A Spectrum of Semi-Perfect Functions in Topology: Classification and Implications

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Abstract: The distinctive features of common issues include large amounts of data and different degrees of uncertainty. Therefore, creating novel mathematical techniques to address problems is essential; it is expected that perfect functions will serve as the most useful tool in this situation. As a result, we look into various set operator methods for producing perfect functions in this study. The relationships between particular classes of perfect functions and associated conventional topologies are linked with symmetry. Through alignment, we can investigate the characteristics and behavior of traditional topological ideas by studying sets. Under the structure of primordial topological spaces, the current investigation proposes and discusses several fresh categories of perfect functions, which include semi-perfect functions and semi-Lindelöf perfect functions. The traits they possess and how they relate to different roles are investigated using instances, alternative examples, and expansions. The study goes over how to calculate the cartesian product of arbitrary unions and finite intersections.

Key-Words: Topological Spaces; Compact Spaces; Perfect Functions; Semi-Lindelöf Spaces; Applications in Topology

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1 Introduction

Common problems are distinguished by their abundance of data and varying levels of ambiguity. Consequently, developing new mathematical methods to solve issues is crucial, and it is anticipated that the perfect functions will be the most helpful instrument in this case. Therefore, in this work, we investigate different set operator strategies for generating perfect functions.

Among the most significant extensions of a topological space are perfect functions. Through the study of general topology, we can infer that closed sets are essential to the development of new set forms and fundamental topological characteristics. Based on a set $\mathscr{M}$ with topology $\mathscr{T}$, consider $(\mathscr{M}, \mathscr{T})$ to represent a topological space. For a subset $\mathscr{A}$ of $\mathscr{M}$, the closure of $\mathscr{A}$ is denoted by $Cl(\mathscr{A})$, while the interior of $\mathscr{A}$ is denoted by $Int(\mathscr{A})$. The topology on $\mathscr{A}$ inherited from $\mathscr{T}$ will be denoted by $\mathscr{T}_{\mathscr{A}}$ respectively.

Numerous mathematicians have created and investigated several types of semi-sets in topological spaces, which include semi-open sets, [1], semi-compact set, [2], semi-lindelöf sets, [3], and semi-continuity, [1]. They have used the concept of semi topological spaces and semi compact, specifically, to introduce various mathematical structures, including semi compact spaces in a neutrosophic crisp topological space, [4], study the structure of difference lindelof̈ topological spaces and their properties, [5], semi normal-spaces, [6], semi topological groups, [7], semi compact sets, [8], soft topological space via semi-open and semi-closed soft sets, [9], soft semi-compact spaces, [10], semi compactness, [11], semi locally connected sets, [12], strongly star semi compactness, [13], generalization and application of locally semi compact spaces, [14], semi compactness with Respect to a Euclidean cone, [15], semi-monotone sets, [16], semi generalized continuous maps, [17], semi continuous mapping, [18], and compact semi topological semi-groups, [19].

This work explores some of the fundamental features of emerging categories of semi functions, which can be described as aspects of the semi sets on primitive topological spaces. We looked further into the emotional connection that exist between them and provide instances of non-compatible relationships. A selection of intriguing perfect function deconstruction theorems is introduced using the new classes of semi-perfect functions that we outline. Ultimately, enlargement is examined through the use of cases and substitute examples. The study covers the computation of the cartesian product of finite intersections and randomized unions. The article is organized as follows: In section 2, We recall definitions and facts because they are important in the content of our paper. In section 3, we study and introduce the concept of semi-perfect functions. Furthermore, we investigate the properties of semi-lindelöf perfect functions. In section 4, we provide additional details and perspectives on different forms of semi-perfect functions. In section 5, we study the types of semi-perfect functions and relationships between them. In section 6, we investigate numerous types of applications, noting how they might improve productivity and promote change in multi-variable models in various of research domains.

2 Definitions and Primary Outcomes

Definition 2.1. [1] (i) If $\mathscr{A} \subseteq Cl(Int(\mathscr{A}))$, then $\mathscr{A}$ is said to be semi-open.

(ii) If $Int(Cl(\mathscr{A})) \subseteq \mathscr{A}$, then $\mathscr{A}$ is said to be semi-closed.

Definition 2.2. [1] If $\mathscr{M}$ and $\mathscr{L}$ are spaces and $\urcorner : \mathscr{M} \to \mathscr{L}$, then the function ϕ is called semi-continuous if the inverse image of each open set of $\mathscr{L}$ is semi-open in $\mathscr{M}$.

Definition 2.3. [20] A function $\urcorner : (\mathscr{M}, \mathscr{T}) \to (\mathscr{L}, \varsigma)$ is called Lindelöf perfect if it satisfies the following conditions: it is continuous, closed, and $\urcorner^{-1}(l)$ is a Lindelöf for every $l \in \mathscr{L}$.

Definition 2.4. A topological space $(\mathscr{M}, \mathscr{T})$ is called :

(i) [2], semi-compact if any semi-open cover of $\mathscr{M}$ has a finite subcover.

(ii) [3], semi-Lindelöf if any semi-open cover has a countable subcover.

Definition 2.5. [21], Consider topological spaces $\mathscr{M}$ and $\mathscr{L}$. If and only if there is a function $\urcorner : \mathscr{M} \to \mathscr{L}$ that is irresolute, one-to-one, onto, and pre-semi-open, then $\mathscr{M}$ and $\mathscr{L}$ are considered semi-homomorphic. We refer to such a $\urcorner$ as a semi-homeomorphism.

Definition 2.6. [21], Given that $\urcorner : \mathscr{M} \to \mathscr{L}$ is irresolute if and only if, for any semi-open subset Z of $\mathscr{L}$, $\urcorner^{-1}(Z)$ is semi-open in $\mathscr{M}$.

Definition 2.7. [21], Given that $\urcorner : \mathscr{M} \to \mathscr{L}$ is pre-semi open if and only if, for all $\mathscr{A} \in SO(\mathscr{M}), \urcorner(\mathscr{A}) \in \mathscr{L}$.

Definition 2.8. [22], A topological space $(\mathscr{M}, \mathscr{T})$ is semi-T_2, If for any two distinct points $p, q \in \mathscr{M}$, there exist semi-open sets $\mathrm{SO}(p)$ and $\mathrm{SO}(q)$ such that $\mathrm{SO}(p) \cap \mathrm{SO}(q) = \emptyset$. Here, $\mathrm{SO}(m)$ denotes a semi-open set contains the provided point m.

Definition 2.9. [6], A space $(\mathscr{M}, \mathscr{T})$ is semi-normal if and only if for any pair of disjoint semi-closed sets $\mathscr{A}$ and $\mathscr{B}$, there exist disjoint semi-open sets U and V such that $\mathscr{A} \subset U$ and $\mathscr{B} \subset V$.

Definition 2.10. [23], A space $(\mathscr{M}, \mathscr{T})$ is semi-regular if and only if for any semi-closed set $\mathscr{A}$ and $m \notin \mathscr{A}$, there exist disjoint semi-open sets U and V such that $m \in U$ and $\mathscr{A} \subset V$.

Definition 2.11. [24], A space $(\mathscr{M}, \mathscr{T})$ is semi-paracompact space if any open cover of $\mathscr{M}$ has a $\mathscr{T}$-locally finite open refinement.

Definition 2.12. [20], Given that $\urcorner : (\mathscr{M}, \mathscr{T}) \to (\mathscr{L}, \varsigma)$ is a perfect function (resp. Lindelöf perfect function), for every compact (resp. Lindelöf) $Z \subset \mathscr{L}$, $\urcorner^{-1}(Z)$ is compact (resp. Lindelöf).

