Vandermonde Determinant for a Certain Sakaguchi Type Function in q-Limaçon Domain

S. BASKARAN[1], G. SARAVANAN[2], S. BULUT[3*]
[1]Department of Mathematics, Agurchand Manmull Jain College, Meenambakkam, Chennai, 600061, Tamilnadu, INDIA
[2] PG and Research Department of Mathematics, Pachaiyappa's College, Chennai-600030, Tamil Nadu, INDIA
[3]Kocaeli University, Faculty of Aviation and Space Sciences, Arslanbey Campus, 41285 Kartepe-Kocaeli, TURKEY

Abstract: This paper defines the pth Vandermonde determinant and determines the coefficient bounds for the second-and third-order Vandermonde determinant for a Sakaguchi type function defined in q-Limaçon a domain.

Key-Words: Analytic function, Univalent function, Sakaguchi type function, q-Derivative, q-Limaçon, Vandermonde determinant.

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1 Introduction

Let $\mathcal{H}$ be the class of analytic functions in the open unit disc $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$ and consider the classes $\mathcal{P}$, $\mathcal{A}$ and $\mathcal{S}$ defined by

$$\begin{aligned} \mathcal{P} &= \{p \in \mathcal{H} : p(0) = 1, \Re(p(z)) > 0 \ (z \in \mathcal{U})\}, \\ \mathcal{A} &= \left\{f \in \mathcal{H} : f(0) = f'(0) - 1 = 0\right\}, \\ \mathcal{S} &= \{f \in \mathcal{A} : f \text{ is univalent in } \mathcal{U}\}, \end{aligned}$$

respectively. It is clear that the function $f \in \mathcal{A}$ can be expressed as

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (z \in \mathcal{U}). \qquad (1)$$

For two functions $f, g \in \mathcal{H}$, we say that the function f is subordinate to g in $\mathcal{U}$, and write

$$f(z) \prec g(z) \qquad (z \in \mathcal{U}),$$

if there exists a Schwarz function

$$\omega \in \Omega := \{\omega \in \mathcal{H} : \omega(0) = 0, |\omega(z)| < 1 \ (z \in \mathcal{U})\},$$

such that

$$f(z) = g(\omega(z)) \quad (z \in \mathcal{U}).$$

The study, [1], introduced a subclass of functions f from the set $\mathcal{A}$ that meet the analytic criterion

$$\Re\left(\frac{zf'(z)}{f(z) - f(-z)}\right) > \alpha \qquad (z \in \mathcal{U}, \quad 0 \le \alpha < 1).$$

These functions are known as Sakaguchi type functions and are starlike with respect to symmetric points (see, [2], [3], [4]).

The study, [5], later generalized the Sakaguchi type class to include functions of the form given by

$$\Re\left(\frac{(s-b)\, zf'(z)}{f(sz) - f(bz)}\right) > \alpha \qquad (z \in \mathcal{U}, \quad 0 \le \alpha < 1)$$

where $s, b \in \mathbb{C}$ with $s \neq b$, $|b| \le 1$, $b \neq 1$.

We shall utilize the notation and terminology from [6]. See, [7], [8], [9], for basic definitions and specific ideas of the q-calculus used in this study.

The q-derivative of a function f, which is defined on a subset of the complex plane $\mathbb{C}$, is expressed as follows:

$$\partial_q(f(z)) = \begin{cases} \frac{f(z) - f(qz)}{(1-q)z} & , \quad z \neq 0 \\ f'(0) & , \quad z = 0 \end{cases}$$

This is under the condition that $f'(0)$ exists. It's important to note that if f is differentiable, then

$$\lim_{q \to 1^-} \partial_q(f(z)) = \lim_{q \to 1^-} \frac{f(z) - f(qz)}{(1-q)z} = f'(z).$$

From this, we can derive that

$$\partial_q(f(z)) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1},$$

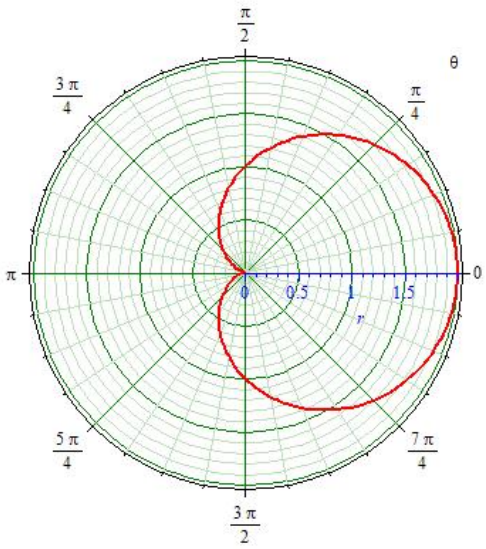
where the notation $[n]_q$ represents the number

$$[n]_q = \frac{1 - q^n}{1 - q}.$$

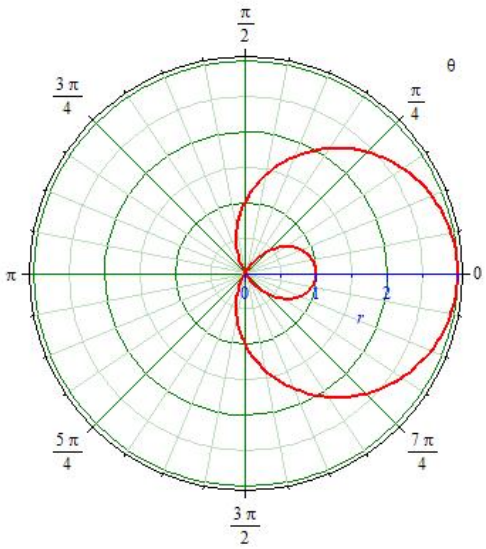
A curve with the form of a limaçon of Pascal, or simply limaçon, has polar coordinates of

$$r = a + b\cos\theta \qquad (a, b \in \mathbb{R}, \quad 0 \le \theta < 2\pi).$$

A limaçon is convex if $a \geq 2b$, and has an indentation bordered by two inflection points if $2b > a > b$. The limaçon degenerates into a cardioid if $b = a$. The limaçon is a trisectrix when $a = \frac{b}{2}$ and has an inner loop if $a < b$. The graphs of $r = a + b\cos\theta$ are depicted in Figure 1.



$a = 1$ and $b = 1$



$a = 1$ and $b = 2$

Fig. 1: The plots of $r = a + b\cos\theta$

The function

$$\mathbb{L}_\xi(z) = (1 + \xi z)^2 \qquad (-1 \leq \xi \leq 1, \quad \xi \neq 0)$$

is the analytic characterization of a limaçon, [10]. It maps the unit disk $\mathcal{U}$ onto a region bounded by the limaçon defined as

$$\partial\mathbb{L}_\xi = \left\{ \mu + i\upsilon \in \mathbb{C} : \left((\mu-1)^2 + \upsilon^2 - \xi^4\right)^2 = 4\xi^2\left(\left(\mu - 1 + \xi^2\right)^2 + \upsilon^2\right)\right\}$$

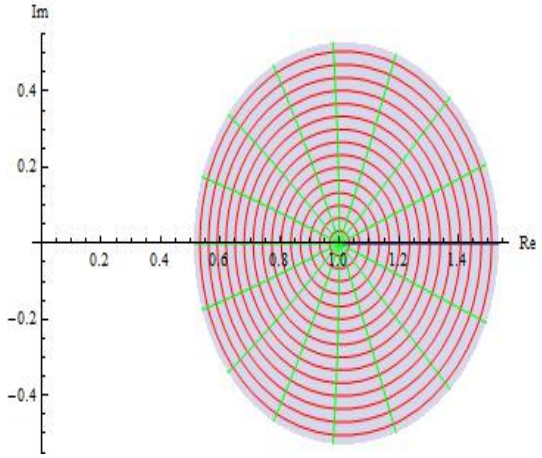
Inspired by the classical definition of the limacon and the principles of q-calculus, several authors have introduced the concept of the q-limacon. For further details, one can refer to [11], [12].

The limaçon's q-analog is known as the q-limaçon, and it is described as

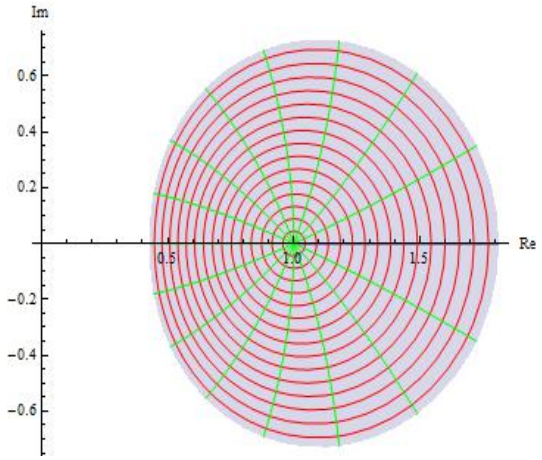
$$\mathbb{L}_{q,\xi}(z) = \left(\frac{2(1+\xi z)}{2+(1-q)\xi z}\right)^2,$$

where $-1 \leq \xi \leq 1,\ \xi \neq 0,\ 0 < q < 1$.

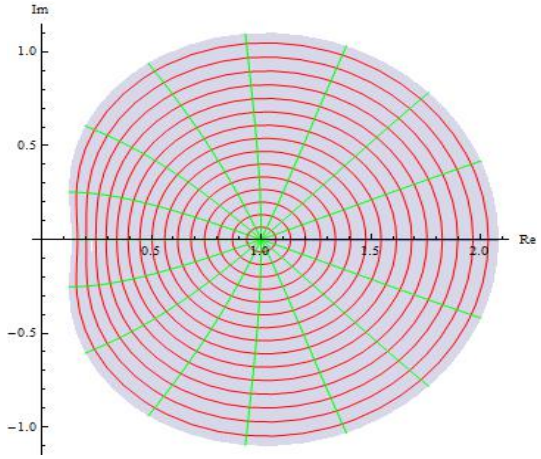
The image depicted in Figure 2 represents the transformation of $\mathcal{U}$ under the function $\mathbb{L}_{q,\xi}(z)$.



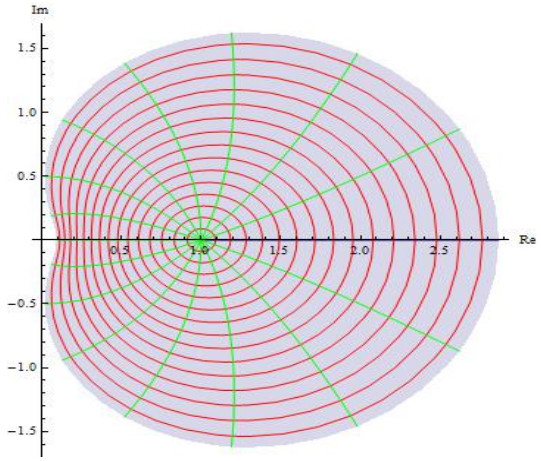
$\xi = \frac{1}{2\sqrt{2}}$ and $q = \frac{1}{2}$



$\xi = \frac{1}{2\sqrt{2}}$ and $q \to 1^-$



$\xi = \frac{1}{\sqrt{2}}$ and $q = \frac{1}{2}$



$\xi = \frac{1}{\sqrt{2}}$ and $q \to 1^{-}$

Fig. 2: The transformation of $\mathcal{U}$ under the function $\mathbb{L}_{q,\xi}(z)$.

A Vandermonde matrix, [13], is a matrix with geometric progression terms in each row. For a $p \times p$ matrix, the pth Vandermonde determinant, $p \geq 1$, and $n \geq 1$ are defined as

$$V_p(n) = \begin{vmatrix} 1 & a_n & a_n^2 & \cdots & a_n^{p-1} \\ 1 & a_{n+1} & a_{n+1}^2 & \cdots & a_{n+1}^{p-1} \\ 1 & a_{n+2} & a_{n+2}^2 & \cdots & a_{n+2}^{p-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_{n+p-1} & a_{n+p-1}^2 & \cdots & a_{n+p-1}^{p-1} \end{vmatrix} = \prod_{n \leq i \leq j \leq n+p-1} (a_j - a_i)$$

where a_i are the Taylor series coefficients in (1); therefore, $a_1 = 1$. Every row or column of a Vandermonde determinant contains a geometric sequence, with the first element being one. The Vandermonde determinant is also referred to as a discriminant. More specifically,

$$V_2(1) = \begin{vmatrix} 1 & a_1 \\ 1 & a_2 \end{vmatrix} = a_2 - a_1,$$

$$V_2(2) = \begin{vmatrix} 1 & a_2 \\ 1 & a_3 \end{vmatrix} = a_3 - a_2$$

and

$$V_3(1) = \begin{vmatrix} 1 & a_1 & a_1^2 \\ 1 & a_2 & a_2^2 \\ 1 & a_3 & a_3^2 \end{vmatrix} = (a_3 - a_2)(a_3 - a_1)(a_2 - a_1).$$

The Vandermonde determinant has a lot of applications in different fields. In particular, the Vandermonde matrix is applied in digital signal processing to compute the Discrete Fourier transform (DFT) and the Inverse Discrete Fourier Transform (IDFT). Vandermonde matrices play an important role in approximation problems.

2 Preliminary Results

In this section, we will present some lemmas utilized in the subsequent section to derive the main results. We remind you that a function u is termed a Schwarz function if it takes the form

$$u(z) = \sum_{n=1}^{\infty} u_n z^n \quad (z \in \mathcal{U}), \tag{2}$$

and satisfies the conditions $u(0) = 0$ and $|u(z)| \leq 1$.

