Some Aspects of the Conjugacy Class Graph of the Direct Product of Special Linear Group and K-metacyclic Group

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Abstract:- Graphs play a crucial role in studying algebraic structures such as groups, as they help reveal various properties of groups by representing them as graphs and vice versa. In this paper, we aim to explore the conjugacy class graphs associated with the direct product of the special linear group $SL(2, q)$, consisting of 2×2 matrices over a field of order q (where q is an odd prime) and the K-metacyclic group of order $p(p-1)$. We begin by determining the conjugacy classes of the direct product using their respective conjugacy classes and then investigate various graph properties, including planarity, connectivity, chromatic number, independence number, and dominating number.

Key-Words: - Conjugacy class graph, K-metacyclic group, special linear group, chromatic number, clique number, independence number, dominating number, complement graph.

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1 Introduction

The combination of graph theory and algebra has led to the development of a fascinating area of mathematics known as algebraic graph theory. This field explores the interaction between graphs and algebraic structures such as groups, rings, fields, and modules. It offers a powerful framework for analyzing and understanding the properties of these algebraic structures by representing them as graphs. Properties of graphs, such as connectivity, planarity, chromatic number, independence number, and dominating number, can be examined through results from group theory. Much of the research on groups and graphs has been concentrated on commutative groups due to their relative simplicity compared to non-commutative groups. However, as non-commutative groups comprise a significant portion of finite groups, they have gained increasing importance and are now widely studied. One effective approach to studying non-abelian groups is by investigating their conjugacy classes. This concept led authors in [1] to introduce the conjugacy class graph of a non-abelian group, where vertices represent the non-central conjugacy classes, and an edge between two vertices, a and b, exists if $\gcd(|a|, |b|) > 1$. Further details on this graph are found in [2], [3], [4], [5] and [6] where the conjugacy class graph of different groups has been investigated and then its properties are obtained. In this paper, we focus on the direct product of the K-metacyclic groups and the special linear groups. Our goal is to derive the conjugacy class graph associated with the direct product of these groups by examining their conjugacy classes. We analyze various graph parameters, including planarity, connectedness, chromatic number, clique number, independence number, and the diameter of these conjugacy class graphs. These results contribute to the broader understanding of algebraic graph theory and provide a framework for further exploration of group-related graphs.

2 Preliminaries

Let us now briefly review some fundamental definitions and related results from group theory and graph theory that are referenced in the following section. The material in this section is sourced from established literature and published papers.

Consider a simple graph $G(V, E)$. Let $v_0, e_0, v_1, e_1, v_2, e_2, \dots, v_{k-1}, e_{k-1}, v_k$ represent an alternating sequence of vertices and edges, where the $v_i's$ are distinct vertices and $e_i's$ is an edge connecting vertices v_i and v_{i+1}. The number of edges connecting two vertices is called the *length* of the path between them. [7] a graph G is said to be *connected* if there exists a path between every pair of distinct vertices; otherwise, G is *disconnected*. [8] the *chromatic number* $\chi(X)$ of a graph X is defined as the smallest number of colors required to color its vertices such that no two adjacent vertices share the same color. A *clique* in a graph X is a subset C of vertices such that the induced subgraph on C forms a complete graph. The *clique number* of the graph X, denoted $\omega(X)$, is the maximum size of a clique in X. [9] an *independent set* in a graph X is a subset Y of its vertices such that the induced subgraph on Y contains no edges. The *independence number* of the graph X, denoted $\alpha(X)$, is the size of the largest independent set in X. [10] consider a subset S of the vertices of a graph Γ and represent the set of vertices in Γ that are either in S or adjacent to a vertex in S as $N_\Gamma[S]$. If $N_\Gamma[S] = V(\Gamma)$, then S is called a *dominating set* for Γ. The *dominating number* $\gamma(\Gamma)$ is the smallest size of a dominating set for the vertices of Γ. Such properties enable us to model and analyze complex systems, as well as address problems across various domains. The properties of a graph associated with a group reveal various characteristics of the group. We have already introduced the groups under investigation in this paper. We will now present some definitions of these groups and related concepts and results.