3 Semi Perfect Function

This section concentrates on the study of semi-perfect functions and introduces and explores semi-lindelöf perfect functions.

Theorem 3.1. *Given that $\urcorner : (\mathscr{M}, \mathscr{T}) \to (\mathscr{L}, \varsigma)$ is a semi-perfect function, then the preimage $\urcorner^{-1}(Z)$ is semi-compact for any semi-compact subset $Z \subset \mathscr{L}$.*

Proof. Consider $\mathscr{P} = \{P_\alpha : \alpha \in \Lambda\}$ be an open cover of $(\mathscr{M}, \mathscr{T})$. Since $\urcorner$ is a semi-perfect function, for every $l \in \mathscr{L}$, the preimage $\urcorner^{-1}(l)$ is semi-compact. This means there exists a finite subset $\Lambda_l \subseteq \Lambda$ such that

$$\urcorner^{-1}(l) \subseteq \bigcup_{\alpha \in \Lambda_l} Q_\alpha,$$

where each Q_α is semi-open in the topology $\mathtt{S}(\mathscr{T})$.

Now, define $O_l = \mathscr{L} - \urcorner(\mathscr{M} - \bigcup Q_\alpha)$. This set O_l is semi-open in $\mathtt{S}(\varsigma)$ and contains l. Additionally, the preimage satisfies

$$\urcorner^{-1}(O_l) \subseteq \bigcup Q_\alpha.$$

Consider the collection $\mathscr{O} = \{O_l : l \in \mathscr{L}\}$. This forms a semi-open cover of $\mathscr{L}$. Since $\mathscr{L}$ is semi-compact, there exists a finite subcover such that

$$\mathscr{L} \subseteq \bigcup_{i=1}^{n} O_{l_i}.$$

Finally, we observe that

$$\daleth^{-1}(\mathscr{L}) \subseteq \bigcup_{i=1}^{n} \daleth^{-1}(O_{l_i}),$$

which is means $\daleth^{-1}(\mathscr{L})$ is semi-compact. □

Corollary 3.1. *A semi-compact space is inverse invariant under semi-perfect function.*

To validate the following theorem, we will adopt a proof technique similar to that employed in the preceding theorem.

Theorem 3.2. *Given that $\daleth : (\mathscr{M}, \mathscr{T}) \to (\mathscr{L}, \varsigma)$ is a semi-Lindelöf perfect function, for each semi-Lindelöf $L \subset \mathscr{L}$, the preimage $\daleth^{-1}(L)$ is semi-Lindelöf.*

Proof. Let $\mathscr{U} = \{U_\alpha : \alpha \in \Lambda\}$ be a semi-open cover of $\daleth^{-1}(L)$ in $(\mathscr{M}, \mathscr{T})$. Since $\daleth$ is a semi-Lindelöf perfect function, the image of each point $l \in \mathscr{L}$ has a preimage $\daleth^{-1}(l)$ that is semi-compact. Thus, for any $l \in \mathscr{L}$, there exists a finite subcollection $\Lambda_l \subseteq \Lambda$ such that

$$\daleth^{-1}(l) \subseteq \bigcup_{\alpha \in \Lambda_l} V_\alpha,$$

where $\{V_\alpha : \alpha \in \Lambda_l\}$ consists of semi-open sets in $\mathtt{S}(\mathscr{T})$.

Now, define

$$G_l = \mathscr{L} - \daleth\left(\mathscr{M} - \bigcup V_\alpha\right),$$

which is semi-open in $\mathtt{S}(\varsigma)$, contains l, and satisfies $\daleth^{-1}(G_l) \subseteq \bigcup V_\alpha$. Let $\mathscr{G} = \{G_l : l \in \mathscr{L}\}$ be a semi-open cover of the subset L in $(\mathscr{L}, \varsigma)$.

Since L is semi-Lindelöf, there exists a countable subcover $\{G_{l_k} : k = 1, 2, \dots\}$ such that

$$L \subseteq \bigcup_{k=1}^{\infty} G_{l_k}.$$

Consequently,

$$\daleth^{-1}(L) \subseteq \bigcup_{k=1}^{\infty} \daleth^{-1}(G_{l_k}),$$

which is a countable subcover of $\mathscr{U}$ for $\daleth^{-1}(L)$. Therefore, $\daleth^{-1}(L)$ is semi-Lindelöf. □

Corollary 3.2. *A semi-Lindelöf space is preserved under semi-Lindelöf perfect function.*

Theorem 3.3. *A perfect function can be expressed as an outcome of combining two perfect functions.*

Proof. let $\daleth : (\mathscr{M}, \mathscr{T}) \to (\mathscr{L}, \varsigma)$ and $\gamma : (\mathscr{L}, \varsigma) \to (Z, \eta)$ be perfect functions, where $(\mathscr{M}, \mathscr{T})$, $(\mathscr{L}, \varsigma)$, and (Z, η) are topological spaces.

Since both $\daleth$ and γ are perfect, they are closed and map compact sets to compact sets. For any compact subset $K \subset Z$, since γ is perfect, the preimage $\gamma^{-1}(K)$ is compact in $\mathscr{L}$. Then, because $\daleth$ is also perfect, the preimage $\daleth^{-1}(\gamma^{-1}(K)) = (\gamma \circ \daleth)^{-1}(K)$ is compact in $\mathscr{M}$. Thus, $\gamma \circ \daleth$ maps compact sets to compact sets.

Since $\daleth$ and γ are closed maps, for any closed set $F \subset \mathscr{M}$, the image $\daleth(F)$ is closed in $\mathscr{L}$, and then $\gamma(\daleth(F)) = (\gamma \circ \daleth)(F)$ is closed in Z. Thus, $\gamma \circ \daleth$ is a closed map.

Since $\gamma \circ \daleth$ is closed and maps compact sets to compact sets, it follows that $\gamma \circ \daleth$ is a perfect function. □

The following illustration demonstrates that the composition of two semi-perfect functions need not be true:

Example 3.1. Consider, $\daleth : (\mathscr{M}, \mathscr{T}) \to (\mathscr{L}, \varsigma)$ and $\gamma : (\mathscr{L}, \varsigma) \to (Z, \eta)$, where:

$$\mathscr{M} = \{k, m, c\}, \quad \mathscr{L} = \{a, e, f\}, \quad Z = \{r, h, i\},$$

with the following topologies:

$$\mathscr{T} = \{\mathscr{M}, \emptyset, \{k\}, \{k, c\}\},$$

$$\varsigma = \{\mathscr{L}, \emptyset, \{a\}, \{e\}, \{a, e\}\},$$

$$\eta = \{Z, \emptyset, \{r\}\}.$$

Define

$$\daleth(k) = a, \quad \daleth(m) = f, \quad \daleth(c) = e,$$

$$\gamma(a) = r, \quad \gamma(e) = i, \quad \gamma(f) = h.$$

It can be verified that both $\daleth$ and γ are semi-continuous; however, $\gamma \circ \daleth$ is not semi-continuous. Specifically,

$$(\gamma \circ \daleth)^{-1}(\{r, i\}) = \{k, c\},$$

which is not a semi-open set in $\mathscr{M}$ (since $\{k, c\} \notin \mathscr{T}$). This shows that the composition of two semi-perfect functions does not necessarily yield a semi-perfect function.