Lemma 1 ([14]). *If the function $u \in \Omega$ is of the form (2), then $|u_n| \leq 1$ for all $n = 1, 2, \ldots$.*

Lemma 2 ([15], p. 25). *If the function $u \in \Omega$ is of the form (2), then $|u_2| \leq 1 - |u_1|^2$.*

Lemma 3 ([16]). *If the function $u \in \Omega$ is of the form (2), then*

$$u_2 = x\left(1 - u_1^2\right)$$

and

$$u_3 = \left(1 - u_1^2\right)\left(1 - |x|^2\right)t - u_1\left(1 - u_1^2\right)x^2,$$

for some x, t with $|x| \leq 1$ and $|t| \leq 1$.

We are now defining the classes $\mathcal{ST}_q^{s,b}(\mathbb{L}_{q,\xi}(z))$ and $\mathcal{CV}_q^{s,b}(\mathbb{L}_{q,\xi}(z))$. These classes map the unit disk, $\mathcal{U}$, onto the region bounded by a Limaçon.

Definition 1. *Let $\mathcal{ST}_q^{s,b}(\mathbb{L}_{q,\xi}(z))$ be the subclass of $\mathcal{A}$ consisting of the function f of the form (1) satisfying the condition*

$$\frac{(s-b)\, z\partial_q f(z)}{f(sz) - f(bz)} \prec \mathbb{L}_{q,\xi}(z) \quad (z \in \mathcal{U}), \tag{3}$$

where $s, b \in \mathbb{C}$ *with* $s \neq b,\ |b| \leq 1,\ b \neq 1, 0 < \xi \leq \frac{1}{\sqrt{2}}$ *and* $0 < q < 1$.

We show that $\mathcal{ST}_q^{s,b}(\mathbb{L}_{q,\xi}(z))$ is non empty. We consider the function $f(z) = z + \frac{a}{4}z^2$, where $a = \frac{(1+q)\xi}{[2]_q-(s+b)}$. A simple calculation shows that

$$\frac{(s-b)\, z\partial_q f(z)}{f(sz) - f(bz)} \prec \mathbb{L}_{q,\xi}(z).$$

Remark 1. (i) *Letting* $q \to 1^-$ *, we get the following new class* $\mathcal{ST}^{s,b}(\mathbb{L}_{\xi}(z))$ *which consists of functions* $f \in \mathcal{A}$ *satisfying*

$$\frac{(s-b)\, zf'(z)}{f(sz) - f(bz)} \prec \mathbb{L}_{\xi}(z) \quad \left(z \in \mathcal{U}, 0 < \xi \leq \tfrac{1}{\sqrt{2}}\right),$$

where $s, b \in \mathbb{C}$ *with* $s \neq b,\ |b| \leq 1,\ b \neq 1$.
(ii) *For* $s = 1$ *and* $b = -1$*, we get the following new class* $\mathcal{ST}_q^{1,-1}(\mathbb{L}_{q,\xi}(z))$ *which consists of functions* $f \in \mathcal{A}$ *satisfying*

$$\frac{2z\partial_q f(z)}{f(z) - f(-z)} \prec \mathbb{L}_{q,\xi}(z) \quad \left(z \in \mathcal{U}, \quad 0 < \xi \leq \frac{1}{\sqrt{2}}\right),$$

where $0 < q < 1$.
(iii) *For* $s = 1$ *and* $b = -1$ *and letting* $q \to 1^-$ *we get the following new class* $\mathcal{ST}^{1,-1}(\mathbb{L}_{\xi}(z))$ *which consists of functions* $f \in \mathcal{A}$ *satisfying*

$$\frac{2zf'(z)}{f(z) - f(-z)} \prec \mathbb{L}_{\xi}(z) \quad \left(z \in \mathcal{U}, \quad 0 < \xi \leq \tfrac{1}{\sqrt{2}}\right).$$

(iv) *For* $s = 1$ *and* $b = 0$*, we get the following new class* $\mathcal{ST}_q^{1,0}(\mathbb{L}_{q,\xi}(z))$ *which consists of functions* $f \in \mathcal{A}$ *satisfying*

$$\frac{z\partial_q f(z)}{f(z)} \prec \mathbb{L}_{q,\xi}(z) \quad \left(z \in \mathcal{U}, 0 < \xi \leq \tfrac{1}{\sqrt{2}}\right),$$

where $0 < q < 1$.
(v) *For* $s = 1$ *and* $b = 0$ *and letting* $q \to 1^-$ *we get the following new class* $\mathcal{ST}^{1,0}(\mathbb{L}_{\xi}(z))$ *which consists of functions* $f \in \mathcal{A}$ *satisfying*

$$\frac{zf'(z)}{f(z)} \prec \mathbb{L}_{\xi}(z) \quad \left(z \in \mathcal{U}, \quad 0 < \xi \leq \frac{1}{\sqrt{2}}\right).$$

Definition 2. *Let* $\mathcal{CV}_q^{s,b}(\mathbb{L}_{q,\xi}(z))$ *be the subclass of* $\mathcal{A}$ *consisting of the function* f *of the form* (1) *satisfying the condition*

$$\frac{(s-b)\, \partial_q(z\partial_q f(z))}{\partial_q(f(sz) - f(bz))} \prec \mathbb{L}_{q,\xi}(z), \quad (z \in \mathcal{U}), \quad (4)$$

where $s, b \in \mathbb{C}$ *with* $s \neq b,\ |b| \leq 1,\ b \neq 1, 0 < \xi \leq \frac{1}{\sqrt{2}}$ *and* $0 < q < 1$.

We show that the class $\mathcal{CV}_q^{s,b}(\mathbb{L}_{q,\xi}(z))$ is non empty. We consider the function $f(z) = z + \frac{a}{4}z^2$, where $a = \frac{\xi}{[2]_q-(s+b)}$. A simple calculation shows that

$$\frac{(s-b)\, \partial_q(z\partial_q f(z))}{\partial_q(f(sz) - f(bz))} \prec \mathbb{L}_{q,\xi}(z).$$

Remark 2. (i) *Letting* $q \to 1^-$ *, we get the following new class* $\mathcal{CV}^{s,b}(\mathbb{L}_{\xi}(z))$ *which consists of functions* $f \in \mathcal{A}$ *satisfying*

$$\frac{(s-b)\,(zf'(z))'}{(f(sz) - f(bz))'} \prec \mathbb{L}_{\xi}(z), \left(z \in \mathcal{U}, 0 < \xi \leq \tfrac{1}{\sqrt{2}}\right),$$

where $s, b \in \mathbb{C}$ *with* $s \neq b,\ |b| \leq 1,\ b \neq 1$.
(ii) *For* $s = 1$ *and* $b = -1$*, we get the following new class* $\mathcal{CV}_q^{1,-1}(\mathbb{L}_{q,\xi}(z))$ *which consists of functions* $f \in \mathcal{A}$ *satisfying*

$$\frac{2\partial_q(z\partial_q f(z))}{\partial_q(f(z) - f(-z))} \prec \mathbb{L}_{q,\xi}(z), \left(z \in \mathcal{U}, 0 < \xi \leq \tfrac{1}{\sqrt{2}}\right),$$

where $0 < q < 1$.
(iii) *For* $s = 1$ *and* $b = -1$ *and letting* $q \to 1^-$ *we get the following new class* $\mathcal{CV}^{1,-1}(\mathbb{L}_{\xi}(z))$ *which consists of functions* $f \in \mathcal{A}$ *satisfying*

$$\frac{2(zf'(z))'}{(f(z) - f(-z))'} \prec \mathbb{L}_{\xi}(z) \quad \left(z \in \mathcal{U}, 0 < \xi \leq \tfrac{1}{\sqrt{2}}\right).$$

(iv) *For* $s = 1$ *and* $b = 0$*, we get the following new class* $\mathcal{CV}_q^{1,0}(\mathbb{L}_{q,\xi}(z))$ *which consists of functions* $f \in \mathcal{A}$ *satisfying*

$$\frac{\partial_q(z\partial_q f(z))}{\partial_q(f(z))} \prec \mathbb{L}_{q,\xi}(z) \quad \left(z \in \mathcal{U}, 0 < \xi \leq \tfrac{1}{\sqrt{2}}\right),$$

where $0 < q < 1$. (v) *For* $s = 1$ *and* $b = 0$ *and letting* $q \to 1^-$ *we get the following new class* $\mathcal{CV}^{1,0}(\mathbb{L}_{\xi}(z))$ *which consists of functions* $f \in \mathcal{A}$ *satisfying*

$$\frac{(zf'(z))'}{f'(z)} \prec \mathbb{L}_{\xi}(z) \quad \left(z \in \mathcal{U}, 0 < \xi \leq \tfrac{1}{\sqrt{2}}\right).$$

3 Main Results

Throughout this paper, unless otherwise stated we assume that $s, b \in \mathbb{C}$ with $s \neq b,\ |b| \leq 1,\ b \neq 1,\ 0 < q < 1$, and $0 < \xi \leq \frac{1}{\sqrt{2}}$.

Theorem 1. *If the function* f *of the form* (1) *belongs to the class* $\mathcal{ST}_q^{s,b}(\mathbb{L}_{q,\xi}(z))$ *and define*

$$\sigma_q^{s,b} = [2]_q - (s+b),$$
$$\tau_q^{s,b} = [3]_q - \left(s^2 + sb + b^2\right). \quad (5)$$

Then
(i) *for* $\sigma_q^{s,b} \neq 0$, $\tau_q^{s,b} \neq 0$, $q \neq \frac{1}{3}$ *and when* $\xi \geq \frac{4\left|\sigma_q^{s,b}\right|}{|3q-1|\left|\sigma_q^{s,b}\right|+4|s+b|(1+q)}$, *we have*

$$\begin{aligned}|V_2(2)| &= |a_3 - a_2| \\ &\leq \frac{(1+q)}{\left|\tau_q^{s,b}\right|}\left(\frac{|3q-1|}{4} + \frac{|s+b|(1+q)}{\left|\sigma_q^{s,b}\right|}\right)\xi^2 \\ &\quad + \frac{1+q}{\left|\sigma_q^{s,b}\right|}\xi. \end{aligned} \tag{6}$$

(ii) *for* $\sigma_q^{s,b} \neq 0$, $\tau_q^{s,b} \neq 0$, $q \neq \frac{1}{3}$ *and when* $\xi < \frac{4\left|\sigma_q^{s,b}\right|}{|3q-1|\left|\sigma_q^{s,b}\right|+4|s+b|(1+q)}$, *we have*

$$|V_2(2)| \leq \begin{cases} \frac{|\tau_q^{s,b}|(1+q)\xi}{4\left|\sigma_q^{s,b}\right|^2\left(1-\left(\frac{|3q-1|}{4}+\frac{|s+b|(1+q)}{\left|\sigma_q^{s,b}\right|}\right)\xi\right)} & \\ \quad + \frac{1+q}{\left|\tau_q^{s,b}\right|}\xi, & \xi < \Xi_0 \\ \frac{(1+q)}{\left|\tau_q^{s,b}\right|}\left(\frac{|3q-1|}{4} + \frac{|s+b|(1+q)}{\left|\sigma_q^{s,b}\right|}\right)\xi^2 & \\ \quad + \frac{1+q}{\left|\sigma_q^{s,b}\right|}\xi, & \xi \geq \Xi_0 \end{cases} \tag{7}$$

where $\Xi_0 = \frac{2(2|\sigma_q^{s,b}|-|\tau_q^{s,b}|)}{|3q-1|\left|\sigma_q^{s,b}\right|+4|s+b|(1+q)}$.
(iii) *for* $\sigma_q^{s,b} \neq 0$, $\tau_q^{s,b} \neq 0$, $q = \frac{1}{3}$, $s \neq -b$ *and when* $\xi \geq \frac{3\left|\sigma_{\frac{1}{3}}^{s,b}\right|}{4|s+b|}$, *we have*

$$|V_2(2)| \leq \frac{16|s+b|}{9\left|\sigma_{\frac{1}{3}}^{s,b}\right|\left|\tau_{\frac{1}{3}}^{s,b}\right|}\xi^2 + \frac{4}{3\left|\sigma_{\frac{1}{3}}^{s,b}\right|}\xi. \tag{8}$$

(iv) *for* $\sigma_q^{s,b} \neq 0$, $\tau_q^{s,b} \neq 0$, $q = \frac{1}{3}$, $s \neq -b$ *and when* $\xi < \frac{3\left|\sigma_{\frac{1}{3}}^{s,b}\right|}{4|s+b|}$, *we have*

$$|V_2(2)| \leq \begin{cases} \frac{\left|\tau_{\frac{1}{3}}^{s,b}\right|\xi}{\left|\sigma_{\frac{1}{3}}^{s,b}\right|\left(3\left|\sigma_{\frac{1}{3}}^{s,b}\right|-|s+b|\xi\right)} + \frac{4}{3\left|\tau_{\frac{1}{3}}^{s,b}\right|}\xi, & \xi < \Xi_1 \\ \frac{16|s+b|}{9\left|\sigma_{\frac{1}{3}}^{s,b}\right|\left|\tau_{\frac{1}{3}}^{s,b}\right|}\xi^2 + \frac{4}{3\left|\sigma_{\frac{1}{3}}^{s,b}\right|}\xi, & \xi \geq \Xi_1 \end{cases} \tag{9}$$

where $\Xi_1 = \frac{6\left|\sigma_{\frac{1}{3}}^{s,b}\right|-3\left|\tau_{\frac{1}{3}}^{s,b}\right|}{8|s+b|}$.
(v) *for* $\sigma_q^{s,b} \neq 0$, $\tau_q^{s,b} \neq 0$, $q = \frac{1}{3}$ *and when* $s = -b$, $(b \neq 0)$, *we have*

$$|V_2(2)| \leq \left(\frac{\left|13-9s^2\right|}{48} + \frac{12}{|13-9s^2|}\right)\xi \tag{10}$$

Proof. For $s, b \in \mathbb{C}$ with $s \neq b$, $|b| \leq 1$, $b \neq 1$ and for $0 < q < 1$, let us define $\sigma_q^{s,b} = [2]_q - (s+b)$ and $\tau_q^{s,b} = [3]_q - \left(s^2 + sb + b^2\right)$.