Definition 2.1: [11] A group G is called K-metacyclic if it has a cyclic normal subgroup N of index k such that G/N is also cyclic. It is of order $p(p-1)$ and is generated by the elements x and y with defining relations:

$$x^p = y^{p-1} = 1;\ y^{-1}xy = x^r; (r-1, p) = 1$$

where r is a primitive root modulo p. Throughout this paper, we shall use G to represent this group.

Definition 2.2: [12] An integer b is a primitive root modulo m if b is coprime to m and the order of $b(\text{mod } m)$ is $\phi(m)$.

Result 2.4: [13] (Kuratowski's): A graph is nonplanar if and only if it contains a subdivision of K_5 or $K_{3,3}$.

3 Main Results

The present section is divided into two parts for clarity and focus. In the first subsection, we derive the conjugacy class graph of the direct product of two special linear groups, as well as the conjugacy class graph of the direct product of a special linear group and a K-metacyclic group. These results include detailed analyses of the graph's connectivity, planarity, chromatic number, independence number, and dominating number. In the second subsection, we investigate the complement graph of the conjugacy class graph of the direct product of a special linear group and a K-metacyclic group. We analyze its structural properties and resulting graph parameters such as independence and chromatic numbers.

3.1 Conjugacy Class Graph of Direct Products

Theorem 3.1.1: The conjugacy class graph of the direct product of $SL(2,q')$ and $SL(2,q)$ denoted by $\Gamma^{cc}_{SL(2,q')\times SL(2,q)}$ is a complete graph of order $(q'+4)(q+4)-4$, where q' and q are odd primes.

Proof: If we consider the direct product $(2,q')\times SL(2,q)$, we get,

- 4 conjugacy classes of order 1
- 8 conjugacy classes of order $\frac{(q+1)(q-1)}{2}$
- $q-1$ conjugacy classes of order $q(q-1)$
- $q-3$ conjugacy classes of order $q(q+1)$
- 8 conjugacy classes of order $\frac{(q\prime+1)(q\prime-1)}{2}$
- 16 conjugacy classes of order $\frac{(q\prime+1)(q\prime-1)(q+1)(q-1)}{4}$
- $2(q-1)$ conjugacy classes of order $\frac{q(q-1)(q\prime+1)(q\prime-1)}{2}$
- $2(q-3)$ conjugacy classes of order $\frac{q(q+1)(q\prime+1)(q\prime-1)}{2}$
- $q'-1$ conjugacy classes of order $q'(q'-1)$
- $2(q'-1)$ conjugacy classes of order $q'(q'-1)(q+1)(q-1)$
- $\frac{(q\prime-1)(q-1)}{4}$ conjugacy classes of order $q'q(q'-1)(q-1)$
- $\frac{(q-3)(q\prime-1)}{4}$ conjugacy classes of order $q'q(q'-1)(q+1)$
- $q'-3$ conjugacy classes of order $q'(q'+1)$
- $2(q'-3)$ conjugacy classes of order $\frac{q\prime(q\prime+1)(q+1)(q-1)}{2}$
- $\frac{(q\prime-3)(q-1)}{4}$ conjugacy class of order $q'q(q'+1)(q-1)$
- $\frac{(q\prime-3)(q-3)}{4}$ conjugacy classes of order $q'q(q'+1)(q+1)$

Since the conjugacy class graph considers only the non-central conjugacy classes, the number of vertices in $\Gamma^{cc}_{SL(2,q\prime)\times SL(2,q)}$ is $(q'+4)(q+4)-4$. Also, given that q' and q are odd primes it follows that $(q'-1), (q-1), (q'+1)$ and $(q+1)$ are even; thus all the cases discussed above are even. Hence all the conjugacy classes have even orders, ensuring that the greatest common divisor gcd of any pair of class orders is always greater than or equal to 2. Consequently, all vertices in the conjugacy class graph are connected, resulting in a complete graph.

Theorem 3.1.2: The conjugacy class graph of $SL(2,q) \times G$ is a connected graph with $p(q+4)-2$ vertices, and the graph structures are shown in Figure 1, Figure 2, Figure 3 and Figure 4.