Corollary 3.3. *The result of combining two semi-perfect functions does not have to be semi-perfect.*

Theorem 3.4. *A lindelöf function can be expressed as an outcome of combining two lindelöf perfect functions.*

Proof. Since the compining of any two lindelöf spaces is lindelöf, also the lindelöf function can be expressed as an outcome of compining two lindelöf perfect functions. $\square$

Theorem 3.5. *Given that $\daleth : (\mathscr{M}, \mathscr{T}) \to (\mathscr{L}, \varsigma)$ is a semi-perfect function and $\gamma : (\mathscr{L}, \varsigma) \to (Z, \eta)$ is a perfect function, the composition $\gamma \circ \daleth$ is a semi-perfect function.*

Proof. Let $\mathscr{A}$ be any η-open set in Z. Since γ is a perfect function, the preimage $\gamma^{-1}(\mathscr{A})$ is a ς-open set in $\mathscr{L}$.

Now, since $\daleth$ is a semi-perfect function, $\daleth^{-1}(\gamma^{-1}(\mathscr{A}))$ is a $\mathscr{T}$-semi-open set in $\mathscr{M}$. Therefore, we conclude that $\gamma \circ \daleth$ is indeed a semi-perfect function. $\square$

Example 3.2. Consider $(\mathscr{M}, \mathscr{T}) = (\mathbb{R}, \tau_s)$, where τ_s is the topology generated by all semi-open intervals of the form $(\mathfrak{c}, \mathfrak{d}) \cup \{\mathfrak{c}\}$, $\mathfrak{c} < \mathfrak{d}$. And let $(\mathscr{L}, \varsigma) = (\mathbb{R}, \tau_u)$.

Define $\daleth : (\mathscr{M}, \mathscr{T}) \to (\mathscr{L}, \varsigma)$ by $\daleth(m) = m$. Since $\daleth$ maps semi-open sets in $(\mathscr{M}, \mathscr{T})$ to open sets in $(\mathscr{L}, \varsigma)$, $\daleth$ is a semi-perfect function.

Now let $(Z, \eta) = (\mathbb{R}, \tau_d)$. Where $\gamma : (\mathscr{L}, \varsigma) \to (Z, \eta)$ by $\gamma(m) = m$. Since every subset of $\mathbb{R}$ is open in the discrete topology, γ is a perfect function.

Finally, consider $\gamma \circ \daleth : (\mathscr{M}, \mathscr{T}) \to (Z, \eta)$, defined by $(\gamma \circ \daleth)(m) = \gamma(\daleth(m)) = m$. For any η-open set $\mathscr{A}$ in (Z, η), $\gamma^{-1}(\mathscr{A}) = \mathscr{A}$ is ς-open in $(\mathscr{L}, \varsigma)$, and $\daleth^{-1}(\gamma^{-1}(\mathscr{A})) = \mathscr{A}$ is $\mathscr{T}$-semi-open in $(\mathscr{M}, \mathscr{T})$. Therefore, $\gamma \circ \daleth$ is a semi-perfect function.

Theorem 3.6. *If $\daleth : (\mathscr{M}, \mathscr{T}) \to (\mathscr{L}, \varsigma)$ is a semi-Lindelöf perfect function and $\gamma : (\mathscr{L}, \varsigma) \to (Z, \eta)$ is a perfect function, then the composition $\gamma \circ \daleth$ is a semi-Lindelöf perfect function.*

Proof. consider $\mathscr{A}$ to be any η-open set in Z. Since γ is a perfect function, $\gamma^{-1}(\mathscr{A})$ is a ς-open set in $\mathscr{L}$. since $\daleth$ is a semi-Lindelöf perfect function, for any open cover $\{U_\alpha\}_{\alpha \in \Lambda}$ of $\daleth^{-1}(\gamma^{-1}(\mathscr{A}))$, there exists a countable subcover $\{U_i\}_{i \in \mathbb{N}}$ such that,

$$\daleth^{-1}(\gamma^{-1}(\mathscr{A})) \subseteq \bigcup_{i \in \mathbb{N}} U_i.$$

Therefore, $\gamma \circ \daleth$ retains the property of being semi-Lindelöf. Thus, we conclude that $\gamma \circ \daleth$ is a semi-Lindelöf perfect function. $\square$

Proposition 3.1. *If $\gamma \circ \daleth$ of a semi-continuous function, $\daleth : (\mathscr{M}, \mathscr{T}) \to (\mathscr{L}, \varsigma)$, and a continuous function $\gamma : (\mathscr{L}, \varsigma) \to (Z, \eta)$ is semi-closed, then $\gamma : (\mathscr{L}, \varsigma) \to (Z, \eta)$ is semi-closed.*

Proof. Consider $\mathscr{A}$ to be a ς-closed set in $\mathscr{L}$, then $\daleth^{-1}(\mathscr{A})$ is $\mathscr{T}$-semi-closed in $\mathscr{M}$. Since $\daleth$ is a semi-continuous function, given that $\gamma \circ \daleth$ is semi-closed, $\gamma(\daleth\daleth^{-1}(\mathscr{A}))$ is η-semi-closed in Z, this implies that $\gamma(\mathscr{A})$ is η-semi-closed in Z. Therefore, γ is semi-closed in $\mathscr{L}$. $\square$

Theorem 3.7. *If the composition $\gamma \circ \daleth$ of a continuous function $\gamma : (\mathscr{L}, \varsigma) \xrightarrow{onto} (\mathscr{Z}, \eta)$ and a semi-continuous function $\daleth : (\mathscr{M}, \mathscr{T}) \to (\mathscr{L}, \varsigma)$ is a semi-perfect function, then $\gamma : (\mathscr{L}, \varsigma) \xrightarrow{onto} (\mathscr{Z}, \eta)$ is semi-perfect.*

Proof. For any $z \in \mathscr{Z}$, consider $\gamma^{-1}(z)$. Since $\gamma \circ \daleth$ is semi-perfect, we know that the set $(\gamma \circ \daleth)^{-1}(z)$ is semi-compact. so,

$$\gamma^{-1}(z) = \daleth((\gamma \circ \daleth)^{-1}(z)),$$

where $(\gamma \circ \daleth)^{-1}(z)$ is semi-compact. Since $\daleth$ is a semi-continuous function, $\daleth^{-1}(\gamma^{-1}(z))$ is also semi-compact. Thus, $\gamma^{-1}(z)$ is semi-compact, consequently γ is semi-perfect. $\square$

Theorem 3.8. *If the composition $\gamma \circ \daleth$ of the continuous function $\gamma : (\mathscr{L}, \varsigma) \xrightarrow{onto} (\mathscr{Z}, \eta)$ and a semi-continuous function $\daleth : (\mathscr{M}, \mathscr{T}) \to (\mathscr{L}, \varsigma)$ is a semi-Lindelöf perfect function, then $\gamma : (\mathscr{L}, \varsigma) \xrightarrow{onto} (\mathscr{Z}, \eta)$ is a semi-Lindelöf perfect function.*

Proof. Consider $\gamma^{-1}(z) = \daleth((\gamma \circ \daleth)^{-1}(z))$ For every $z \in Z$, which is semi-Lindelöf because the function $\gamma \circ \daleth$ is semi-Lindelöf perfect. Since γ is semi-closed by proposition 3.1, it follows that γ is semi-Lindelöf perfect. $\square$

Theorem 3.9. *If $\daleth : (\mathscr{M}, \mathscr{T}) \to (\mathscr{L}, \varsigma)$ is a semi-closed function, then for any subset $\mathfrak{E} \subset \mathscr{L}$, the restriction $\daleth_{\mathfrak{E}} : \daleth^{-1}(\mathfrak{E}) \to \mathfrak{E}$ is semi-closed.*