Now for $\sigma_q^{s,b} \neq 0$, $\tau_q^{s,b} \neq 0$, $q \neq \frac{1}{3}$ and for $0 < \xi \leq \frac{1}{\sqrt{2}}$, let $f \in \mathcal{ST}_q^{s,b}(\mathbb{L}_{q,\xi}(z))$ given by (1). Then there exists an analytic function $u(z)$ of the form (2) such that

$$\begin{aligned}\frac{(s-b)\,z\partial_q f(z)}{f(sz)-f(bz)} &= \mathbb{L}_{q,\xi}(u(z)) \\ &= \left(\frac{2\left(1+\xi u(z)\right)}{2+(1-q)\,\xi u(z)}\right)^2. \end{aligned} \tag{11}$$

From the above equation, we have

$$\begin{aligned}&1 + \sigma_q^{s,b}a_2 z \\ &\quad + \left(\tau_q^{s,b}a_3 - (s+b)\,\sigma_q^{s,b}a_2^2\right)z^2 + \cdots = \\ &1 + h_1(q)\,\xi u_1 z \\ &\quad + \left(h_1(q)\xi u_2 + h_2(q)\xi^2 u_1^2\right)z^2 + \cdots, \end{aligned} \tag{12}$$

where $h_1(q) = 1+q$ and $h_2(q) = \frac{(3q-1)(1+q)}{4}$.

Equating the coefficients of z^n on both sides, we get

$$\begin{aligned}a_2 &= \frac{h_1(q)}{\sigma_q^{s,b}}\xi u_1, \\ a_3 &= \frac{h_1(q)}{\tau_q^{s,b}}\xi u_2 + \frac{h_2(q)}{\tau_q^{s,b}}\xi^2 u_1^2 \\ &\quad + \frac{(s+b)\left(h_1(q)\right)^2}{\sigma_q^{s,b}\tau_q^{s,b}}\xi^2 u_1^2. \end{aligned} \tag{13}$$

Using Lemma 2 and taking $|u_1| = c$ we obtain

$$\begin{aligned}|V_2(2)| &= |a_3 - a_2| \\ &\leq \frac{h_1(q)}{\left|\tau_q^{s,b}\right|}\xi|u_2| + \frac{|h_2(q)|}{\left|\tau_q^{s,b}\right|}\xi^2|u_1|^2 \\ &+ \frac{|s+b|(h_1(q))^2}{\left|\sigma_q^{s,b}\right|\left|\tau_q^{s,b}\right|}\xi^2|u_1|^2 + \frac{h_1(q)}{\left|\sigma_q^{s,b}\right|}\xi|u_1| \\ &\leq \frac{h_1(q)}{\left|\tau_q^{s,b}\right|}\xi\left(1-|u_1|^2\right) + \frac{|h_2(q)|}{\left|\tau_q^{s,b}\right|}\xi^2|u_1|^2 \\ &+ \frac{|s+b|(h_1(q))^2}{\left|\sigma_q^{s,b}\right|\left|\tau_q^{s,b}\right|}\xi^2|u_1|^2 + \frac{h_1(q)}{\left|\sigma_q^{s,b}\right|}\xi|u_1| \\ &= \frac{h_1(q)}{\left|\tau_q^{s,b}\right|}\xi\left(1-c^2\right) + \frac{|h_2(q)|}{\left|\tau_q^{s,b}\right|}\xi^2c^2 \\ &+ \frac{|s+b|(h_1(q))^2}{\left|\sigma_q^{s,b}\right|\left|\tau_q^{s,b}\right|}\xi^2c^2 + \frac{h_1(q)}{\left|\sigma_q^{s,b}\right|}\xi c.\end{aligned}$$

That is,

$$\begin{aligned}|V_2(2)| &\leq \left(\frac{|h_2(q)|}{\left|\tau_q^{s,b}\right|}\xi^2 + \frac{|s+b|(h_1(q))^2}{\left|\sigma_q^{s,b}\right|\left|\tau_q^{s,b}\right|}\xi^2 - \frac{h_1(q)}{\left|\tau_q^{s,b}\right|}\xi\right)c^2 \\ &+ \frac{h_1(q)}{\left|\sigma_q^{s,b}\right|}\xi c + \frac{h_1(q)}{\left|\tau_q^{s,b}\right|}\xi =: \alpha_0(c).\end{aligned}$$

As $|u_1| \leq 1$ by Lemma 1, we observe that $c \in [0,1]$. Hence, we get

Case 1. When $\xi \geq \frac{h_1(q)\left|\sigma_q^{s,b}\right|}{|h_2(q)|\left|\sigma_q^{s,b}\right| + |s+b|(h_1(q))^2}$, $\alpha_0(c)$ is an increasing function of c in $[0,1]$, since

$$\begin{aligned}\alpha_0'(c) =& 2\left(\frac{|h_2(q)|}{\left|\tau_q^{s,b}\right|}\xi^2 + \frac{|s+b|(h_1(q))^2}{\left|\sigma_q^{s,b}\right|\left|\tau_q^{s,b}\right|}\xi^2 - \frac{h_1(q)}{\left|\tau_q^{s,b}\right|}\xi\right)c \\ &+ \frac{h_1(q)}{\left|\sigma_q^{s,b}\right|}\xi \geq 0 \quad \text{for} \quad 0 \leq c \leq 1.\end{aligned}$$

Hence $\alpha_0(c)$ attains its maximum value at $c = 1$. Thus,

$$|V_2(2)| \leq \alpha_0(1).$$

Case 2. When $\xi < \frac{h_1(q)\left|\sigma_q^{s,b}\right|}{|h_2(q)|\left|\sigma_q^{s,b}\right| + |s+b|(h_1(q))^2}$, $\alpha_0(c)$ attains its maximum value either at c_0 for some $c_0 \in (0,1)$ or at $c = 1$. Indeed, when $\xi < \frac{h_1(q)\left(2\left|\sigma_q^{s,b}\right| - \left|\tau_q^{s,b}\right|\right)}{2\left(|h_2(q)|\left|\sigma_q^{s,b}\right| + |s+b|(h_1(q))^2\right)}$, $\alpha_0(c)$ attains its maximum value at

$$c_0 = \frac{\frac{h_1(q)}{\left|\sigma_q^{s,b}\right|}}{2\left(\frac{h_1(q)}{\left|\tau_q^{s,b}\right|} - \left(\frac{|h_2(q)|}{\left|\tau_q^{s,b}\right|} + \frac{|s+b|(h_1(q))^2}{\left|\sigma_q^{s,b}\right|\left|\tau_q^{s,b}\right|}\right)\xi\right)}, \quad (0 < c_0 < 1),$$

since $\alpha_0'(c_0) = 0$ and $\alpha_0''(c_0) < 0$. Thus,

$$|V_2(2)| \leq \alpha_0(c_0).$$

On the other hand, when $\xi \geq \frac{h_1(q)\left(2\left|\sigma_q^{s,b}\right| - \left|\tau_q^{s,b}\right|\right)}{2\left(|h_2(q)|\left|\sigma_q^{s,b}\right| + |s+b|(h_1(q))^2\right)}$, $\alpha_0(c)$ is an increasing function of c in $[0,1]$, since

$$\begin{aligned}\alpha_0'(c) = 2&\left(\frac{|h_2(q)|}{\left|\tau_q^{s,b}\right|}\xi^2 + \frac{|s+b|(h_1(q))^2}{\left|\sigma_q^{s,b}\right|\left|\tau_q^{s,b}\right|}\xi^2 - \frac{h_1(q)}{\left|\tau_q^{s,b}\right|}\xi\right)c \\ &+ \frac{h_1(q)}{\left|\sigma_q^{s,b}\right|}\xi \geq 0 \quad \text{for} \quad 0 \leq c \leq 1.\end{aligned}$$

Hence $\alpha_0(c)$ attains its maximum value at $c = 1$. Thus,

$$|V_2(2)| \leq \alpha_0(1).$$

Now for $\sigma_q^{s,b} \neq 0$, $\tau_q^{s,b} \neq 0$, $q = \frac{1}{3}$ and when $s \neq -b$, the Taylor series expansion will become,

$$\begin{aligned}&1 + \sigma_{\frac{1}{3}}^{s,b}a_2z \\ &+ \left(\tau_{\frac{1}{3}}^{s,b}a_3 - (s+b)\sigma_{\frac{1}{3}}^{s,b}a_2^2\right)z^2 + \cdots \\ &= 1 + \frac{4}{3}\xi u_1 z + \frac{4}{3}\xi u_2 z^2 + \cdots, \end{aligned} \tag{14}$$

Equating the coefficients of z^n on both sides, we get

$$\begin{aligned}a_2 &= \frac{4}{3\sigma_{\frac{1}{3}}^{s,b}}\xi u_1, \\ a_3 &= \frac{4}{3\tau_{\frac{1}{3}}^{s,b}}\xi u_2 + \frac{16(s+b)}{9\sigma_{\frac{1}{3}}^{s,b}\tau_{\frac{1}{3}}^{s,b}}\xi^2u_1^2. \end{aligned} \tag{15}$$

Using Lemma 2 and taking $|u_1| = c$ we obtain

$$\begin{aligned}|V_2(2)| \leq &\left(\frac{16|s+b|}{9\left|\sigma_{\frac{1}{3}}^{s,b}\right|\left|\tau_{\frac{1}{3}}^{s,b}\right|}\xi^2 - \frac{4}{3\left|\tau_{\frac{1}{3}}^{s,b}\right|}\xi\right)c^2 \\ &+ \frac{4}{3\left|\sigma_{\frac{1}{3}}^{s,b}\right|}\xi c + \frac{4}{3\left|\tau_{\frac{1}{3}}^{s,b}\right|}\xi =: \alpha_1(c).\end{aligned}$$

Case 3. When $\xi \geq \frac{3\left|\sigma_{\frac{1}{3}}^{s,b}\right|}{4|s+b|}$, $\alpha_1(c)$ is an increasing function of c in $[0,1]$. Hence,

$$|V_2(2)| \leq \alpha_1(1).$$

Case 4. When $\xi < \frac{3\left|\sigma_{\frac{1}{3}}^{s,b}\right|}{4|s+b|}$, $\alpha_1(c)$ attains its maximum value either at some $c_1 \in (0,1)$ or at $c = 1$. Hence,

when $\xi < \frac{6\left|\sigma_{\frac{1}{3}}^{s,b}\right|-3\left|\tau_{\frac{1}{3}}^{s,b}\right|}{8|s+b|}$, $|V_2(2)| \leq \alpha_1(c_1)$, where $c_1 = \frac{3\left|\tau_{\frac{1}{3}}^{s,b}\right|}{6\left|\sigma_{\frac{1}{3}}^{s,b}\right|-8|s+b|\xi}, (0 < c_1 < 1)$.

On the other hand, when $\xi \geq \frac{6\left|\sigma_{\frac{1}{3}}^{s,b}\right|-3\left|\tau_{\frac{1}{3}}^{s,b}\right|}{8|s+b|}$, $\alpha_1(c)$ attains its maximum value at $c = 1$. Thus,

$$|V_2(2)| \leq \alpha_1(1).$$

Now for $\sigma_q^{s,b} \neq 0$, $\tau_q^{s,b} \neq 0$, $q = \frac{1}{3}$ and when $s = -b$, $(b \neq 0)$, the Taylor series expansion will become,

$$\begin{aligned} &1 + \frac{4}{3}a_2 z + \left(\frac{13}{9} - s^2\right)a_3 z^2 + \cdots \\ &= 1 + \frac{4}{3}\xi u_1 z + \frac{4}{3}\xi u_2 z^2 + \cdots, \end{aligned} \tag{16}$$

Equating the coefficients of z^n on both sides, we get

$$a_2 = \xi u_1 \qquad \text{and} \qquad a_3 = \frac{12}{13 - 9s^2}\xi u_2. \tag{17}$$

Using Lemma 2 and taking $|u_1| = c$ we obtain

$$|V_2(2)| \leq \frac{-12}{|13 - 9s^2|}\xi c^2 + \xi c + \frac{12}{|13 - 9s^2|}\xi =: \alpha_2(c).$$

Case 5. $\alpha_2(c)$ attains its maximum value at $c_2 = \frac{|13-9s^2|}{24}, (0 < c_2 < 1)$, since $\alpha_2'(c_2) = 0$ and $\alpha_2''(c_2) < 0$. Thus, $|V_2(2)| \leq \alpha_2(c_2)$. □

Theorem 2. *Given the same conditions as in Theorem 1, when $\sigma_q^{s,b} \neq 0$, it follows that*

$$|V_2(1)| = |a_2 - a_1| \leq \frac{1+q}{\left|\sigma_q^{s,b}\right|}\xi + 1. \tag{18}$$

Proof. By means of the proof of the Theorem 1, we have

$$|V_2(1)| = |a_2 - a_1| = \left|\frac{1+q}{\sigma_q^{s,b}}\xi u_1 - 1\right|.$$

Using Lemma 2 and taking $|u_1| = c$ we obtain

$$|V_2(1)| \leq \frac{1+q}{\left|\sigma_q^{s,b}\right|}\xi c + 1 =: \alpha_3(c).$$