Proof: $SL(2,q)$ consists of $q+4$ conjugacy classes while G consists of p conjugacy classes, [11]. Hence, $SL(2,q) \times G$ consists of a total of $p(q+4)$ conjugacy classes. 2 of the conjugacy classes are of order 1, 2 are of order $p-1$, $2(p-2)$ are of order p, 4 are of order $\frac{(q+1)(q-1)}{2}$, 4 are of order $\frac{(p-1)(q+1)(q-1)}{2}$, $4(p-2)$ are of order $\frac{p(q+1)(q-1)}{2}$, $\frac{q-1}{2}$ are of order $q(q-1)$, $\frac{q-1}{2}$ are of order $q(q-1)(p-1)$, $\frac{(q-1)(p-2)}{2}$ are of order $q(q-1)p$, $\frac{q-3}{2}$ are of order $q(q+1)$, $\frac{q-3}{2}$ are of order $q(q+1)(p-1)$, $\frac{(q-3)(p-2)}{2}$ are of order $pq(q+1)$. Since the conjugacy class graph considers only the non-central conjugacy classes, the number of vertices in $\Gamma^{cc}_{SL(2,q)\times G}$ is $p(q+4)-2$. In the following figures: Figure 1, Figure 2, Figure 3 and Figure 4, each v_i represents a complete graph in which each vertex is a conjugacy class with specific orders which are listed as follows:

- v_1 represents the complete graph $K_{2(p-2)}$ in which each vertex is the conjugacy class of order p
- v_2 represents the complete graph K_4 in which each vertex is the conjugacy class of order $\frac{(q+1)(q-1)}{2}$
- v_3 represents the complete graph K_4 in which each vertex is the conjugacy class of order $\frac{(p-1)(q+1)(q-1)}{2}$
- v_4 represents the complete graph $K_{4(p-2)}$ in which each vertex is the conjugacy class of order $\frac{p(q+1)(q-1)}{2}$
- v_5 represents the complete graph $K_{\frac{q-1}{2}}$ in which each vertex is the conjugacy class of order $q(q-1)$
- v_6 represents the complete graph $K_{\frac{q-1}{2}}$ in which each vertex is the conjugacy class of order $q(q-1)(p-1)$
- v_7 represents the complete graph $K_{\frac{(q-1)(p-2)}{2}}$ in which each vertex is the conjugacy class of order $pq(q-1)$
- v_8 represents the complete graph $K_{\frac{q-3}{2}}$ in which each vertex is the conjugacy class of order $q(q+1)$
- v_9 represents the complete graph $K_{\frac{q-3}{2}}$ in which each vertex is the conjugacy class of order $q(q+1)(p-1)$
- v_{10} represents the complete graph $K_{\frac{(q-3)(p-2)}{2}}$ in which each vertex is the conjugacy class of order $pq(q+1)$
- v_{11} represents the complete graph K_2 in which each vertex is the conjugacy class of order $p-1$.

Vertices between two distinct v_i and v_j indicate that all the vertices in v_i are adjacent to all the vertices in v_j, also all the vertices in a particular v_i are all adjacent to one another. The shaded region is used to imply that all the vertices are connected and hence represent a complete graph. The graph structure of $\Gamma^{cc}_{SL(2,q)\times G}$ is discussed below:-

Case 1: When $p \neq q$: Three sub-cases arise.

Case (i): $gcd(p,q+1) > 1$.

q being an odd prime, $q+1$ and $q-1$ are consecutive even integers and thus $\gcd(q+1, q-1) = 2$. The condition $gcd(p,q+1) > 1$ implies that p divides $q+1$. Since p is an odd prime and $\gcd(q+1,q-1) = 2$, it is easy to verify that p cannot divide $q-1$. Hence $gcd(p,q-1) = 1$. In this case, the graph of $\Gamma^{cc}_{SL(2,q)\times G}$ is as given in Figure 1.