Proof. Consider $\mathfrak{E} \subset \mathscr{L}$, and $\daleth : (\mathscr{M}, \mathscr{T}) \to (\mathscr{L}, \varsigma)$, let A be $\mathscr{T}$-semi-closed. Then

$$\daleth_{\mathfrak{E}}(\mathscr{A} \cap \daleth^{-1}(\mathfrak{E})) = \daleth(\mathscr{A}) \cap \mathfrak{E}$$

is ς-semi-closed in $\mathfrak{E}$. Thus, $\daleth_{\mathfrak{E}} : \daleth^{-1}(\mathfrak{E}) \to \mathfrak{E}$ is semi-closed. $\square$

Theorem 3.10. *Given that $\daleth : (\mathscr{M}, \mathscr{T}) \xrightarrow{onto} (\mathscr{L}, \varsigma)$ is semi-perfect, for any $\mathfrak{E} \subset \mathscr{L}$, the restriction $\daleth_{\mathfrak{E}} : \daleth^{-1}(\mathfrak{E}) \to \daleth$ is semi-perfect.*

Proof. Assume $\daleth : (\mathscr{M}, \mathscr{T}) \to (\mathscr{L}, \varsigma)$ is semi-perfect, then the preimage of any semi-compact subset of $\mathscr{L}$ is semi-compact in $\mathscr{M}$. Now, consider $\mathfrak{E} \subset \mathscr{L}$. To demonstrate that $\daleth_{\mathfrak{E}}$ is semi-perfect, we need to show that for any semi-compact subset G of $\mathfrak{E}$, $\daleth_{\mathfrak{E}}^{-1}(G) = \daleth^{-1}(G)$ is semi-compact in $\daleth^{-1}(\mathfrak{E})$.

Since $\urcorner$ is semi-perfect on $(\mathscr{M}, \mathscr{T})$, $\urcorner^{-1}(G)$ is semi-compact in $\mathscr{M}$ for any semi-compact $G \subset \mathscr{L}$. Therefore, for any semi-compact $G \subset \mathfrak{E}$, $\urcorner_{\mathfrak{E}}^{-1}(G)$ is semi-compact in $\urcorner^{-1}(\mathfrak{E})$. Moreover, by Theorem 3.9, the restriction $\urcorner_{\mathfrak{E}}$ is semi-closed. Thus, $\urcorner_{\mathfrak{E}}$ is semi-perfect. □

Theorem 3.11. *Given that $\urcorner : (\mathscr{M}, \mathscr{T}) \xrightarrow{onto} (\mathscr{L}, \varsigma)$ is semi-perfect, where $(\mathscr{M}, \mathscr{T})$ is semi-compact, and $(\mathscr{L}, \varsigma)$ is semi-Hausdorff, then $\urcorner$ is semi-closed.*

Proof. If $\mathscr{A}$ is a $\mathscr{T}$-semi-closed subset of $(\mathscr{M}, \mathscr{T})$, then it is $\mathscr{T}$-semi-compact, because $(\mathscr{M}, \mathscr{T})$ is semi-compact. Since $\urcorner$ is semi-continuous, $\urcorner(\mathscr{A})$ is a ς-semi-compact subset of $(\mathscr{L}, \varsigma)$. Given that $(\mathscr{L}, \varsigma)$ is semi-Hausdorff, it follows that every semi-compact subset of $(\mathscr{L}, \varsigma)$ is ς-semi-closed. Thus, $\urcorner(\mathscr{A})$ is ς-semi-closed. □

Theorem 3.12. *Given that $\urcorner : (\mathscr{M}, \mathscr{T}) \xrightarrow{onto} (\mathscr{L}, \varsigma)$ is a semi-Lindelöf perfect function, where $(\mathscr{M}, \mathscr{T})$ is semi-Lindelöf and $(\mathscr{L}, \varsigma)$ is semi-Hausdorff, then $\urcorner$ is semi closed.*

Proof. If $\mathscr{A}$ is a $\mathscr{T}$-semi-closed subset of $(\mathscr{M}, \mathscr{T})$, it is $\mathscr{T}$-semi-Lindelöf, as $(\mathscr{M}, \mathscr{T})$ is semi-Lindelöf. Since $\urcorner$ is semi-continuous and semi-Lindelöf perfect, $\urcorner(\mathscr{A})$ is a ς-semi-Lindelöf subset of $(\mathscr{L}, \varsigma)$. Given that $(\mathscr{L}, \varsigma)$ is semi-Hausdorff, every semi-Lindelöf subset of $(\mathscr{L}, \varsigma)$ is ς-semi-closed. Thus, $\urcorner(\mathscr{A})$ is ς-semi-closed. □

Theorem 3.13. *Consider $\urcorner : (\mathscr{M}, \mathscr{T}) \to (\mathscr{L}, \varsigma)$ as a bijective semi-continuous function. If $(\mathscr{L}, \varsigma)$ is a semi-Hausdorff space, $(\mathscr{M}, \mathscr{T})$ is semi-compact, then $\urcorner$ is a semi-homeomorphism function.*

Proof. It is sufficient to demonstrate that $\urcorner$ be a semi-closed. Consider F to be a $\mathscr{T}$-closed proper included in $\mathscr{M}$, and hence F a proper $\mathscr{T}$-semi-compact set, by using theorem 3.12. Consequently, $\urcorner(F)$ is ς-semi-compact, but since $(\mathscr{L}, \varsigma)$ is a semi-Hausdorff space, $\urcorner(F)$ is ς-semi-closed. It means $\urcorner$ is a semi-homeomorphism function. □

Theorem 3.14. *Consider $\urcorner : (\mathscr{M}, \mathscr{T}) \to (\mathscr{L}, \varsigma)$ as a semi-continuous bijection function. If $(\mathscr{L}, \varsigma)$ is semi-Hausdorff and $(\mathscr{M}, \mathscr{T})$ is semi-Lindelöf, then $\urcorner$ is a semi-homomorphism function.*

Proof. To establish that $\urcorner$ is a semi-homomorphism, it's suffices to show that $\urcorner$ is semi-closed. Given that $(\mathscr{M}, \mathscr{T})$ is semi-Lindelöf, F is a semi-Lindelöf set with $\mathscr{T}$ closure. Since $\urcorner$ is semi-continuous and bijective, $\urcorner(F)$ is a ς-semi-Lindelöf subset of $(\mathscr{L}, \varsigma)$. $(\mathscr{L}, \varsigma)$ is semi-Hausdorff, hence $\urcorner(F)$ is ς-semi-closed. So, $\urcorner$ is a semi-homeomorphism function. □

Theorem 3.15. *The semi-Hausdorff property is preserved under semi-perfect functions.*

Proof. Consider $(\mathscr{M}, \mathscr{T})$ to be a semi-Hausdorff space, and $\urcorner : (\mathscr{M}, \mathscr{T}) \to (\mathscr{L}, \varsigma)$ to be a semi-perfect function. Let $l_1 \neq l_2$ in $(\mathscr{L}, \varsigma)$. Then $\urcorner^{-1}(l_1)$ and $\urcorner^{-1}(l_2)$ are disjoint and semi-compact subsets of $(\mathscr{M}, \mathscr{T})$. Since $(\mathscr{M}, \mathscr{T})$ is a semi-Hausdorff space, there exist $\mathscr{T}$-semi-open sets P, Q of $\mathscr{M}$ such that

$$\urcorner^{-1}(l_1) \subseteq P, \urcorner^{-1}(l_2) \subseteq Q, \text{and } P \cap Q = \emptyset.$$

Consider the sets $\mathscr{L} - \urcorner(\mathscr{M} - P)$, which contains z_1, and $\mathscr{L} - \urcorner(\mathscr{M} - Q)$, which contains l_2, as ς-semi-open sets in $(\mathscr{L}, \varsigma)$.