Since $\alpha_3'(c) > 0$, $\alpha_3(c)$ attains its maximum value at $c = 1$ in $[0, 1]$.
Hence,

$$|V_2(1)| \leq \alpha_3(1).$$

□

Theorem 3. *Given the same conditions as in Theorem 1, when $\sigma_q^{s,b} \neq 0$ and $\tau_q^{s,b} \neq 0$, it follows that*

$$|a_3 - a_1| \leq \begin{cases} \frac{(1+q)}{\left|\tau_q^{s,b}\right|}\left(\frac{|3q-1|}{4} + \frac{|s+b|(1+q)}{\left|\sigma_q^{s,b}\right|}\right)\xi^2 \\ +1, & \xi \geq \Xi_2 \\ \frac{1+q}{\left|\tau_q^{s,b}\right|}\xi + 1, & \xi < \Xi_2 \end{cases} \tag{19}$$

where $\Xi_2 = \frac{4\left|\sigma_q^{s,b}\right|}{|3q-1|\left|\sigma_q^{s,b}\right|+4|s+b|(1+q)}$.
(ii) if $q = \frac{1}{3}$ and $s \neq -b$, then

$$|a_3 - a_1| \leq \begin{cases} \frac{16|s+b|}{9\left|\sigma_{\frac{1}{3}}^{s,b}\right|\left|\tau_{\frac{1}{3}}^{s,b}\right|}\xi^2 + 1 & , \quad \xi \geq \frac{3\left|\sigma_{\frac{1}{3}}^{s,b}\right|}{4|s+b|} \\ \frac{4}{3\left|\tau_{\frac{1}{3}}^{s,b}\right|}\xi + 1 & , \quad \xi < \frac{3\left|\sigma_{\frac{1}{3}}^{s,b}\right|}{4|s+b|} \end{cases} \tag{20}$$

(iii) if $q = \frac{1}{3}$ and $s = -b, (b \neq 0)$, then

$$|a_3 - a_1| \leq \frac{12}{|13 - 9s^2|}\xi + 1. \tag{21}$$

Proof. By means of the proof of the Theorem 1, for $q \neq \frac{1}{3}$ and using Lemma 2 and taking $|u_1| = c$ we obtain

$$\begin{aligned} |a_3 - a_1| \leq &\left(\frac{|h_2(q)|}{\left|\tau_q^{s,b}\right|}\xi^2 + \frac{|s+b|(h_1(q))^2}{\left|\sigma_q^{s,b}\right|\left|\tau_q^{s,b}\right|}\xi^2 - \frac{h_1(q)}{\left|\tau_q^{s,b}\right|}\xi\right)c^2 \\ &+ \frac{h_1(q)}{\left|\tau_q^{s,b}\right|}\xi + 1 =: \alpha_4(c). \end{aligned}$$

As $|u_1| \leq 1$ by Lemma 1, we observe that $c \in [0, 1]$. Hence, we get

Case 1. When $\xi \geq \frac{h_1(q)\left|\sigma_q^{s,b}\right|}{|h_2(q)|\left|\sigma_q^{s,b}\right|+|s+b|(h_1(q))^2}$, $\alpha_4(c)$ is an increasing function of c in $[0, 1]$, since

$$\alpha_4'(c) = 2\left(\frac{|h_2(q)|}{\left|\tau_q^{s,b}\right|}\xi^2 + \frac{|s+b|(h_1(q))^2}{\left|\sigma_q^{s,b}\right|\left|\tau_q^{s,b}\right|}\xi^2 - \frac{h_1(q)}{\left|\tau_q^{s,b}\right|}\xi\right)c \geq 0,$$

for $0 \leq c \leq 1$.

Hence $\alpha_4(c)$ attains its maximum value at $c = 1$. Thus,

$$|a_3 - a_1| \leq \alpha_4(1).$$

Case 2. When $\xi < \frac{h_1(q)\left|\sigma_q^{s,b}\right|}{|h_2(q)|\left|\sigma_q^{s,b}\right|+|s+b|(h_1(q))^2}$, $\alpha_4(c)$ is a decreasing function of c in $[0, 1]$, since

$$\alpha_4'(c) = 2\left(\frac{|h_2(q)|}{\left|\tau_q^{s,b}\right|}\xi^2 + \frac{|s+b|(h_1(q))^2}{\left|\sigma_q^{s,b}\right|\left|\tau_q^{s,b}\right|}\xi^2 - \frac{h_1(q)}{\left|\tau_q^{s,b}\right|}\xi\right)c \leq 0,$$

for $0 \leq c \leq 1$.

Hence $\alpha_4(c)$ attains its maximum value at $c = 0$. Thus,

$$|a_3 - a_1| \leq \alpha_4(0).$$

Now for $q = \frac{1}{3}$ and when $s \neq -b$, we obtain

$$|a_3 - a_1| \leq \left(\frac{16|s+b|}{9\left|\sigma_{\frac{1}{3}}^{s,b}\right|\left|\tau_{\frac{1}{3}}^{s,b}\right|}\xi^2 - \frac{4}{3\left|\tau_{\frac{1}{3}}^{s,b}\right|}\xi \right) c^2 + \frac{4}{3\left|\tau_{\frac{1}{3}}^{s,b}\right|}\xi + 1 =: \alpha_5(c).$$

Case 3. When $\xi \geq \frac{3\left|\sigma_{\frac{1}{3}}^{s,b}\right|}{4|s+b|}$, $\alpha_5(c)$ is an increasing function of c in $[0,1]$. Hence $|a_3 - a_1| \leq \alpha_5(1)$. On the other hand, when $\xi < \frac{3\left|\sigma_{\frac{1}{3}}^{s,b}\right|}{4|s+b|}$, $\alpha_5(c)$ is a decreasing function of c in $[0,1]$. Hence $|a_3 - a_1| \leq \alpha_5(0)$.
Now for $q = \frac{1}{3}$ and when $s = -b$, $(b \neq 0)$, we obtain

$$|a_3 - a_1| \leq \frac{-12}{|13 - 9s^2|}\xi c^2 + \frac{12}{|13 - 9s^2|}\xi + 1 =: \alpha_6(c).$$

Case 4. Since $\alpha_6(c)$ is a decreasing function of c in $[0,1]$, we have $|a_3 - a_1| \leq \alpha_6(0)$. □

Theorem 4. *Given the same conditions as in Theorem 1, when $\sigma_q^{s,b} \neq 0$ and $\tau_q^{s,b} \neq 0$, it follows that*
(i) if $q \neq \frac{1}{3}$ and $\xi \geq \frac{4\left|\sigma_q^{s,b}\right|}{|3q-1|\left|\sigma_q^{s,b}\right| + 4|s+b|(1+q)}$, then

$$|V_3(1)| \leq \left(\frac{(1+q)}{\left|\tau_q^{s,b}\right|}\left(\frac{|3q-1|}{4} + \frac{|s+b|(1+q)}{\left|\sigma_q^{s,b}\right|} \right)\xi^2 + \frac{1+q}{\left|\sigma_q^{s,b}\right|}\xi \right) \times \left(\frac{(1+q)}{\left|\tau_q^{s,b}\right|}\left(\frac{|3q-1|}{4} + \frac{|s+b|(1+q)}{\left|\sigma_q^{s,b}\right|} \right)\xi^2 + 1 \right) \times \left(\frac{1+q}{\left|\sigma_q^{s,b}\right|}\xi + 1 \right). \quad (22)$$

(ii) if $q \neq \frac{1}{3}$ and $\xi < \frac{4\left|\sigma_q^{s,b}\right|}{|3q-1|\left|\sigma_q^{s,b}\right| + 4|s+b|(1+q)}$, then

$$|V_3(1)| \leq \begin{cases} \left(\frac{|\tau_q^{s,b}|(1+q)\xi}{4\left|\sigma_q^{s,b}\right|^2\left(1 - \left(\frac{|3q-1|}{4} + \frac{|s+b|(1+q)}{\left|\sigma_q^{s,b}\right|}\right)\xi\right)} + \frac{1+q}{\left|\tau_q^{s,b}\right|}\xi \right)\left(\frac{1+q}{\left|\sigma_q^{s,b}\right|}\xi + 1 \right) \times \left(\frac{1+q}{\left|\tau_q^{s,b}\right|}\xi + 1 \right), & \xi < \Xi_0 \\ \left(\frac{(1+q)}{\left|\tau_q^{s,b}\right|}\left(\frac{|3q-1|}{4} + \frac{|s+b|(1+q)}{\left|\sigma_q^{s,b}\right|} \right)\xi^2 + \frac{1+q}{\left|\sigma_q^{s,b}\right|}\xi \right)\left(\frac{1+q}{\left|\sigma_q^{s,b}\right|}\xi + 1 \right) \times \left(\frac{1+q}{\left|\tau_q^{s,b}\right|}\xi + 1 \right), & \xi \geq \Xi_0 \end{cases} \quad (23)$$

where Ξ_0 is given in Theorem 1.

(iii) if $q = \frac{1}{3}$, $s \neq -b$ and $\xi \geq \frac{3\left|\sigma_{\frac{1}{3}}^{s,b}\right|}{4|s+b|}$, then

$$|V_3(1)| \leq \left(\frac{16|s+b|}{9\left|\sigma_{\frac{1}{3}}^{s,b}\right|\left|\tau_{\frac{1}{3}}^{s,b}\right|}\xi^2 + \frac{4}{3\left|\sigma_{\frac{1}{3}}^{s,b}\right|}\xi \right) \times \left(\frac{16|s+b|}{9\left|\sigma_{\frac{1}{3}}^{s,b}\right|\left|\tau_{\frac{1}{3}}^{s,b}\right|}\xi^2 + 1 \right) \times \left(\frac{4}{3\left|\sigma_{\frac{1}{3}}^{s,b}\right|}\xi + 1 \right). \quad (24)$$

(iv) if $q=\frac{1}{3}$, $s\neq -b$ *and* $\xi<\frac{3\left|\sigma_{\frac{1}{3}}^{s,b}\right|}{4|s+b|}$, *then*

$$|V_3(1)|\leq\begin{cases}\left(\frac{\left|\tau_{\frac{1}{3}}^{s,b}\right|\xi}{\left|\sigma_{\frac{1}{3}}^{s,b}\right|\left(3\left|\sigma_{\frac{1}{3}}^{s,b}\right|-|s+b|\xi\right)}+\frac{4}{3\left|\tau_{\frac{1}{3}}^{s,b}\right|}\xi\right)\\ \times\left(\frac{4}{3\left|\sigma_{\frac{1}{3}}^{s,b}\right|}\xi+1\right)\\ \times\left(\frac{4}{3\left|\tau_{\frac{1}{3}}^{s,b}\right|}\xi+1\right), & \xi<\frac{6\left|\sigma_{\frac{1}{3}}^{s,b}\right|-3\left|\tau_{\frac{1}{3}}^{s,b}\right|}{8|s+b|}\\ \left(\frac{16|s+b|}{9\left|\sigma_{\frac{1}{3}}^{s,b}\right|\left|\tau_{\frac{1}{3}}^{s,b}\right|}\xi^2+\frac{4}{3\left|\sigma_{\frac{1}{3}}^{s,b}\right|}\xi\right)\\ \times\left(\frac{4}{3\left|\sigma_{\frac{1}{3}}^{s,b}\right|}\xi+1\right)\\ \times\left(\frac{4}{3\left|\tau_{\frac{1}{3}}^{s,b}\right|}\xi+1\right), & \xi\geq\frac{6\left|\sigma_{\frac{1}{3}}^{s,b}\right|-3\left|\tau_{\frac{1}{3}}^{s,b}\right|}{8|s+b|}\end{cases}\tag{25}$$

(v) if $q=\frac{1}{3}$ *and* $s=-b$, $(b\neq 0)$, *then*

$$|V_3(1)|\leq\left(\frac{\left|13-9s^2\right|}{48}+\frac{12}{|13-9s^2|}\right)\times\xi(\xi+1)\left(\frac{12}{|13-9s^2|}\xi+1\right).\tag{26}$$

Proof. Utilizing the results from Theorem 1 to Theorem 3, we obtain the desired result. □

Letting $q\to 1^-$ in Theorems 1-4, we get the following results.

Corollary 1. *If the function f of the form* (1) *belongs to the class* $\mathcal{ST}^{s,b}(\mathbb{L}_\xi(z))$ *and define*

$$\sigma^{s,b}=2-(s+b)$$
$$\tau^{s,b}=3-\left(s^2+sb+b^2\right).\tag{27}$$

Then
(i) for $\sigma^{s,b}\neq 0$, $\tau^{s,b}\neq 0$ *and when* $\xi\geq\frac{2|\sigma^{s,b}|}{|\sigma^{s,b}|+4|s+b|}$, *we have*

$$|V_2(2)|\leq\frac{2}{|\tau^{s,b}|}\left(\frac{1}{2}+\frac{2|s+b|}{|\sigma^{s,b}|}\right)\xi^2+\frac{2}{|\sigma^{s,b}|}\xi\tag{28}$$

(ii) for $\sigma^{s,b}\neq 0$, $\tau^{s,b}\neq 0$ *and when* $\xi<\frac{2|\sigma^{s,b}|}{|\sigma^{s,b}|+4|s+b|}$, *we have*

$$|V_2(2)|\leq\begin{cases}\frac{|\tau^{s,b}|\xi}{2|\sigma^{s,b}|^2\left(1-\left(\frac{1}{2}+\frac{2|s+b|}{|\sigma^{s,b}|}\right)\xi\right)}\\ +\frac{2}{|\tau^{s,b}|}\xi, & \xi<\frac{2|\sigma^{s,b}|-|\tau^{s,b}|}{|\sigma^{s,b}|+4|s+b|}\\ \frac{2}{|\tau^{s,b}|}\left(\frac{1}{2}+\frac{2|s+b|}{|\sigma^{s,b}|}\right)\xi^2\\ +\frac{2}{|\sigma^{s,b}|}\xi, & \xi\geq\frac{2|\sigma^{s,b}|-|\tau^{s,b}|}{|\sigma^{s,b}|+4|s+b|}\end{cases}\tag{29}$$