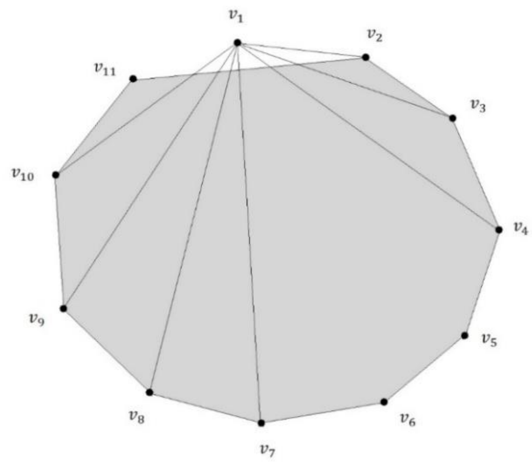

Fig. 1: $\Gamma^{cc}_{SL(2,q)\times G}$ when $p \neq q$ and $gcd(p,q+1) > 1$

Case (ii): $gcd(p,q-1) > 1$.

With similar arguments as in case 1, we obtain that $gcd(p, q+1) = 1$ when $gcd(p, q-1) > 1$, and the graph of $\Gamma^{cc}_{SL(2,q)\times G}$ in this case is as given in Figure 2.

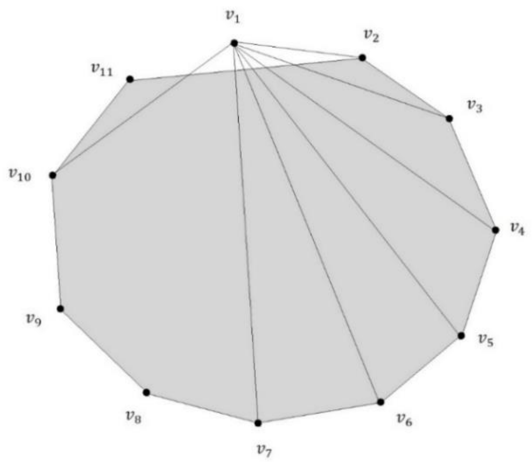

Fig. 2: $\Gamma^{cc}_{SL(2,q)\times G}$ when $p \neq q$ and $gcd(p, q-1) > 1$

Case (iii): $gcd(p, q+1) = gcd(p, q-1) = 1$.
The graph of $\Gamma^{cc}_{SL(2,q)\times G}$ is given in Figure 3.

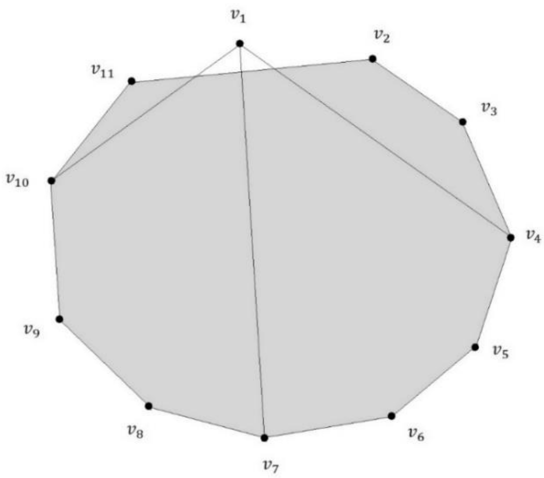

Fig. 3: $\Gamma^{cc}_{SL(2,q)\times G}$ when $p \neq q$ and $gcd(p, q+1) = gcd(p, q-1) = 1$

Case 2: When $p = q$: The graph of $\Gamma^{cc}_{SL(2,q)\times G}$ is given in Figure 4.

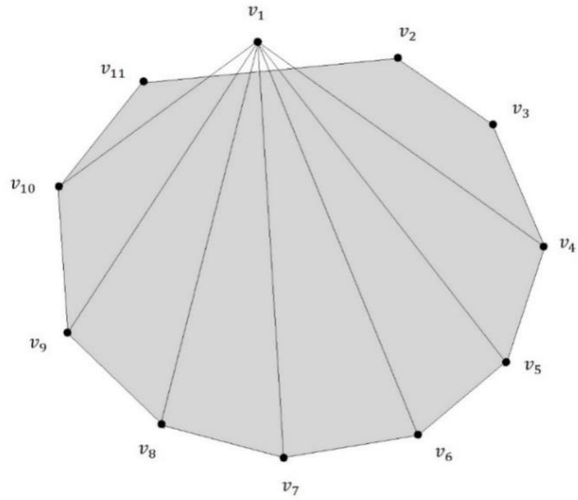

Fig. 4: $\Gamma^{cc}_{SL(2,q)\times G}$ when $p = q$

From Figure 1, Figure 2, Figure 3 and Figure 4, we see that $\Gamma^{cc}_{SL(2,q)\times G}$ is always a connected graph. Hence the theorem.