Then,

$$\begin{aligned} \mathscr{L} - (\urcorner(\mathscr{M} - P) \cup \urcorner(\mathscr{M} - Q)) \\ = \mathscr{L} - \urcorner(\mathscr{M} - (P \cap Q)) \\ = \mathscr{L} - \urcorner(\mathscr{M}) \\ = \emptyset. \end{aligned}$$

Hence, $(\mathscr{L}, \varsigma)$ is a semi-Hausdorff space. □

We will use a similar proof technique as in the previous theorem to establish the following.

Theorem 3.16. *The semi-Hausdorff property is preserved under semi-lindelöf perfect function.*

Proof. Let $(\mathscr{L}, \varsigma)$ be a semi-Hausdorff space and $\gamma : (\mathscr{L}, \varsigma) \to (Z, \eta)$ a semi-Lindelöf perfect function. Assume $z_1 \neq z_2$ in (Z, η). The preimages $\gamma^{-1}(z_1)$ and $\gamma^{-1}(z_2)$ are disjoint semi-Lindelöf subsets of $(\mathscr{L}, \varsigma)$. Because $(\mathscr{L}, \varsigma)$ is semi-Hausdorff, we can find semi-open sets A and B in $\mathscr{L}$ such that

$$\gamma^{-1}(z_1) \subseteq A, \quad \gamma^{-1}(z_2) \subseteq B, \quad \text{and} \quad A \cap B = \emptyset.$$

Next, we examine the sets $Z - \gamma(\mathscr{L} - A)$ and $Z - \gamma(\mathscr{L} - B)$. The former contains z_1, while the latter contains w_2. Both of these sets are η-semi-open in (Z, η). Then,

$$\begin{aligned} Z - (\gamma(\mathscr{L} - A) \cup \gamma(\mathscr{L} - B)) \\ = Z - \gamma(\mathscr{L} - (A \cap B)) \\ = Z - \gamma(\mathscr{L}) \\ = \emptyset. \end{aligned}$$

Thus, it follows that (Z, η) is a semi-Hausdorff space. □

Using the same technique, the following remarks have been formulated.

Remark 3.1. The semi-Hausdorff property that is preserved by the inverse images of semi-perfect functions.

Remark 3.2. The semi-Hausdorff property that is preserved by the inverse images of semi-Lindelöf perfect functions.

4 More Results about Types of Semi Perfect Functions

This section provides further details and perspectives about various types of semi perfect functions.

Lemma 4.1. *Consider $\mathscr{M}$ be a semi-regular space, and let $\mathscr{A}$ be a $\mathscr{T}$-semi-compact subset of $\mathscr{M}$. Then for any $\mathscr{T}$-semi-neighbourhood U of $\mathscr{A}$, there exists a $\mathscr{T}$-semi-open set W such that $\mathscr{A} \subset W \subset s\,Cl\,W^{\mathscr{T}} \subset U$.*

Proof. Since $\mathscr{M}$ is a semi-regular space, for every point $a \in \mathscr{M}$ and every $\mathscr{T}$-semi-neighborhood U of a, there exists a $\mathscr{T}$-semi-open set $V(a)$ such that:

$$a \in V(a) \subset s\,Cl(V(a))^{\mathscr{T}} \subset U.$$

Now consider $a \in \mathscr{A}$, where $\mathscr{A}$ is a $\mathscr{T}$-semi-compact subset of $\mathscr{M}$. Since U is given as a $\mathscr{T}$-semi-neighborhood of $\mathscr{A}$, every $a \in \mathscr{A}$ has U as a $\mathscr{T}$-semi-neighborhood. By the semi-regularity of $\mathscr{M}$, for each $a \in \mathscr{A}$, there exists a $\mathscr{T}$-semi-open set $V(a)$ such that:

$$a \in V(a) \quad \text{and} \quad s\,Cl(V(a))^{\mathscr{T}} \subset U.$$

Thus,

$$\mathscr{A} \subset \bigcup_{k=1}^{n} V(a_k) \subset U$$

so

$$\mathscr{A} \subset \bigcup_{k=1}^{n} V(a_k) \subset s\,Cl\left(\bigcup_{k=1}^{n} V(a_k)\right)^{\mathscr{T}}.$$

Consider $W = \bigcup_{k=1}^{n} V(a_k)$, then W is $\mathscr{T}$-semi-open, but

$$s\,Cl\,W^{\mathscr{T}} = s\,Cl \bigcup_{k=1}^{n} V(a_k)^{\mathscr{T}} = s\,Cl\left(\bigcup_{k=1}^{n} V(a_k)\right)^{\mathscr{T}}.$$

Hence,

$$\mathscr{A} \subset W \subset Cl\,W^{\mathscr{T}} \subset U.$$

□

The semi-compactness of subsets within semi-regular spaces implies the semi-Lindelöf property. Therefore, Lemma 4.1 can be expanded to include $\mathscr{T}$-semi-Lindelöf subsets.

Lemma 4.2. *Consider $\mathscr{M}$ be a semi-regular space, and let $\mathscr{A}$ be a $\mathscr{T}$-semi-Lindelöf subset of $\mathscr{M}$. For any $\mathscr{T}$-semi-neighbourhood U of $\mathscr{A}$, there exists a $\mathscr{T}$-semi-open set W such that $\mathscr{A} \subset W \subset sclW^{\mathscr{T}} \subset U$.*

Proof. Since $\mathscr{A}$ is a $\mathscr{T}$-semi-Lindelöf subset, there exists a countable collection of $\mathscr{T}$-semi-open sets, $\{V(a_i)\}_{i=1}^{\infty}$, that covers $\mathscr{A}$.

By the semi-Lindelöf property, we can select a finite $\{V(a_{i_k})\}_{k=1}^{n}$ from this cover so that

$$\mathscr{A} \subset \bigcup_{k=1}^{n} V(a_{i_k}) \subset U.$$

Let $W = \bigcup_{k=1}^{n} V(a_{i_k})$; then W is a $\mathscr{T}$-semi-open set that includes $\mathscr{A}$.
Additionally,

$$s\,Cl\,W^{\mathscr{T}} = s\,Cl \bigcup_{k=1}^{n} V(a_{i_k})^{\mathscr{T}} = s\,Cl\left(\bigcup_{k=1}^{n} V(a_{i_k})\right)^{\mathscr{T}}.$$

Hence,

$$\mathscr{A} \subset W \subset s\,Cl\,W^{\mathscr{T}} \subset U$$

. □

Theorem 4.1. *Consider $\daleth : (\mathscr{M}, \mathscr{T}) \to (\mathscr{L}, \varsigma)$ be a semi-perfect function, and suppose $(\mathscr{M}, \mathscr{T})$ is semi-regular. Then $(\mathscr{L}, \varsigma)$ is semi open.*

Proof. Given a ς-semi-open set V and $l \in \mathscr{L}$, $\daleth^{-1}(l) \in \daleth^{-1}(V)$ in $\mathscr{M}$. Since $\mathscr{M}$ is semi-regular, there exists a $\mathscr{T}$-semi-open set U (by using Lemma 4.2)
suchthat

$$\daleth^{-1}(l) \in s\,Cl\,(\left(\bigcup_{k=1}^{n} U_k\right)^{\mathscr{T}}) \subset \daleth^{-1}(V).$$

Since $\daleth$ is $\mathscr{T}$-semi, $\exists$ ς-semi-neighbourhood W of l such that

$$\daleth^{-1}(l) \in \daleth^{-1}(W) \subset V.$$

Moreover,

$$W \subset \daleth(s\,Cl\,U^{\mathscr{T}}) \subset V$$

since $\daleth(s\,Cl\,U^{\mathscr{T}})$ is ς-semi-closed.
Thus,

$$l \in W \subset s\,Cl\,W^{\mathscr{S}} \subset (s\,Cl\,U^{\mathscr{T}}) \subset V$$

Hence, $\mathscr{L}$ is semi-regular. □

By employing the same method, we obtain the following corollaries.

