty} w_k = \lim_{k\to\infty} d(a_k, b_k) = \epsilon > 0$. So, by Lemma 3.4, (γ_2) and condition 6, we get

$$\epsilon^2 < \liminf_{k\to\infty} \gamma(w_k, a_k, b_k) \le \max\{\epsilon, 0\},$$

a contradiction. Hence, (l_n) is Cauchy, so there is $l \in \mathcal{L}$ such that (l_n) converges to l. Now, by Remark 4.2, we have

$[\mathcal{D}(fl, l_{n+1})]^2$
$\leq \max\{[\mathcal{D}(l, l_n)]^2, [\mathcal{D}(l, fl)]^a[\mathcal{D}(l_n, l_{n+1})]^{2-a}\}$.
So, if we pass $n \to +\infty$, we get $l_n \to fl$, and so, then by uniqueness of the limit, we get $l = fl$.

The uniqueness follows from Lemma 4.3. □

Example 4.6. *Assume that the function* $f : [0,1] \to [0,1]$ *be defined as:*

$$fl = 1 - \frac{l^m}{\Gamma + l^m}\ , where\ \Gamma^2 \geq m,\ m \geq 1.$$

Then $\mathcal{F}_f$ *consists of only one element on* $[0,1]$.
To show this, let $\mathcal{L} = [0,1]$ *and define the following mappings:*
$\mathcal{D} : \mathcal{L} \to \mathcal{L}$ *via* $\mathcal{D}(l_1, l_2) = |l_1 - l_2|$. *Then* $(\mathcal{L}, \mathcal{D})$ *is complete metric space. Suppose that* $\gamma(t, l_1, l_2) = \left(\frac{t^2+1}{2}\right)\mathcal{D}(l_1, l_2)$. *Then,*$\gamma$ *is a* γ*-distance over* $(\mathcal{L}, \mathcal{D})$. *For all* $l_1, l_2 \in \mathcal{L}$ *with* $l_1 \neq l_2$ *we have*

$$\begin{aligned}&\gamma(\mathcal{D}(fl_1, fl_2), fl_1, fl_2)\\ &= \left(\frac{\mathcal{D}^2(fl_1, fl_2)+1}{2}\right)\mathcal{D}(fl_1, fl_2)\\ &= \left(\frac{|fl_1 - fl_2|^2+1}{2}\right)|fl_1 - fl_2|\end{aligned}$$

Now,

$$\begin{aligned}&|fl_1 - fl_2|\\ &= |1 - \frac{l_1^m}{\Gamma + l_1^m} - 1 - \frac{l_2^m}{\Gamma + l_2^m}|\\ &= \frac{1}{(\Gamma + l_1^m)(\Gamma + l_2^m)}|l_1^m - l_2^m|\\ &\leq \frac{m}{\Gamma^2}|l_1 - l_2|\\ &= \frac{m}{\Gamma^2}\mathcal{D}(l_1, l_2)\\ &\leq \mathcal{D}(l_1, l_2).\end{aligned}$$

Hence,

$$\begin{aligned}&\gamma(\mathcal{D}(fl_1, fl_2), fl_1, fl_2)\\ &= \left(\frac{|fl_1 - fl_2|^2+1}{2}\right)|fl_1 - fl_2|\\ &\leq \frac{1}{2}[|l_1 - l_2|^2 + 1]|l_1 - l_2|\\ &= \frac{1}{2}[|l_1 - l_2|^3 + |l_1 - l_2|]\\ &\leq |l_1 - l_2|\\ &= \mathcal{D}(l_1, l_2).\end{aligned}$$

Consequently, all conditions of γ*-interpolative contractions are met by* f. *Theorem 3.5 confirms that* $\mathcal{F}_f$ *cosists of only one element.*

5 Application

To construct our application, we will refer to Example 2.3 Let $m, \Gamma \in \mathbb{R}$ with $m \geq 1$ and $\Gamma \geq m^2$. The following equation:

$$l^{m+1} + \Gamma l - \Gamma = 0, \tag{17}$$

where m,Γ $\in \mathbb{R}$ with $m \geq 1$ and $\Gamma \geq m^2$ has a unique solution in the unit interval $[0,1]$.
As an equivalent proof, it can be demonstrated that the following mapping has a unique fixed point within the interval [0,1].

$$fl = 1 - \frac{l^m}{\Gamma + l^m}\ ,$$

where $m, \Gamma \in \mathbb{R}$ with $m \geq 1$ and $\Gamma \geq m^2$.
Example 4.6 confirms that $\mathcal{F}_f$ consists of one element 0n [0,1]. Hence, equation 17 has a unique solution.

6 Conclusion

Fixed point theory is fundamental in both applied and abstract mathematics, as well as in various scientific disciplines. In this research, we introduce several novel contractions that are noteworthy. These contractions emerge from the integration of two concepts: γ-distance and interpolative contraction. To demonstrate the importance of our findings, we present an application along with numerical examples.

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