Example 3.1: $\Gamma^{cc}_{SL(2,q)\times G}$ when $p = q = 3$ is a connected graph with 19 vertices. Since $p = q = 3$, the graph follows from Figure 4 and is as shown in Figure 5.

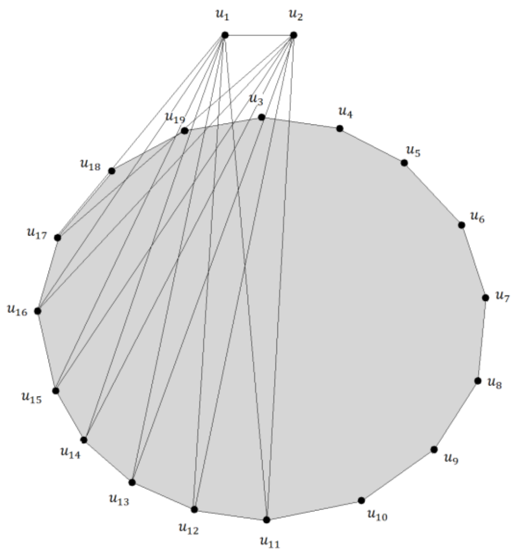

Fig. 5: $\Gamma^{cc}_{SL(2,q)\times G}$ when $p = q = 3$

In Figure 5, each u_i is a vertex of $\Gamma^{cc}_{SL(2,q)\times G}$ when $p = q = 3$ and the shaded region represents a complete graph.

- u_1 and u_2 are the conjugacy classes of order 3 (they are represented by v_1 in Figure 4)

- u_3, u_4, u_5 and u_6 are the conjugacy classes of order 4 (they are represented by v_2 in Figure 4)
- u_7, u_8, u_9 and u_{10} are the conjugacy classes of order 8 (they are represented by v_3 in Figure 4)
- u_{11}, u_{12}, u_{13} and u_{14} are the conjugacy classes of order 12 (they are represented by v_4 in Figure 4)
- u_{15} is the conjugacy class of order 6 (they are represented by v_5 in Figure 4)
- u_{16} is the conjugacy class of order 12 (they are represented by v_6 in Figure 4)
- u_{17} is the conjugacy class of order 18 (they are represented by v_7 in Figure 4)
- u_{18} and u_{19} are the conjugacy classes of order 2 (they are represented by v_{11} in Figure 4).

Corollary 3.1.3: $\Gamma^{cc}_{SL(2,q)\times G}$ is non-planar for all p and q.

Proof: $\Gamma^{cc}_{SL(2,q)\times G}$ always contains a subdivision of K_5 for all p and q and hence is non-planar.

Theorem 3.1.4: $\Gamma^{cc}_{SL(2,q)\times G}$ is Eulerian for all p and q.

Proof: To prove that a graph is Eulerian, it is sufficient to show that all the vertices are of even degree. We verify this property for $\Gamma^{cc}_{SL(2,q)\times G}$ based on the degree of vertices shown in Figure 1, Figure 2, Figure 3 and Figure 4. For the degree of vertices in Figure 1, we have as follows:
Degree of each vertex in $v_1 = 2p - 5 + 4 + 4 + 4p - 8 + \frac{(q-1)(p-2)}{2} + \frac{q-3}{2} + \frac{q-3}{2} + \frac{(q-3)(p-2)}{2} = (q+4)(p-1)$ which is even.

Degree of each vertex in $v_2, v_3, v_4, v_7, v_8, v_9$, and $v_{10} = p(q+4) - 3$ which is even.