Corollary 4.1. *Consider $\urcorner : (\mathscr{M}, \mathscr{T}) \to (\mathscr{L}, \varsigma)$ to be semi-Lindelöf perfect, and suppose $(\mathscr{M}, \mathscr{T})$ is semi-regular. Then $(\mathscr{M}, \mathscr{T})$ is semi open.*

Corollary 4.2. *The semi-regular property is preserved under semi-perfect.*

Corollary 4.3. *The semi-regular property is preserved under semi-lindelöf perfect.*

Theorem 4.2. *Consider $\urcorner : (\mathscr{M}, \mathscr{T}) \to (\mathscr{L}, \varsigma)$ to be a semi-perfect function, and suppose $(\mathscr{M}, \mathscr{T})$ is semi-regular. Then $(\mathscr{L}, \varsigma)$ is semi open.*

Proof. Let $\urcorner : (\mathscr{M}, \mathscr{T}) \to (\mathscr{L}, \varsigma)$ be a semi-perfect function, and assume $(\mathscr{M}, \mathscr{T})$ is semi-regular.

Consider any ς-semi-open set V in $(\mathscr{L}, \varsigma)$ and an element $l \in V$. Then, $\urcorner^{-1}(V)$ is a $\mathscr{T}$-semi-open subset of $(\mathscr{M}, \mathscr{T})$ containing $\urcorner^{-1}(l)$.

Since $(\mathscr{M}, \mathscr{T})$ is semi-regular, we know that for any $\mathscr{T}$-semi-neighborhood U of $\urcorner^{-1}(l)$, there exists a $\mathscr{T}$-semi-open set W such that

$$\urcorner^{-1}(l) \subset W \subset s\,Cl\,W^{\mathscr{T}} \subset U.$$

Applying $\urcorner$ to this containment, we obtain

$$\urcorner(\urcorner^{-1}(l)) \subset \urcorner(W) \subset \urcorner(s\,Cl\,W^{\mathscr{T}}) \subset \urcorner(U).$$

Since $\urcorner$ is semi-perfect, $\urcorner(s\,Cl\,W^{\mathscr{T}})$ is ς-semi-closed, which implies that $\urcorner(W)$ is ς-semi-open and forms a ς-semi-neighborhood of l within V.

Therefore, for every point $l \in V$, there exists a ς-semi-neighborhood contained in V, which implies that $(\mathscr{L}, \varsigma)$ is semi-open. □

Theorem 4.3. *Consider $(\mathscr{M}, \mathscr{T})$ and $(\mathscr{L}, \varsigma)$ as arbitrary topological spaces. Given that $(\mathscr{M}, \mathscr{T})$ is semi-compact, then $\zeta : (\mathscr{M} \times \mathscr{L}, \mathscr{T} \times \varsigma) \to (\mathscr{L}, \varsigma)$ is semi-closed.*

Proof. Let $\zeta : (\mathscr{M} \times \mathscr{L}, \mathscr{T} \times \varsigma) \to (\mathscr{L}, \varsigma)$ be the projection function, where $(\mathscr{M}, \mathscr{T})$ is a semi-compact topological space. Consider an arbitrary semi-closed set $\mathscr{A} \subset \mathscr{M} \times \mathscr{L}$ in the $\mathscr{T} \times \varsigma$ topology. This implies that $\mathscr{A}$ is semi-compact in $(\mathscr{M} \times \mathscr{L}, \mathscr{T} \times \varsigma)$, as it is a closed subset of a semi-compact space. Since $(\mathscr{M}, \mathscr{T})$ is semi-compact and the projection of function is closed function, so the projection of $\mathscr{A}$ onto $\mathscr{L}$, $\zeta(\mathscr{A}) = \{l \in \mathscr{L} : \exists m \in \mathscr{M}$ such that $(m, l) \in \mathscr{A}\}$, is also semi-compact in $(\mathscr{L}, \varsigma)$.

Given that a semi-compact subset of $(\mathscr{L}, \varsigma)$ is also semi-closed, it follows that $\zeta(A)$ is semi-closed in $(\mathscr{L}, \varsigma)$. Hence, ζ is a semi-closed function. □

Theorem 4.4. *Consider $(\mathscr{M}, \mathscr{T})$ and $(\mathscr{L}, \varsigma)$ as arbitrary topological spaces. Given that $(\mathscr{M}, \mathscr{T})$ is semi-compact, then $\zeta : (\mathscr{M} \times \mathscr{L}, \mathscr{T} \times \varsigma) \to (\mathscr{L}, \varsigma)$ is semi-closed.*

Proof. Let $\zeta : (\mathscr{M} \times \mathscr{L}, \mathscr{T} \times \varsigma) \to (\mathscr{L}, \varsigma)$ be the projection function, where $(\mathscr{M}, \mathscr{T})$ is a semi-compact topological space. Consider an arbitrary semi-closed set $\mathscr{A} \subset \mathscr{M} \times \mathscr{L}$ in the $\mathscr{T} \times \varsigma$ topology. This implies that $\mathscr{A}$ is semi-compact in $(\mathscr{M} \times \mathscr{L}, \mathscr{T} \times \varsigma)$, as it is a semi-closed subset of a semi-compact space.

Since $(\mathscr{M}, \mathscr{T})$ is semi-compact, for any $l \in \zeta(\mathscr{A})$, there exists at least one $m \in \mathscr{M}$ such that $(m, l) \in \mathscr{A}$. The projection of $\mathscr{A}$ onto $\mathscr{L}$, defined as:

$$\zeta(\mathscr{A}) = \{l \in \mathscr{L} : \exists m \in \mathscr{M} \text{ such that } (m, l) \in \mathscr{A}\},$$

is the image of a semi-compact set under the projection map ζ. Since projections preserve semi-compactness, $\zeta(\mathscr{A})$ is semi-compact in $(\mathscr{L}, \varsigma)$.

Furthermore, in $(\mathscr{L}, \varsigma)$, every semi-compact subset is semi-closed. Therefore, $\zeta(\mathscr{A})$ is semi-closed in $(\mathscr{L}, \varsigma)$.

Hence, ζ is a semi-closed function. □

Theorem 4.5. *Consider $\urcorner : (\mathscr{M}, \mathscr{T}) \to (\mathscr{L}, \varsigma)$ to be a semi-perfect function, and suppose $(\mathscr{L}, \varsigma)$ be a semi-paracompact. Then, $(\mathscr{M}, \mathscr{T})$ is semi-open.*

Proof. Consider $U = \{U_\alpha : \alpha \in \Lambda\}$ to be a semi-open cover of $(\mathscr{M}, \mathscr{T})$. Since $\urcorner$ is a semi-perfect function, $\forall l \in \mathscr{L}$, the preimage $\urcorner^{-1}(l)$ is semi-compact. Therefore, $\exists$ a finite subset $\Lambda_l \subseteq \Lambda$ such that

$$\urcorner^{-1}(l) \subseteq \bigcup_{\alpha \in \Lambda_l} U_\alpha,$$

where U_α are semi-open sets in $(\mathscr{M}, \mathscr{T})$ containing $\urcorner^{-1}(l)$.