Corollary 2. *Given the same conditions as in Corollary 1, when* $\sigma^{s,b}\neq 0$, *it follows that*

$$|V_2(1)|\leq\frac{2\xi}{|\sigma^{s,b}|}+1.\tag{30}$$

Corollary 3. *Given the same conditions as in Corollary 1, when* $\sigma^{s,b}\neq 0$, *it follows that*

$$|a_3-a_1|\leq\begin{cases}\frac{2}{|\tau^{s,b}|}\left(\frac{1}{2}+\frac{2|s+b|}{|\sigma^{s,b}|}\right)\xi^2\\ +1, & \xi\geq\frac{2|\sigma^{s,b}|}{|\sigma^{s,b}|+4|s+b|}\\ \frac{2\xi}{|\tau^{s,b}|}+1, & \xi<\frac{2|\sigma^{s,b}|}{|\sigma^{s,b}|+4|s+b|}\end{cases}\tag{31}$$

Corollary 4. *Given the same conditions as in Corollary 1, when* $\sigma^{s,b}\neq 0$, *it follows that*
(i) if $\sigma^{s,b}\neq 0$, $\tau^{s,b}\neq 0$ *and* $\xi\geq\frac{2|\sigma^{s,b}|}{|\sigma^{s,b}|+4|s+b|}$, *we have*

$$|V_3(1)|\leq\left(\frac{2}{|\tau^{s,b}|}\left(\frac{1}{2}+\frac{2|s+b|}{|\sigma^{s,b}|}\right)\xi^2+\frac{2}{|\sigma^{s,b}|}\xi\right)\times\left(\frac{2}{|\tau^{s,b}|}\left(\frac{1}{2}+\frac{2|s+b|}{|\sigma^{s,b}|}\right)\xi^2+1\right)\times\left(\frac{2\xi}{|\sigma^{s,b}|}+1\right).\tag{32}$$

(ii) if $\sigma^{s,b}\neq 0$, $\tau^{s,b}\neq 0$ *and* $\xi<\frac{2|\sigma^{s,b}|}{|\sigma^{s,b}|+4|s+b|}$, *we have*

$$|V_3(1)|\leq\begin{cases}\left(\frac{|\tau^{s,b}|\xi}{2|\sigma^{s,b}|^2\left(1-\left(\frac{1}{2}+\frac{2|s+b|}{|\sigma^{s,b}|}\right)\xi\right)}+\frac{2}{|\tau^{s,b}|}\xi\right)\\ \times\left(\frac{2\xi}{|\sigma^{s,b}|}+1\right)\\ \times\left(\frac{2}{|\tau^{s,b}|}\xi+1\right), & \xi<\frac{2|\sigma^{s,b}|-|\tau^{s,b}|}{|\sigma^{s,b}|+4|s+b|}\\ \left(\frac{2}{|\tau^{s,b}|}\left(\frac{1}{2}+\frac{2|s+b|}{|\sigma^{s,b}|}\right)\xi^2+\frac{2}{|\sigma^{s,b}|}\xi\right)\\ \times\left(\frac{2\xi}{|\sigma^{s,b}|}+1\right)\\ \times\left(\frac{2}{|\tau^{s,b}|}\xi+1\right), & \xi\geq\frac{2|\sigma^{s,b}|-|\tau^{s,b}|}{|\sigma^{s,b}|+4|s+b|}\end{cases}\tag{33}$$

Letting $s=1$ and $b=-1$ in Theorems 1-4, we get the following results.

Corollary 5. *If the function f of the form* (1) *belongs to the class* $\mathcal{ST}_q^{1,-1}\left(\mathbb{L}_{q,\xi}\left(z\right)\right)$*, then*

$$|V_2\left(2\right)| \leq \frac{q\xi}{4-|3q-1|\,\xi} + \frac{\xi}{q}. \tag{34}$$

Corollary 6. *Given the same conditions as in Corollary 5, it follows that*

$$|V_2\left(1\right)| \leq \xi + 1. \tag{35}$$

Corollary 7. *Given the same conditions as in Corollary 5, it follows that*

$$|a_3 - a_1| \leq \frac{\xi}{q} + 1. \tag{36}$$

Corollary 8. *Given the same conditions as in Corollary 5, it follows that*

$$|V_3\left(1\right)| \leq \left(\frac{q\xi}{4-|3q-1|\xi} + \frac{\xi}{q}\right)(\xi+1)\left(\frac{\xi}{q}+1\right). \tag{37}$$

Letting $s = 1$, $b = -1$ and $q \to 1^-$ in Theorems 1-4, we get the following results.

Corollary 9. *If the function f of the form* (1) *belongs to the class* $\mathcal{ST}^{1,-1}\left(\mathbb{L}_{\xi}\left(z\right)\right)$*, then*

$$|V_2\left(2\right)| \leq \frac{\xi}{4-2\xi} + \xi. \tag{38}$$

Corollary 10. *Given the same conditions as in Corollary 9, it follows that*

$$|V_2\left(1\right)| \leq \xi + 1. \tag{39}$$

Corollary 11. *Given the same conditions as in Corollary 9, it follows that*

$$|a_3 - a_1| \leq \xi + 1. \tag{40}$$

Corollary 12. *Given the same conditions as in Corollary 9, it follows that*

$$|V_3\left(1\right)| \leq \left(\frac{\xi}{4-2\xi} + \xi\right)(\xi+1)^2. \tag{41}$$

Letting $s = 1$ and $b = 0$ in Theorems 1-4, we get the following results.

Corollary 13. *If the function f of the form* (1) *belongs to the class* $\mathcal{ST}_q^{1,0}\left(\mathbb{L}_{q,\xi}\left(z\right)\right)$*, then*
(i) *when* $\xi \geq \frac{4q}{|3q-1|q+4(1+q)}$*, we have*

$$|V_2\left(2\right)| \leq \left(\frac{|3q-1|}{4q} + \frac{1+q}{q^2}\right)\xi^2 + \frac{1+q}{q}\xi. \tag{42}$$

The inequality is sharp for the function
(ii) *when* $\xi < \frac{4q}{|3q-1|q+4(1+q)}$*, we have*

$$|V_2\left(2\right)| \leq \begin{cases} \frac{(1+q)^2\xi}{4q-(|3q-1|q+4(1+q))\xi} \\ +\frac{\xi}{q}, \qquad \xi < \frac{2q(1-q)}{|3q-1|q+4(1+q)} \\ \\ \left(\frac{|3q-1|}{4q} + \frac{1+q}{q^2}\right)\xi^2 \\ +\frac{1+q}{q}\xi, \quad \xi \geq \frac{2q(1-q)}{|3q-1|q+4(1+q)} \end{cases} \tag{43}$$

Corollary 14. *Given the same conditions as in Corollary 13, it follows that*

$$|V_2\left(1\right)| \leq \frac{1+q}{q}\xi + 1. \tag{44}$$

Corollary 15. *Given the same conditions as in Corollary 13, it follows that*

$$|a_3 - a_1| \leq \begin{cases} \left(\frac{|3q-1|}{4q} + \frac{1+q}{q^2}\right)\xi^2 \\ +1, \qquad \xi \geq \frac{4q}{|3q-1|q+4(1+q)} \\ \\ \frac{\xi}{q} + 1, \quad \xi < \frac{4q}{|3q-1|q+4(1+q)} \end{cases} \tag{45}$$

Corollary 16. *Given the same conditions as in Corollary 13, it follows that*
(i) *when* $\xi \geq \frac{4q}{|3q-1|q+4(1+q)}$*, we have*

$$\begin{aligned} |V_3\left(1\right)| \leq & \left(\left(\frac{|3q-1|}{4q} + \frac{1+q}{q^2}\right)\xi^2 + \frac{1+q}{q}\xi\right) \\ & \times \left(\left(\frac{|3q-1|}{4q} + \frac{1+q}{q^2}\right)\xi^2 + 1\right) \\ & \times \left(\frac{1+q}{q}\xi + 1\right). \end{aligned} \tag{46}$$

(ii) *when* $\xi < \frac{4q}{|3q-1|q+4(1+q)}$*, we have*

$$|V_2\left(2\right)| \leq \begin{cases} \left(\frac{(1+q)^2\xi}{4q-(|3q-1|q+4(1+q))\xi} + \frac{\xi}{q}\right) \\ \times \left(\frac{1+q}{q}\xi + 1\right) \\ \times \left(\frac{\xi}{q} + 1\right), \qquad \xi < \frac{2q(1-q)}{|3q-1|q+4(1+q)} \\ \\ \left(\left(\frac{|3q-1|}{4q} + \frac{1+q}{q^2}\right)\xi^2 + \frac{1+q}{q}\xi\right) \\ \times \left(\frac{1+q}{q}\xi + 1\right) \\ \times \left(\frac{\xi}{q} + 1\right), \qquad \xi \geq \frac{2q(1-q)}{|3q-1|q+4(1+q)} \end{cases} \tag{47}$$

Letting $s = 1$, $b = 0$ and $q \to 1^-$ in Theorems 1-4, we get the following results.

Corollary 17. *If the function f of the form* (1) *belongs to the class* $\mathcal{ST}^{1,0}\left(\mathbb{L}_{\xi}\left(z\right)\right)$*, then*

$$|V_2\left(2\right)| \leq \frac{5}{2}\xi^2 + 2\xi. \tag{48}$$

The inequality is sharp for the function $f_0(z) = z\mathrm{e}^{\frac{1}{2}\xi\, z(\xi\, z-4)}$.

Corollary 18. *Given the same conditions as in Corollary 17, it follows that*

$$|V_2(1)| \leq 2\xi + 1. \tag{49}$$

The inequality is sharp for the function $f_0(z)$.

Corollary 19. *Given the same conditions as in Corollary 17, it follows that*

$$|a_3 - a_1| \leq \begin{cases} \frac{5}{2}\xi^2 + 1 & , \quad \xi \geq \frac{2}{5} \\ \xi + 1 & , \quad \xi < \frac{2}{5} \end{cases} \tag{50}$$

The inequality is sharp for the function $f_1(z) = z\mathrm{e}^{\frac{1}{2}i\xi\, z(4+i\xi\, z)}$ *if* $\xi \geq \frac{2}{5}$ *and for* $f_2(z) = \left(f_0(z^2)\right)^{\frac{1}{2}} = z\mathrm{e}^{\frac{1}{4}\xi\, z^2(\xi\, z^2-4)}$ *if* $\xi < \frac{2}{5}$.

Corollary 20. *Given the same conditions as in Corollary 17, it follows that*

$$|V_3(1)| \leq \begin{cases} \left(\frac{5}{2}\xi^2 + 2\xi\right)(2\xi+1) \\ \times\left(\frac{5}{2}\xi^2 + 1\right), & \xi \geq \frac{2}{5} \\ \left(\frac{5}{2}\xi^2 + 2\xi\right)(2\xi+1) \\ \times(\xi+1), & \xi < \frac{2}{5} \end{cases} \tag{51}$$

We are now going to calculate the results for class $\mathcal{CV}_q^{s,b}\left(\mathbb{L}_{q,\xi}(z)\right)$.

Theorem 5. *If the function f of the form* (1) *belongs to the class* $\mathcal{CV}_q^{s,b}\left(\mathbb{L}_{q,\xi}(z)\right)$ *and* $\sigma_q^{s,b}, \tau_q^{s,b}$ *are defined as in Theorem 1, then*
(i) *for* $\sigma_q^{s,b} \neq 0$, $\tau_q^{s,b} \neq 0$, $q \neq \frac{1}{3}$ *and when* $\xi \geq \frac{4\left|\sigma_q^{s,b}\right|}{|3q-1|\left|\sigma_q^{s,b}\right|+4|s+b|(1+q)}$, *we have*

$$\begin{aligned} |V_2(2)| &= |a_3 - a_2| \\ &\leq \frac{(1+q)}{[3]_q\left|\tau_q^{s,b}\right|}\left(\frac{|3q-1|}{4} + \frac{|s+b|(1+q)}{\left|\sigma_q^{s,b}\right|}\right)\xi^2 \\ &\quad + \frac{1}{\left|\sigma_q^{s,b}\right|}\xi \end{aligned} \tag{52}$$

(ii) *for* $\sigma_q^{s,b} \neq 0$, $\tau_q^{s,b} \neq 0$, $q \neq \frac{1}{3}$ *and when* $\xi < \frac{4\left|\sigma_q^{s,b}\right|}{|3q-1|\left|\sigma_q^{s,b}\right|+4|s+b|(1+q)}$, *we have*

$$|V_2(2)| \leq \begin{cases} \frac{[3]_q\left|\tau_q^{s,b}\right|\xi}{4\left|\sigma_q^{s,b}\right|^2(1+q)\left(1-\left(\frac{|3q-1|}{4}+\frac{|s+b|(1+q)}{\left|\sigma_q^{s,b}\right|}\right)\xi\right)} \\ \quad + \frac{1+q}{[3]_q\left|\tau_q^{s,b}\right|}\xi, \quad \xi < \Xi_3 \\ \frac{(1+q)}{[3]_q\left|\tau_q^{s,b}\right|}\left(\frac{|3q-1|}{4} + \frac{|s+b|(1+q)}{\left|\sigma_q^{s,b}\right|}\right)\xi^2 \\ \quad + \frac{1}{\left|\sigma_q^{s,b}\right|}\xi, \qquad \xi \geq \Xi_3 \end{cases} \tag{53}$$

where $\Xi_3 = \frac{4(1+q)|\sigma_q^{s,b}|-2[3]_q|\tau_q^{s,b}|}{|3q-1|(1+q)\left|\sigma_q^{s,b}\right|+4|s+b|(1+q)^2}$.
(iii) *for* $\sigma_q^{s,b} \neq 0$, $\tau_q^{s,b} \neq 0$, $q = \frac{1}{3}$, $s \neq -b$ *and when* $\xi \geq \frac{3\left|\sigma_{\frac{1}{3}}^{s,b}\right|}{4|s+b|}$, *we have*

$$|V_2(2)| \leq \frac{16\,|s+b|}{13\left|\sigma_{\frac{1}{3}}^{s,b}\right|\left|\tau_{\frac{1}{3}}^{s,b}\right|}\xi^2 + \frac{1}{\left|\sigma_{\frac{1}{3}}^{s,b}\right|}\xi. \tag{54}$$