Consider $\mathscr{O} = \{O_l : l \in \mathscr{L}\}$ to be a semi-open cover of $\mathscr{L}$ since $(\mathscr{L}, \varsigma)$ is semi-paracompact. Then, $\mathscr{O}$ has a locally finite open refinement. Let $\mathscr{H} = \{H_{\mathfrak{E}} : \mathfrak{E} \in \Gamma\}$ where $\{H_{\mathfrak{E}} : \mathfrak{E} \in \Gamma\}$ is ς-locally finite paracompact of O_l.

Now, Consider $G = \{\urcorner^{-1}(H_{\mathfrak{E}}) \cap V_\alpha \mid \alpha \in \Lambda_l\}$ is a $\mathscr{T}$-semi-open locally finite refinement of $V = \{V_\alpha : \alpha \in \Lambda\}$, so G is a semi-locally semi-open refinement of $\mathscr{U}$, since

$$\bigcup_{\alpha \in \Lambda_l} V_\alpha \subset \mathscr{U}$$

Hence, $(\mathscr{M}, \varsigma)$ is a semi-paracompact space. □

Corollary 4.4. *If $\urcorner : (\mathscr{M}, \mathscr{T}) \to (\mathscr{L}, \varsigma)$ is a semi-Lindelöf perfect function and $(\mathscr{L}, \varsigma)$ is semi-paracompact, then $(\mathscr{M}, \mathscr{T})$ is semi-open.*

5 Four Types of Semi Perfect Functions

Definition 5.1. A function $\daleth : (\mathscr{M}, \mathscr{T}) \to (\mathscr{L}, \varsigma)$ is referred to as *semi-perfect* if $\daleth$ is semi-closed, semi-continuous, and for any $l \in \mathscr{L}$, $\daleth^{-1}(l)$ is semi-compact.

Definition 5.2. A function $\daleth : (\mathscr{M}, \mathscr{T}) \to (\mathscr{L}, \varsigma)$ is referred to as *semi-Lindelöf perfect* if $\daleth$ is semi-closed, semi-continuous, $\daleth^{-1}(l)$ is semi-Lindelöf for any $l \in \mathscr{L}$.

Theorem 5.1. *Every semi-perfect function is perfect function, but the converse need not to be true.*

Proof. consider any semi-compact space is also a compact space by definition, as a semi-compact space is defined as one where every semi-open cover has a finite subcover. Since a perfect function requires the preimage of each compact set to be compact, and every semi-compact subset is compact, it follows that a semi-perfect function, which preserves semi-compactness, will also preserve compactness. Therefore, every semi-perfect function is indeed a perfect function. □

However, as we will demonstrate, the opposite does not necessarily hold.

Example 5.1. Consider an infinite set $\mathscr{U}$ and $c \in \mathscr{U}$. Define the topology $\varsigma = \{\mathscr{U}, \emptyset, \{c\}\}$ on $\mathscr{U}$. In this topology, $(\mathscr{U}, \varsigma)$ is compact because any open cover has a finite subcover. However, it is not semi-compact since the collection $\{\{u, c\} : u \in \mathscr{U}\}$ forms a semi-open cover of $\mathscr{U}$ without a finite subcover.

Theorem 5.2. *Every perfect function is lindelöf perfect spaces, but the reverse isn't necessarily true.*

Proof. Since every compact space is Lindelöf, any perfect function, which is continuous and maps compact spaces to compact spaces, leads to a Lindelöf perfect space. Specifically, if $\daleth : (\mathscr{M}, \mathscr{T}) \to (\mathscr{L}, \varsigma)$ is a perfect function, then for every $l \in \mathscr{L}$,$\daleth^{-1}(l)$ is compact and therefore Lindelöf. □

As illustrated by an upcoming example, the opposite not necessarily true.

Example 5.2. Consider $\{U_\alpha\}_{\alpha \in \mathscr{A}}$ to be an open cover of $\mathbb{R}$, where $\mathbb{R}$ represents the real line. There exists a countable subcover. Consider the set $U_k = (k, k+2), k \in \mathbb{Z}$, and $\mathbb{R} = \bigcup_{k \in \mathbb{Z}}(k, k+2)$ covers the entire real line, demonstrating that $\mathbb{R}$ is Lindelöf.

However, consider $\{(k, k+2)\}_{k \in \mathbb{Z}}$ as an open cover of $\mathbb{R}$. Suppose there exists a finite subcover, say $\{(k_i, k_i+2)\}_{i=1}^{n}, k_i \in \mathbb{Z}$. In this case, no finite subcover can cover $\mathbb{R}$, demonstrating that $\mathbb{R}$ is not compact.

Theorem 5.3. *Every semi-lindelöf perfect function is lindelöf but the converse is not always true.*

Proof. Since every semi-Lindelöf space is Lindelöf, a semi-Lindelöf perfect function must also produce a Lindelöf space. Specifically, if $\daleth : (\mathscr{M}, \mathscr{T}) \to (\mathscr{L}, \varsigma)$ is a semi-Lindelöf perfect function, it implies $\daleth^{-1}(l)$ are semi-Lindelöf for every $l \in \mathscr{L}$. This further leads to that $\mathscr{L}$ is Lindelöf. □

Example 5.3. Consider $(\mathbb{R}, \mathscr{T}_s)$, the real line with the standard topology. Let $\forall U = \{U_\alpha\}_{\alpha \in \mathscr{A}}$, where $\mathscr{A}$ is an arbitrary index set, $\bigcup\{U_\alpha : \alpha \in \mathscr{A}\} = \mathbb{R}$, and $U_\alpha \in \mathscr{T}_s \ \forall \alpha \in \mathscr{A}$. There exists $\{U_n : n \in \mathbb{N}\} \subseteq U$ such that $\bigcup\{U_n : n \in \mathbb{N}\} = \mathbb{R}$. Define $\daleth : \mathbb{Z} \to \mathbb{N}$ as a bijection between integers and natural numbers. $\forall k \in \mathbb{Z}$, take $\alpha_k \in \mathscr{A}$ such that $\daleth(k) \in U_{\alpha_k}$. Consider $V = \{U_{\alpha_k} : k \in \mathbb{Z}\}$ as a countable subcover. This shows that $\mathbb{R}$ is Lindelöf.

Now, consider $\mathfrak{C} = \{C_m : m \in \mathbb{R}\}$, where $C_m = \{m, m+1\}$ for all $m \in \mathbb{R}$. $\mathfrak{C}$ is a semi-open cover since $\exists U_m = \{m, m+1\} \in \mathfrak{T}_s$ such that $U_m \subseteq C_m \subseteq \overline{U_m}$. $\mathfrak{C}$ covers $\mathbb{R}$, and $\forall l \in \mathbb{R}$, $l \in C_l$.

Assume that $\mathfrak{C}$ has a countable subcover $\mathfrak{T}$, where $\mathfrak{T} = \{C_{m_n} : n \in \mathbb{N}\} \subseteq \mathfrak{C}$ and $\bigcup\{C_{m_n} : n \in \mathbb{N}\} = \mathbb{R}$. Define $\mathscr{L} = \sup\{m_n : n \in \mathbb{N}\}$.

Now, consider examining this:

Case 1: Let $l \in \mathbb{R}$ and consider $\mathfrak{A} = l + 1$. $\forall n \in \mathbb{N}$, $m_n \leq l < \mathfrak{A}$. Thus, $\mathfrak{A} \notin [m_n, x_{n+1}) = C_{m_n}$. Therefore, $\mathfrak{A} \notin \bigcup\{C_{m_n} : n \in \mathbb{N}\} = \mathbb{R}$, which is a contradiction.

Case 2: Let $l = \infty^+$. Consider $\mathfrak{A} = 0$. $\forall n \in \mathbb{N}$, $m_n > \mathfrak{A}$ (since $\sup\{m_n\} = \infty^+$), then $\mathfrak{A} \notin \{m_n, m_{n+1}\} = C_{m_n}$. Therefore, $\mathfrak{A} \notin \bigcup\{C_{m_n} : n \in \mathbb{N}\} = \mathbb{R}$, which is a contradiction.

Therefore, $\mathfrak{C}$ does not have a countable subcover. Thus, $\mathbb{R}$ with the standard topology is not semi-Lindelöf.

Theorem 5.4. *Every semi-perfect function is semi-lindelöf perfect function but the converse need not to be true.*

Proof. A function $\daleth : (\mathscr{M}, \mathscr{T}) \to (\mathscr{L}, \varsigma)$ is said to be semi-perfect if the preimage of every semi-compact subset in $(\mathscr{L}, \varsigma)$ is semi-compact in $(\mathscr{M}, \mathscr{T})$. Since every semi-compact space is also semi-Lindelöf, it follows that the preimage under a semi-perfect function of any semi-compact subset is semi-Lindelöf. Hence, $\daleth$ is a semi-Lindelöf perfect function.

□

However, as the following example demonstrates, that the opposite is not necessarily true:

Example 5.4. In $(\mathbb{R}, \mathscr{T}_s)$, consider $\{U_\alpha\}_{\alpha \in \mathscr{A}}$ to be a semi-open cover of $\mathbb{R}$, where U_α is semi-open in

$\mathbb{R}$. A set $V \subseteq \mathbb{R}$ is semi-open if there exists an open set $M \subseteq \mathbb{R}$ such that $M \subseteq V \subseteq \overline{M}$. For any U_α, there exists M_α such that $M_\alpha \subseteq U_\alpha \subseteq \overline{M_\alpha}$. Since $\mathbb{R}$ is Lindelöf, the open cover $\{M_\alpha\}_{\alpha \in \mathbb{A}}$ has a countable subcover $\{M_{\alpha_i}\}_{i \in \mathbb{N}}$. The corresponding semi-open sets $\{U_{\alpha_i}\}_{i \in \mathbb{N}}$ form a countable subcover of $\mathbb{R}$. Therefore, $\mathbb{R}$ is semi-Lindelöf with respect to the perfect function property.

However, consider the semi-open cover $\{(k, k+1) \cup \{k+1\}\}_{k \in \mathbb{Z}}$ of $\mathbb{R}$. Any set $(k, k+1) \cup \{k+1\}$ is semi-open because it can be expressed as $(k, k+1) \cup \{k+1\}$, where $(k, k+1)$ is open and $\{k+1\}$ is a singleton. No finite subcover of this semi-open cover can cover all of $\mathbb{R}$. Therefore, $\mathbb{R}$ is not semi-compact with respect to the perfect function property.

The following figure (Figure 1) summarizes the relationships between these perfect functions:

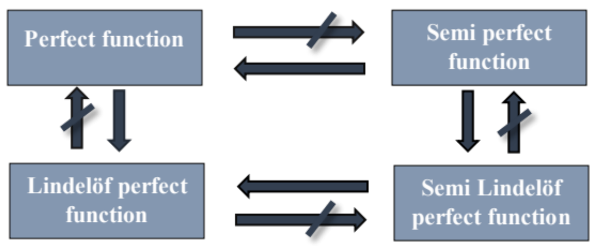

Fig. 1: The relationship between perfect functions

6 Potential Applications

The concept of semi- perfect defined in this study is an innovative idea with applicability in a wide range of scientific and technological sectors, making it a significant starting point for potential studies across multiple domains.

This article investigates numerous applications of semi perfect and semi-lindelöf perfect across diverse disciplines, showing its ability to inspire innovation and efficiency in various sectors.

Functional Analysis: The analysis of certain operators and functions may benefit from the use of semi perfect areas. Their ability to provide a balance between compact and non-compact lindelöf and semi-lindelöf areas is useful for studying the convergence properties of functions or sequences.

Fixed Point Theory: Semi perfect spaces can be used to generalize some fixed point theorems that typically require compactness. This allows us to forecast the occurrence of fixed points for specific types of mappings over a broader range of spaces.

Resource Allocation and Economics: The concept of semi-compactness in economics can be extended to market analysis and resource allocation.

Market Coverage: Inventory and supply chain management can be improved by ensuring that a small number of products or services can meet the needs of the whole market.

Resource Distribution: Semi compact perfect and semi-lindelöf perfect can help identify important areas where resource allocation will have the greatest impact in scenarios where resources need to be distributed efficiently.

Data science and big data analytics rely heavily on ensuring that a finite sample of data points accurately represents the entire dataset. Sampling Techniques; when establishing sampling strategies, semi compactness can help ensure that a finite subset of samples is chosen for analysis and prediction, implying that the data points' distributions cover the whole dataset.

Subsets of representative data: When dealing with large datasets, effective predictive models can be created by selecting a finite sample of data points that encapsulate the data's variability.

7 Conclusions

The associations among semi-lindelöf perfect functions, perfect spaces, lindelöf perfect functions, and semi-lindelöf perfect functions in the topological spaces from which those functions originate were examined in this work. Employing the model of semi-perfect functions that is offered here, the study demonstrated the prerequisites for synchronizing semi sets and continuous closed functions. We evaluated the hyperlinks among these thoughts and expressed them with various functions. The subsequent goal of the study was to highlight specific features of the intricate properties of semi-perfect functions and some peculiarities in the cartesian multiplication of these functions under specific conditions. Additionally, the key features of these ideas as well as a few sample scenarios were thoroughly examined. We outlined their essential traits collectively and clarified what was needed to establish equal relationships between them. The study also emphasized the features of these functions and provided multiple illustrations of them. These duties will act as a springboard for further investigation into the potential futures of all of these functions. Subsequent investigations could potentially examine additional variations of these functionalities include: (1) Define and study soft semi-lindelöf perfect functions; (2) Define and study pairwise semi perfect functions; (3) Define and study fuzzy semi perfect function; (4) Finding a use for our new results of semi perfect functions in Machine Learning Optimization, Data Sampling and Coverage, Functional Analysis, Fuzzy Set Theory, and Generalized Continuity.

In physics and mathematics, topology is one of the most important fields of study. According to [25], [26], [27], [28], [29], [30], [31], [32], [33], [34], [35], [36], its application can be advantageous for a variety of mathematical fields, including algebra, matter physics, Riemann integration, quantum field theory, operations research, physical cosmology, game theory, fuzzy sets, and soft sets.

Use of AI tools declaration

The authors wrote, reviewed and edited the content as needed and They have not utilised artificial intelligence (AI) tools. The authors take full responsibility for the content of the publication.

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Contribution of individual authors to the creation of a scientific article (ghostwriting policy)

Ali A. Atoom: Initiated the primary concept for the study, developed the title, provided essential definitions, along with formulating and proving core theorems, and oversaw the overall research process.

Hamza Qoqazeh: Wrote the introduction and contributed additional theories and proofs, enhancing the theoretical framework of the study.

Mohammad A. Bani abdelrahman: Compiled the research summary and supplemented the study with various illustrative examples to clarify concepts, and explored applications of the research.

Eman Hussein: Reviewed prior studies referenced in the research, documented key sources, and examined the depth of scientific inference to ensure rigor.

Diana Amin Mahmoud: Conducted a detailed scientific audit to verify the correctness of theorem statements and their proofs.

Anas A. Owledat: Undertook a thorough language review, focusing on grammar and overall clarity.

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