(iv) *for* $\sigma_q^{s,b} \neq 0$, $\tau_q^{s,b} \neq 0$, $q = \frac{1}{3}$, $s \neq -b$ *and when* $\xi < \frac{3\left|\sigma_{\frac{1}{3}}^{s,b}\right|}{4|s+b|}$, *we have*

$$|V_2(2)| \leq \begin{cases} \frac{13\left|\tau_{\frac{1}{3}}^{s,b}\right|\xi}{16\left|\sigma_{\frac{1}{3}}^{s,b}\right|\left(3\left|\sigma_{\frac{1}{3}}^{s,b}\right|-4|s+b|\xi\right)} \\ \quad + \frac{12}{13\left|\tau_{\frac{1}{3}}^{s,b}\right|}\xi, \qquad \xi < \frac{24\left|\sigma_{\frac{1}{3}}^{s,b}\right|-13\left|\tau_{\frac{1}{3}}^{s,b}\right|}{32|s+b|} \\ \frac{16|s+b|}{13\left|\sigma_{\frac{1}{3}}^{s,b}\right|\left|\tau_{\frac{1}{3}}^{s,b}\right|}\xi^2 \\ \quad + \frac{1}{\left|\sigma_{\frac{1}{3}}^{s,b}\right|}\xi, \qquad \xi \geq \frac{24\left|\sigma_{\frac{1}{3}}^{s,b}\right|-13\left|\tau_{\frac{1}{3}}^{s,b}\right|}{32|s+b|} \end{cases} \tag{55}$$

(v) *for* $\sigma_q^{s,b} \neq 0$, $\tau_q^{s,b} \neq 0$, $q = \frac{1}{3}$ *and when* $s = -b$, $(b \neq 0)$, *we have*

$$|V_2(2)| \leq \left(\frac{13|13-9s^2|}{768} + \frac{108}{13|13-9s^2|}\right)\xi. \tag{56}$$

Proof. For $s, b \in \mathbb{C}$ with $s \neq b$, $|b| \leq 1$, $b \neq 1$ and for $0 < q < 1$, let $\sigma_q^{s,b}$ and $\tau_q^{s,b}$ be as defined in Theorem 1. Now for $\sigma_q^{s,b} \neq 0$, $\tau_q^{s,b} \neq 0$, $q \neq \frac{1}{3}$ and for $0 < \xi \leq \frac{1}{\sqrt{2}}$, let $f \in \mathcal{CV}_q^{s,b}\left(\mathbb{L}_{q,\xi}(z)\right)$ given by (1). Then there exists an analytic function $v(z)$ of the

form (2) such that

$$\frac{(s-b)\,\partial_q\left(z\partial_q f(z)\right)}{\partial_q\left(f(sz)-f(bz)\right)} = \mathbb{L}_{q,\xi}(v(z)) = \left(\frac{2\left(1+\xi v(z)\right)}{2+(1-q)\,\xi v(z)}\right)^2. \tag{57}$$

From the above equation, we have

$$1+[2]_q\sigma_q^{s,b}a_2z + \left([3]_q\tau_q^{s,b}a_3-(s+b)\,[2]_q^2\sigma_q^{s,b}a_2^2\right)z^2+\cdots = 1+h_1\left(q\right)\xi v_1z+\left(h_1(q)\xi v_2+h_2(q)\xi^2v_1^2\right)z^2+\cdots, \tag{58}$$

where $h_1(q)$ and $h_2(q)$ are as defined in Theorem 1. Equating the coefficients of z^n on both sides, we get

$$\begin{aligned} a_2 =& \frac{1}{\sigma_q^{s,b}}\xi v_1 \\ a_3 =& \frac{h_1\left(q\right)}{[3]_q\tau_q^{s,b}}\xi v_2+\frac{h_2\left(q\right)}{[3]_q\tau_q^{s,b}}\xi^2v_1^2 \\ &+\frac{(s+b)\left(h_1\left(q\right)\right)^2}{[3]_q\sigma_q^{s,b}\tau_q^{s,b}}\xi^2v_1^2. \end{aligned} \tag{59}$$

Using Lemma 2 and taking $|v_1|=d$ we obtain

$$\begin{aligned} |V_2\left(2\right)| =& |a_3-a_2| \\ \leq& \frac{h_1\left(q\right)}{[3]_q\left|\tau_q^{s,b}\right|}\xi\left(1-d^2\right)+\frac{|h_2\left(q\right)|}{[3]_q\left|\tau_q^{s,b}\right|}\xi^2d^2 \\ &+\frac{|s+b|\left(h_1\left(q\right)\right)^2}{[3]_q\left|\sigma_q^{s,b}\right|\left|\tau_q^{s,b}\right|}\xi^2d^2+\frac{1}{\left|\sigma_q^{s,b}\right|}\xi d. \end{aligned}$$

That is,

$$|V_2\left(2\right)| \leq \left(\frac{|h_2(q)|}{[3]_q\left|\tau_q^{s,b}\right|}\xi^2+\frac{|s+b|(h_1(q))^2}{[3]_q\left|\sigma_q^{s,b}\right|\left|\tau_q^{s,b}\right|}\xi^2-\frac{h_1(q)}{[3]_q\left|\tau_q^{s,b}\right|}\xi\right)d^2 + \frac{1}{\left|\sigma_q^{s,b}\right|}\xi d+\frac{h_1(q)}{[3]_q\left|\tau_q^{s,b}\right|}\xi =: \beta_0\left(d\right).$$

As $|v_1|\leq 1$ by Lemma 1, we observe that $d\in[0,1]$. Hence, we get

Case 1. When $\xi \geq \frac{h_1(q)|\sigma_q^{s,b}|}{|h_2(q)|\left|\sigma_q^{s,b}\right|+|s+b|(h_1(q))^2}$, $\beta_0\left(d\right)$ is an increasing function of d in $[0,1]$, since

$$\beta_0'\left(d\right) = 2\left(\frac{|h_2(q)|}{[3]_q\left|\tau_q^{s,b}\right|}\xi^2+\frac{|s+b|(h_1(q))^2}{[3]_q\left|\sigma_q^{s,b}\right|\left|\tau_q^{s,b}\right|}\xi^2-\frac{h_1(q)}{[3]_q\left|\tau_q^{s,b}\right|}\xi\right)d + \frac{1}{\left|\sigma_q^{s,b}\right|}\xi \geq 0$$

for $0\leq d\leq 1$.
Hence $\beta_0\left(d\right)$ attains its maximum value at $d=1$. Thus,

$$|V_2\left(2\right)| \leq \beta_0\left(1\right).$$

Case 2. When $\xi < \frac{h_1(q)|\sigma_q^{s,b}|}{|h_2(q)|\left|\sigma_q^{s,b}\right|+|s+b|(h_1(q))^2}$, $\beta_0\left(d\right)$ attains its maximum value either at d_0 for some $d_0\in(0,1)$ or at $d=1$. Indeed, when $\xi < \frac{2h_1(q)|\sigma_q^{s,b}|-[3]_q|\tau_q^{s,b}|}{2\left(|h_2(q)|\left|\sigma_q^{s,b}\right|+|s+b|(h_1(q))^2\right)}$, $\beta_0\left(d\right)$ attains its maximum value at

$$d_0 = \frac{1}{2\left|\sigma_q^{s,b}\right|\left(\frac{h_1(q)}{[3]_q\left|\tau_q^{s,b}\right|}-\left(\frac{|h_2(q)|}{[3]_q\left|\tau_q^{s,b}\right|}+\frac{|s+b|(h_1(q))^2}{[3]_q\left|\sigma_q^{s,b}\right|\left|\tau_q^{s,b}\right|}\right)\xi\right)},$$

$(0<d_0<1)$, since $\beta_0'\left(d_0\right)=0$ and $\beta_0''\left(d_0\right)<0$. Thus,

$$|V_2\left(2\right)| \leq \beta_0\left(d_0\right).$$

On the other hand, when $\xi \geq \frac{2h_1(q)|\sigma_q^{s,b}|-[3]_q|\tau_q^{s,b}|}{2\left(|h_2(q)|\left|\sigma_q^{s,b}\right|+|s+b|(h_1(q))^2\right)}$, $\beta_0\left(d\right)$ is an increasing function of d in $[0,1]$, since

$$\beta_0'\left(d\right) = 2\left(\frac{|h_2(q)|\xi^2}{[3]_q\left|\tau_q^{s,b}\right|}+\frac{|s+b|(h_1(q))^2\xi^2}{[3]_q\left|\sigma_q^{s,b}\right|\left|\tau_q^{s,b}\right|}-\frac{h_1(q)\xi}{[3]_q\left|\tau_q^{s,b}\right|}\right)d + \frac{1}{\left|\sigma_q^{s,b}\right|}\xi \geq 0$$

for $0\leq d\leq 1$.
Hence $\beta_0\left(d\right)$ attains its maximum value at $d=1$. Thus,

$$|V_2\left(2\right)| \leq \beta_0\left(1\right).$$

Now for $\sigma_q^{s,b}\neq 0$, $\tau_q^{s,b}\neq 0$, $q=\frac{1}{3}$ and when $s\neq -b$, the Taylor series expansion will become,

$$1+\frac{4}{3}\sigma_{\frac{1}{3}}^{s,b}a_2z + \left(\frac{13}{9}\tau_{\frac{1}{3}}^{s,b}a_3-\frac{16}{9}\left(s+b\right)\sigma_{\frac{1}{3}}^{s,b}a_2^2\right)z^2+\cdots = 1+\frac{4}{3}\xi v_1z+\frac{4}{3}\xi v_2z^2+\cdots, \tag{60}$$

Equating the coefficients of z^n on both sides, we get

$$\begin{aligned} a_2 =& \frac{1}{\sigma_{\frac{1}{3}}^{s,b}}\xi v_1 \\ a_3 =& \frac{12}{13\tau_{\frac{1}{3}}^{s,b}}\xi v_2+\frac{16\left(s+b\right)}{13\sigma_{\frac{1}{3}}^{s,b}\tau_{\frac{1}{3}}^{s,b}}\xi^2v_1^2. \end{aligned} \tag{61}$$

Using Lemma 2 and taking $|v_1| = d$ we obtain

$$|V_2(2)| \leq \left(\frac{16|s+b|}{13\left|\sigma_{\frac{1}{3}}^{s,b}\right|\left|\tau_{\frac{1}{3}}^{s,b}\right|}\xi^2 - \frac{12}{13\left|\tau_{\frac{1}{3}}^{s,b}\right|}\xi \right) d^2 + \frac{1}{\left|\sigma_{\frac{1}{3}}^{s,b}\right|}\xi d + \frac{12}{13\left|\tau_{\frac{1}{3}}^{s,b}\right|}\xi =: \beta_1(d).$$

Case 3. When $\xi \geq \frac{3\left|\sigma_{\frac{1}{3}}^{s,b}\right|}{4|s+b|}$, $\beta_1(d)$ is an increasing function of d in $[0,1]$. Hence,

$$|V_2(2)| \leq \beta_1(1).$$

Case 4. When $\xi < \frac{3\left|\sigma_{\frac{1}{3}}^{s,b}\right|}{4|s+b|}$, $\beta_1(d)$ attains its maximum value either at some $d_1 \in (0,1)$ or at $d = 1$. Hence, when $\xi < \frac{24\left|\sigma_{\frac{1}{3}}^{s,b}\right| - 13\left|\tau_{\frac{1}{3}}^{s,b}\right|}{32|s+b|}$, $|V_2(2)| \leq \beta_1(d_1)$, where $d_1 = \frac{13\left|\tau_{\frac{1}{3}}^{s,b}\right|}{24\left|\sigma_{\frac{1}{3}}^{s,b}\right| - 32|s+b|\xi}, (0 < d_1 < 1)$.

On the other hand, when $\xi \geq \frac{24\left|\sigma_{\frac{1}{3}}^{s,b}\right| - 13\left|\tau_{\frac{1}{3}}^{s,b}\right|}{32|s+b|}$, $\beta_1(d)$ attains its maximum value at $d = 1$. Thus,

$$|V_2(2)| \leq \beta_1(1).$$

Now for $\sigma_q^{s,b} \neq 0$, $\tau_q^{s,b} \neq 0$, $q = \frac{1}{3}$ and when $s = -b$, $(b \neq 0)$, the Taylor series expansion will become,

$$1 + \frac{16}{9}a_2 z + \frac{13}{9}\left(\frac{13}{9} - s^2\right)a_3 z^2 + \cdots = 1 + \frac{4}{3}\xi v_1 z + \frac{4}{3}\xi v_2 z^2 + \cdots, \tag{62}$$

Equating the coefficients of z^n on both sides, we get

$$a_2 = \frac{3}{4}\xi v_1$$
$$a_3 = \frac{108}{13(13 - 9s^2)}\xi v_2. \tag{63}$$

Using Lemma 2 and taking $|v_1| = d$ we obtain

$$|V_2(2)| \leq \frac{-108\xi d^2}{13|13-9s^2|} + \frac{3\xi d}{4} + \frac{108\xi}{13|13-9s^2|} =: \beta_2(d).$$

Case 5. $\beta_2(d)$ attains its maximum value at $d_2 = \frac{13|13-9s^2|}{288}, (0 < d_2 < 1)$, since $\beta_2'(d_2) = 0$ and $\beta_2''(d_2) < 0$. Thus, $|V_2(2)| \leq \beta_2(d_2)$. □

Theorem 6. *Given the same conditions as in Theorem 5, when $\sigma_q^{s,b} \neq 0$, it follows that*

$$|V_2(1)| = |a_2 - a_1| \leq \frac{1}{\left|\sigma_q^{s,b}\right|}\xi + 1. \tag{64}$$

Proof. By means of the proof of the Theorem 5, we have

$$|V_2(1)| = |a_2 - a_1| = \left|\frac{1}{\sigma_q^{s,b}}\xi v_1 - 1\right|.$$

Using Lemma 2 and taking $|v_1| = d$ we obtain

$$|V_2(1)| \leq \frac{1}{\left|\sigma_q^{s,b}\right|}\xi d + 1 =: \beta_3(d).$$

Since $\beta_3'(d) > 0$, $\beta_3(d)$ attains its maximum value at $d = 1$ in $[0,1]$.
Hence,

$$|V_2(1)| \leq \beta_3(1).$$

□

Theorem 7. *Given the same conditions as in Theorem 5, when $\sigma_q^{s,b} \neq 0$ and $\tau_q^{s,b} \neq 0$, it follows that*

$$|a_3 - a_1| \leq \begin{cases} \frac{(1+q)}{[3]_q\left|\tau_q^{s,b}\right|}\left(\frac{|3q-1|}{4} + \frac{|s+b|(1+q)}{\left|\sigma_q^{s,b}\right|}\right)\xi^2 \\ +1, \qquad \xi \geq \Xi_2 \\ \\ \frac{1+q}{[3]_q\left|\tau_q^{s,b}\right|}\xi + 1, \quad \xi < \Xi_2 \end{cases} \tag{65}$$

where Ξ_2 is given in Theorem 3.
(ii) if $q = \frac{1}{3}$ and $s \neq -b$, then

$$|a_3 - a_1| \leq \begin{cases} \frac{16|s+b|}{13\left|\sigma_{\frac{1}{3}}^{s,b}\right|\left|\tau_{\frac{1}{3}}^{s,b}\right|}\xi^2 + 1, \quad \xi \geq \frac{3\left|\sigma_{\frac{1}{3}}^{s,b}\right|}{4|s+b|} \\ \\ \frac{12}{13\left|\tau_{\frac{1}{3}}^{s,b}\right|}\xi + 1, \quad \xi < \frac{3\left|\sigma_{\frac{1}{3}}^{s,b}\right|}{4|s+b|} \end{cases} \tag{66}$$

(iii) if $q = \frac{1}{3}$ and $s = -b, (b \neq 0)$, then

$$|a_3 - a_1| \leq \frac{108}{13|13 - 9s^2|}\xi + 1. \tag{67}$$

Proof. By means of the proof of the Theorem 5, for $q \neq \frac{1}{3}$ and using Lemma 2 and taking $|v_1| = d$ we obtain

$$|a_3 - a_1| \leq \left(\frac{|h_2(q)|\xi^2}{[3]_q\left|\tau_q^{s,b}\right|} + \frac{|s+b|(h_1(q))^2\xi^2}{[3]_q\left|\sigma_q^{s,b}\right|\left|\tau_q^{s,b}\right|} - \frac{h_1(q)\xi}{[3]_q\left|\tau_q^{s,b}\right|}\right) d^2 + \frac{h_1(q)}{[3]_q\left|\tau_q^{s,b}\right|}\xi + 1 =: \beta_4(d).$$

As $|v_1| \leq 1$ by Lemma 1, we observe that $d \in [0,1]$. Hence, we get

Case 1. When $\xi \geq \frac{h_1(q)|\sigma_q^{s,b}|}{|h_2(q)|\left|\sigma_q^{s,b}\right|+|s+b|(h_1(q))^2}$, $\beta_4(d)$ is an increasing function of d in $[0,1]$, since

$$\beta_4'(d) = 2\left(\frac{|h_2(q)|\xi^2}{[3]_q\left|\tau_q^{s,b}\right|} + \frac{|s+b|(h_1(q))^2\xi^2}{[3]_q\left|\sigma_q^{s,b}\right|\left|\tau_q^{s,b}\right|} - \frac{h_1(q)\xi}{[3]_q\left|\tau_q^{s,b}\right|}\right) d \geq 0$$

for $0 \leq d \leq 1$.
Hence $\beta_4(d)$ attains its maximum value at $d = 1$. Thus,

$$|a_3 - a_1| \leq \beta_4(1).$$

Case 2. When $\xi < \frac{h_1(q)|\sigma_q^{s,b}|}{|h_2(q)|\left|\sigma_q^{s,b}\right|+|s+b|(h_1(q))^2}$, $\beta_4(d)$ is a decreasing function of d in $[0,1]$, since

$$\beta_4'(d) = 2\left(\frac{|h_2(q)|\xi^2}{[3]_q\left|\tau_q^{s,b}\right|} + \frac{|s+b|(h_1(q))^2\xi^2}{[3]_q\left|\sigma_q^{s,b}\right|\left|\tau_q^{s,b}\right|} - \frac{h_1(q)\xi}{[3]_q\left|\tau_q^{s,b}\right|}\right) d \leq 0$$

for $0 \leq d \leq 1$.
Hence $\beta_4(d)$ attains its maximum value at $d = 0$. Thus,

$$|a_3 - a_1| \leq \beta_4(0).$$

Now for $q = \frac{1}{3}$ and when $s \neq -b$, we obtain

$$|a_3 - a_1| \leq \left(\frac{16|s+b|}{13\left|\sigma_{\frac{1}{3}}^{s,b}\right|\left|\tau_{\frac{1}{3}}^{s,b}\right|}\xi^2 - \frac{12}{13\left|\tau_{\frac{1}{3}}^{s,b}\right|}\xi\right) d^2 + \frac{12}{13\left|\tau_{\frac{1}{3}}^{s,b}\right|}\xi + 1 =: \beta_5(d).$$

Case 3. When $\xi \geq \frac{3\left|\sigma_{\frac{1}{3}}^{s,b}\right|}{4|s+b|}$, $\beta_5(d)$ is an increasing function of d in $[0,1]$. Hence $|a_3 - a_1| \leq \beta_5(1)$. On the other hand, when $\xi < \frac{3\left|\sigma_{\frac{1}{3}}^{s,b}\right|}{4|s+b|}$, $\beta_5(d)$ is a decreasing function of d in $[0,1]$. Hence $|a_3 - a_1| \leq \beta_5(0)$.
Now for $q = \frac{1}{3}$ and when $s = -b$, $(b \neq 0)$, we obtain

$$|a_3 - a_1| \leq \frac{-108}{13\,|13 - 9s^2|}\xi d^2 + \frac{108}{13\,|13 - 9s^2|}\xi + 1 =: \beta_6(d).$$

Case 4. Since $\beta_6(d)$ is a decreasing function of d in $[0,1]$, we have $|a_3 - a_1| \leq \beta_6(0)$. □

Theorem 8. *Given the same conditions as in Theorem 5, when $\sigma_q^{s,b} \neq 0$ and $\tau_q^{s,b} \neq 0$, it follows that (i) if $q \neq \frac{1}{3}$ and $\xi \geq \frac{4|\sigma_q^{s,b}|}{|3q-1|\left|\sigma_q^{s,b}\right|+4|s+b|(1+q)}$, then*

$$|V_3(1)| \leq \left(\frac{(1+q)}{[3]_q\left|\tau_q^{s,b}\right|}\left(\frac{|3q-1|}{4} + \frac{|s+b|(1+q)}{\left|\sigma_q^{s,b}\right|}\right)\xi^2 + \frac{1}{\left|\sigma_q^{s,b}\right|}\xi\right) \times \left(\frac{(1+q)}{[3]_q\left|\tau_q^{s,b}\right|}\left(\frac{|3q-1|}{4} + \frac{|s+b|(1+q)}{\left|\sigma_q^{s,b}\right|}\right)\xi^2 + 1\right) \times \left(\frac{1}{\left|\sigma_q^{s,b}\right|}\xi + 1\right). \quad (68)$$

(ii) if $q \neq \frac{1}{3}$ and $\xi < \frac{4|\sigma_q^{s,b}|}{|3q-1|\left|\sigma_q^{s,b}\right|+4|s+b|(1+q)}$, then

$$|V_3(1)| \leq \begin{cases} \left(\frac{[3]_q\left|\tau_q^{s,b}\right|\xi}{4\left|\sigma_q^{s,b}\right|^2(1+q)\left(1-\left(\frac{|3q-1|}{4}+\frac{|s+b|(1+q)}{\left|\sigma_q^{s,b}\right|}\right)\xi\right)} + \frac{1+q}{[3]_q\left|\tau_q^{s,b}\right|}\xi\right)\left(\frac{1}{\left|\sigma_q^{s,b}\right|}\xi + 1\right) \times \left(\frac{1+q}{[3]_q\left|\tau_q^{s,b}\right|}\xi + 1\right), & \xi < \Xi_3 \\ \left(\frac{(1+q)}{[3]_q\left|\tau_q^{s,b}\right|}\left(\frac{|3q-1|}{4} + \frac{|s+b|(1+q)}{\left|\sigma_q^{s,b}\right|}\right)\xi^2 + \frac{1}{\left|\sigma_q^{s,b}\right|}\xi\right)\left(\frac{1}{\left|\sigma_q^{s,b}\right|}\xi + 1\right) \times \left(\frac{1+q}{[3]_q\left|\tau_q^{s,b}\right|}\xi + 1\right), & \xi \geq \Xi_3 \end{cases} \quad (69)$$

where Ξ_3 is given in Theorem 5.

(iii) if $q = \frac{1}{3}$, $s \neq -b$ and $\xi \geq \frac{3\left|\sigma_{\frac{1}{3}}^{s,b}\right|}{4|s+b|}$, then

$$|V_3(1)| \leq \left(\frac{16|s+b|}{13\left|\sigma_{\frac{1}{3}}^{s,b}\right|\left|\tau_{\frac{1}{3}}^{s,b}\right|}\xi^2 + \frac{1}{\left|\sigma_{\frac{1}{3}}^{s,b}\right|}\xi\right)\left(\frac{1}{\left|\sigma_{\frac{1}{3}}^{s,b}\right|}\xi + 1\right) \times \left(\frac{16|s+b|}{13\left|\sigma_{\frac{1}{3}}^{s,b}\right|\left|\tau_{\frac{1}{3}}^{s,b}\right|}\xi^2 + 1\right). \quad (70)$$

(iv) if $q=\frac{1}{3}$, $s\neq -b$ *and* $\xi<\frac{3\left|\sigma_{\frac{1}{3}}^{s,b}\right|}{4|s+b|}$, *then*

$$|V_3(1)|\leq\begin{cases}\left(\frac{13\left|\tau_{\frac{1}{3}}^{s,b}\right|\xi}{16\left|\sigma_{\frac{1}{3}}^{s,b}\right|\left(3\left|\sigma_{\frac{1}{3}}^{s,b}\right|-4|s+b|\xi\right)}+\frac{12\xi}{13\left|\tau_{\frac{1}{3}}^{s,b}\right|}\right)\\ \times\left(\frac{1}{\left|\sigma_{\frac{1}{3}}^{s,b}\right|}\xi+1\right)\\ \times\left(\frac{12\xi}{13\left|\tau_{\frac{1}{3}}^{s,b}\right|}+1\right), \quad \xi<\frac{24\left|\sigma_{\frac{1}{3}}^{s,b}\right|-13\left|\tau_{\frac{1}{3}}^{s,b}\right|}{32|s+b|}\\ \left(\frac{16|s+b|}{13\left|\sigma_{\frac{1}{3}}^{s,b}\right|\left|\tau_{\frac{1}{3}}^{s,b}\right|}\xi^2+\frac{\xi}{\left|\sigma_{\frac{1}{3}}^{s,b}\right|}\right)\\ \times\left(\frac{1}{\left|\sigma_{\frac{1}{3}}^{s,b}\right|}\xi+1\right)\\ \times\left(\frac{12\xi}{13\left|\tau_{\frac{1}{3}}^{s,b}\right|}+1\right), \quad \xi\geq\frac{24\left|\sigma_{\frac{1}{3}}^{s,b}\right|-13\left|\tau_{\frac{1}{3}}^{s,b}\right|}{32|s+b|}\end{cases}\tag{71}$$

(v) if $q=\frac{1}{3}$ *and* $s=-b$, $(b\neq 0)$, *then*

$$|V_3(1)|\leq\left(\frac{13|13-9s^2|}{768}+\frac{108}{13|13-9s^2|}\right)\times\xi\left(\frac{3}{4}\xi+1\right)\left(\frac{108}{13|13-9s^2|}\xi+1\right).\tag{72}$$

Proof. Utilizing the results from Theorem 5 to Theorem 7, we obtain the desired result. □

Letting $q\to 1^-$ in Theorems 5-8, we get the following results.

Corollary 21. *If the function f of the form* (1) *belongs to the class* $\mathcal{CV}^{s,b}(\mathbb{L}_\xi(z))$ *and* $\sigma^{s,b}$, $\tau^{s,b}$ *are defined as in Corollary 1, then*
(i) for $\sigma^{s,b}\neq 0$, $\tau^{s,b}\neq 0$ *and when* $\xi\geq\frac{2|\sigma^{s,b}|}{|\sigma^{s,b}|+4|s+b|}$, *we have*

$$|V_2(2)|\leq\frac{2}{3|\tau^{s,b}|}\left(\frac{1}{2}+\frac{2|s+b|}{|\sigma^{s,b}|}\right)\xi^2+\frac{1}{|\sigma^{s,b}|}\xi\tag{73}$$

(ii) for $\sigma^{s,b}\neq 0$, $\tau^{s,b}\neq 0$ *and when* $\xi<\frac{2|\sigma^{s,b}|}{|\sigma^{s,b}|+4|s+b|}$, *we have*

$$|V_2(2)|\leq\begin{cases}\frac{3|\tau^{s,b}|\xi}{8|\sigma^{s,b}|^2\left(1-\left(\frac{1}{2}+\frac{2|s+b|}{|\sigma^{s,b}|}\right)\xi\right)}\\ +\frac{2}{3|\tau^{s,b}|}\xi, \quad \xi<\frac{4|\sigma^{s,b}|-3|\tau^{s,b}|}{2|\sigma^{s,b}|+8|s+b|}\\ \frac{2}{3|\tau^{s,b}|}\left(\frac{1}{2}+\frac{2|s+b|}{|\sigma^{s,b}|}\right)\xi^2\\ +\frac{1}{|\sigma^{s,b}|}\xi, \quad \xi\geq\frac{4|\sigma^{s,b}|-3|\tau^{s,b}|}{2|\sigma^{s,b}|+8|s+b|}\end{cases}\tag{74}$$

Corollary 22. *Given the same conditions as in Corollary 21, when* $\sigma^{s,b}\neq 0$, *it follows that*

$$|V_2(1)|\leq\frac{\xi}{|\sigma^{s,b}|}+1.\tag{75}$$

Corollary 23. *Given the same conditions as in Corollary 21, when* $\sigma^{s,b}\neq 0$, *it follows that*

$$|a_3-a_1|\leq\begin{cases}\frac{2}{3|\tau^{s,b}|}\left(\frac{1}{2}+\frac{2|s+b|}{|\sigma^{s,b}|}\right)\xi^2\\ +1, \qquad \xi\geq\frac{2|\sigma^{s,b}|}{|\sigma^{s,b}|+4|s+b|}\\ \frac{2}{3|\tau^{s,b}|}\xi+1, \quad \xi<\frac{2|\sigma^{s,b}|}{|\sigma^{s,b}|+4|s+b|}\end{cases}\tag{76}$$

Corollary 24. *Given the same conditions as in Corollary 21, when* $\sigma^{s,b}\neq 0$, *it follows that*
(i) if $\sigma^{s,b}\neq 0$, $\tau^{s,b}\neq 0$ *and* $\xi\geq\frac{2|\sigma^{s,b}|}{|\sigma^{s,b}|+4|s+b|}$, *we have*

$$|V_3(1)|\leq\left(\frac{2}{3|\tau^{s,b}|}\left(\frac{1}{2}+\frac{2|s+b|}{|\sigma^{s,b}|}\right)\xi^2+\frac{1}{|\sigma^{s,b}|}\xi\right)\times\left(\frac{2}{3|\tau^{s,b}|}\left(\frac{1}{2}+\frac{2|s+b|}{|\sigma^{s,b}|}\right)\xi^2+1\right)\times\left(\frac{\xi}{|\sigma^{s,b}|}+1\right).\tag{77}$$

(ii) if $\sigma^{s,b}\neq 0$, $\tau^{s,b}\neq 0$ *and* $\xi<\frac{2|\sigma^{s,b}|}{|\sigma^{s,b}|+4|s+b|}$, *we have*

$$|V_3(1)|\leq\begin{cases}\left(\frac{3|\tau^{s,b}|\xi}{8|\sigma^{s,b}|^2\left(1-\left(\frac{1}{2}+\frac{2|s+b|}{|\sigma^{s,b}|}\right)\xi\right)}+\frac{2}{3|\tau^{s,b}|}\xi\right)\\ \times\left(\frac{\xi}{|\sigma^{s,b}|}+1\right)\\ \times\left(\frac{2\xi}{3|\tau^{s,b}|}+1\right), \quad \xi<\frac{4|\sigma^{s,b}|-3|\tau^{s,b}|}{2|\sigma^{s,b}|+8|s+b|}\\ \left(\frac{2}{3|\tau^{s,b}|}\left(\frac{1}{2}+\frac{2|s+b|}{|\sigma^{s,b}|}\right)\xi^2+\frac{1}{|\sigma^{s,b}|}\xi\right)\\ \times\left(\frac{\xi}{|\sigma^{s,b}|}+1\right)\\ \times\left(\frac{2\xi}{3|\tau^{s,b}|}+1\right), \quad \xi\geq\frac{4|\sigma^{s,b}|-3|\tau^{s,b}|}{2|\sigma^{s,b}|+8|s+b|}\end{cases}\tag{78}$$

Letting $s=1$ and $b=-1$ in Theorems 5-8, we get the following results.

Corollary 25. *If the function f of the form* (1) *belongs to the class* $\mathcal{CV}_q^{1,-1}(\mathbb{L}_{q,\xi}(z))$, *then*
(i) for $q\neq\frac{1}{3}$ *we have*

$$|V_2(2)|\leq\begin{cases}\frac{q(1+q+q^2)\xi}{(1+q)^2(4-|3q-1|\xi)}\\ +\frac{\xi}{q(1+q+q^2)}, \quad \xi<\frac{4+2q-2q^2-2q^3}{|3q-1|(1+q)}\\ \frac{|3q-1|\xi^2}{4q(1+q+q^2)}\\ +\frac{\xi}{1+q}, \qquad \xi\geq\frac{4+2q-2q^2-2q^3}{|3q-1|(1+q)}\end{cases}\tag{79}$$

(ii) for $q=\frac{1}{3}$ *we have*

$$|V_2(2)| \leq \frac{5353}{2496}\xi. \tag{80}$$

Corollary 26. *Given the same conditions as in Corollary 25, it follows that*

$$|V_2(1)| \leq \frac{\xi}{1+q}+1. \tag{81}$$

Corollary 27. *Given the same conditions as in Corollary 25, it follows that*

$$|a_3-a_1| \leq \frac{\xi}{q\left(1+q+q^2\right)}+1. \tag{82}$$

Corollary 28. *Given the same conditions as in Corollary 25, it follows that*
(i) for $q \neq \frac{1}{3}$ *we have*

$$|V_3(1)| \leq \begin{cases} \left(\frac{q(1+q+q^2)\xi}{(1+q)^2(4-|3q-1|\xi)}+\frac{\xi}{q(1+q+q^2)}\right) \\ \times\left(\frac{\xi}{1+q}+1\right) \\ \times\left(\frac{\xi}{q(1+q+q^2)}+1\right), & \xi<\frac{4+2q-2q^2-2q^3}{|3q-1|(1+q)} \\ \left(\frac{|3q-1|\xi^2}{4q(1+q+q^2)}+\frac{\xi}{1+q}\right) \\ \times\left(\frac{\xi}{1+q}+1\right) \\ \times\left(\frac{\xi}{q(1+q+q^2)}+1\right), & \xi \geq \frac{4+2q-2q^2-2q^3}{|3q-1|(1+q)} \end{cases} \tag{83}$$

(ii) for $q=\frac{1}{3}$ *we have*

$$|V_3(1)| \leq \left(\frac{5353}{2496}\xi\right)\left(\frac{3}{4}\xi+1\right)\left(\frac{27}{13}\xi+1\right). \tag{84}$$

Letting $s=1$, $b=-1$ and $q \to 1^-$ in Theorems 5-8, we get the following results.

Corollary 29. *If the function f of the form* (1) *belongs to the class* $\mathcal{CV}^{1,-1}\left(\mathbb{L}_{\xi}(z)\right)$*, then*

$$|V_2(2)| \leq \begin{cases} \frac{3\xi}{8(2-\xi)}+\frac{\xi}{3} & , \quad \xi<\frac{1}{2} \\ \frac{\xi^2}{6}+\frac{\xi}{2} & , \quad \xi \geq \frac{1}{2} \end{cases} \tag{85}$$

Corollary 30. *Given the same conditions as in Corollary 29, it follows that*

$$|V_2(1)| \leq \frac{\xi}{2}+1. \tag{86}$$

Corollary 31. *Given the same conditions as in Corollary 29, it follows that*

$$|a_3-a_1| \leq \frac{\xi}{3}+1. \tag{87}$$

Corollary 32. *Given the same conditions as in Corollary 29, it follows that*

$$|V_3(1)| \leq \begin{cases} \left(\frac{3\xi}{8(2-\xi)}+\frac{\xi}{3}\right)\left(\frac{\xi}{2}+1\right) \\ \times\left(\frac{\xi}{3}+1\right), & \xi<\frac{1}{2} \\ \left(\frac{\xi^2}{6}+\frac{\xi}{2}\right)\left(\frac{\xi}{2}+1\right) \\ \times\left(\frac{\xi}{3}+1\right), & \xi \geq \frac{1}{2} \end{cases} \tag{88}$$

Letting $s=1$ and $b=0$ in Theorems 5-8, we get the following results.

Corollary 33. *If the function f of the form* (1) *belongs to the class* $\mathcal{CV}_q^{1,0}\left(\mathbb{L}_{q,\xi}(z)\right)$*, then*
(i) when $\xi \geq \frac{4q}{|3q-1|q+4(1+q)}$*, we have*

$$|V_2(2)| \leq \frac{1}{q(1+q+q^2)}\left(\frac{|3q-1|}{4}+\frac{1+q}{q}\right)\xi^2+\frac{\xi}{q}. \tag{89}$$

(ii) when $\xi < \frac{4q}{|3q-1|q+4(1+q)}$*, we have*

$$|V_2(2)| \leq \begin{cases} \frac{(1+q+q^2)\xi}{4q-(|3q-1|q+4(1+q))\xi} \\ +\frac{\xi}{q(1+q+q^2)}, & \xi<\frac{2q(1-q-q^2)}{|3q-1|q+4(1+q)} \\ \frac{1}{q(1+q+q^2)}\left(\frac{|3q-1|}{4}+\frac{1+q}{q}\right)\xi^2 \\ +\frac{1}{q}\xi, & \xi \geq \frac{2q(1-q-q^2)}{|3q-1|q+4(1+q)} \end{cases} \tag{90}$$

Corollary 34. *Given the same conditions as in Corollary 33, it follows that*

$$|V_2(1)| \leq \frac{1}{q}\xi+1. \tag{91}$$

Corollary 35. *Given the same conditions as in Corollary 33, it follows that*

$$|a_3-a_1| \leq \begin{cases} \frac{1}{q(1+q+q^2)}\left(\frac{|3q-1|}{4}+\frac{1+q}{q}\right)\xi^2 \\ +1, & \xi \geq \frac{4q}{|3q-1|q+4(1+q)} \\ \frac{\xi}{q(1+q+q^2)} \\ +1, & \xi<\frac{4q}{|3q-1|q+4(1+q)} \end{cases} \tag{92}$$

Corollary 36. *Given the same conditions as in Corollary 33, it follows that*
(i) when $\xi \geq \frac{4q}{|3q-1|q+4(1+q)}$*, we have*

$$\begin{aligned} |V_3(1)| \leq & \left(\frac{1}{q(1+q+q^2)}\left(\frac{|3q-1|}{4}+\frac{1+q}{q}\right)\xi^2+\frac{1}{q}\xi\right) \\ & \times\left(\frac{1}{q(1+q+q^2)}\left(\frac{|3q-1|}{4}+\frac{1+q}{q}\right)\xi^2+1\right) \\ & \times\left(\frac{1}{q}\xi+1\right). \end{aligned} \tag{93}$$

(ii) when $\xi < \frac{4q}{|3q-1|q+4(1+q)}$*, we have*

$$|V_2(2)| \leq \begin{cases} \left(\frac{(1+q+q^2)\xi}{4q-(|3q-1|q+4(1+q))\xi} + \frac{\xi}{q(1+q+q^2)}\right) \\ \times \left(\frac{\xi}{q(1+q+q^2)} + 1\right) \\ \times \left(\frac{1}{q}\xi + 1\right), \quad \xi < \frac{2q(1-q-q^2)}{|3q-1|q+4(1+q)} \\ \left(\frac{1}{q(1+q+q^2)}\left(\frac{|3q-1|}{4} + \frac{1+q}{q}\right)\xi^2 + \frac{1}{q}\xi\right) \\ \times \left(\frac{\xi}{q(1+q+q^2)} + 1\right) \\ \times \left(\frac{1}{q}\xi + 1\right), \quad \xi \geq \frac{2q(1-q-q^2)}{|3q-1|q+4(1+q)} \end{cases} \tag{94}$$

Letting $s = 1$, $b = 0$ and $q \to 1^-$ in Theorems 5-8, we get the following results.

Corollary 37. *If the function f of the form* (1) *belongs to the class* $\mathcal{CV}^{1,0}(\mathbb{L}_\xi(z))$*, then*

$$|V_2(2)| \leq \frac{5}{6}\xi^2 + \xi. \tag{95}$$

Corollary 38. *Given the same conditions as in Corollary 37, it follows that*

$$|V_2(1)| \leq \xi + 1. \tag{96}$$

Corollary 39. *Given the same conditions as in Corollary 37, it follows that*

$$|a_3 - a_1| \leq \begin{cases} \frac{5}{6}\xi^2 + 1 & , \quad \xi \geq \frac{2}{5} \\ \frac{\xi}{3} + 1 & , \quad \xi < \frac{2}{5} \end{cases} \tag{97}$$

Corollary 40. *Given the same conditions as in Corollary 37, it follows that*

$$|V_3(1)| \leq \begin{cases} \left(\frac{5}{6}\xi^2 + \xi\right)(\xi + 1) \\ \times \left(\frac{5}{6}\xi^2 + 1\right) & , \quad \xi \geq \frac{2}{5} \\ \left(\frac{5}{6}\xi^2 + \xi\right)(\xi + 1) \\ \times \left(\frac{\xi}{3} + 1\right) & , \quad \xi < \frac{2}{5} \end{cases} \tag{98}$$

It should be noted that the classes $\mathcal{ST}^{1,-1}(\mathbb{L}\xi(z))$ and $\mathcal{CV}^{1,-1}(\mathbb{L}\xi(z))$ were introduced by [17]. Our findings, as presented in Corollaries 29 to 32, improve upon those results.

4 Conclusion

In conclusion, this study successfully defined the p-th Vandermonde determinant and established the coefficient bounds for the second- and third-order Vandermonde determinants for a Sakaguchi type function within the q-Limaçon domain. The results obtained provide significant insights into the behavior and properties of these determinants, contributing to the broader understanding of analytic functions in complex domains. Future research could explore higher-order Vandermonde determinants and their applications in other mathematical contexts. Univalent functions have numerous applications, including quantum calculus (q-calculus), special functions, signal processing, and image processing. The results presented in this paper can be utilized to further extend these applications, leading to new advancements in our field.

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