Degree of each vertex in v_5, v_6 and $v_{11} = pq + 2p + 1$ which is even. i.e. the degree of each vertex in Figure 1 is even.

Similarly, from Figure 2, Figure 3 and Figure 4, we can deduce that all the vertices of $\Gamma^{cc}_{SL(2,q)\times G}$ have an even degree for all p and q and hence $\Gamma^{cc}_{SL(2,q)\times G}$ is Eulerian.

Theorem 3.1.5: The independence number α of $\Gamma^{cc}_{SL(2,q)\times G}$ is 2 for all p and q.

Proof: Case1: When $p \neq q$ and $gcd(p, q+1) > 1$.
From Figure 1, we see that a maximum of 2 vertices can be taken from the graph so that the induced subgraph is an empty graph. One of the vertices is taken from v_1 and the other vertex is taken from either v_5 or v_6 or v_{11}. Hence, $\alpha\left(\Gamma^{cc}_{SL(2,q)\times G}\right) = 2$.

Case 2: When $p \neq q$ and $gcd(p, q-1) > 1$.
From Figure 2, we can take one vertex from v_1 and the other vertex from either v_8 or v_9 or v_{11} so that the induced subgraph is an empty graph. Hence, $\alpha\left(\Gamma^{cc}_{SL(2,q)\times G}\right) = 2$.

Case 3: When $p \neq q$ and $gcd(p, q+1) = gcd(p, q-1) = 1$.
Similarly, from Figure 3, we can take one vertex from v_1 and the other vertex from either v_2 or v_3 or v_5 or v_6 or v_8 or v_9 or v_{11} so that the induced subgraph is an empty graph. Hence, $\alpha\left(\Gamma^{cc}_{SL(2,q)\times G}\right) = 2$.

Case 4: When $p = q$.
From Figure 4, we can take one vertex from v_1 and the other vertex from either v_2 or v_3 or v_{11} so that the induced subgraph is an empty graph. Hence, $\alpha\left(\Gamma^{cc}_{SL(2,q)\times G}\right) = 2$.

Theorem 3.1.6: The dominating number γ of $\Gamma^{cc}_{SL(2,q)\times G}$ is 1 for all p and q.

Proof: Firstly, in Theorem 3.12, we have already obtained that $\Gamma^{cc}_{SL(2,q)\times G}$ is a connected graph and so there are no isolated vertices. Also, from Figure 1, Figure 2, Figure 3 and Figure 4, we see that each figure consists of at least one v_i which is adjacent to all other vertices in the graph. Hence, if we take any vertex from these v_i's, we get the minimum size of a dominating set and hence, $\gamma(\Gamma^{cc}_{SL(2,q)\times G}) = 1$ for all p and q.

Theorem 3.1.7: $\chi\left(\Gamma^{cc}_{SL(2,q)\times G}\right) = \omega\left(\Gamma^{cc}_{SL(2,q)\times G}\right) = pq + 2p + 2$ when $p \neq q$ and $gcd(p, q+1) > 1$.

Proof: From Figure 1, the two largest complete subgraphs that can be formed are the subgraphs obtained by joining the vertices of $v_1, v_2, v_3, v_4, v_7, v_8, v_9, v_{10}$ and another by joining $v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}$. The former complete subgraph gives the complete subgraph $K_{pq+4p-q-3}$ and the latter complete subgraph gives the complete subgraph $K_{pq+2p+2}$. Thus the clique number will be the greater one between $pq + 2p + 2$ and $pq + 4p - q - 3$. Now, take $A = pq + 2p + 2$ and $B = pq + 4p - q - 3$.

$$A - B = -2p + q + 5$$

Thus, $A > B$ if $q > 2p - 5$ and $A < B$ if $q < 2p - 5$.
But, we have taken $gcd(p, q+1) > 1$. Thus, $p | q + 1 \Rightarrow q + 1 = kp$, where k is an integer. Since p and

q are odd primes, $q+1$ is even and thus k must be even.

We can now conclude that, $q+1=kp \geq 2p \Rightarrow q \geq 2p-1 > 2p-5$. So, $A<B$ is not possible and hence $A>B$, which implies $\omega\left(\Gamma_{SL(2,q)\times G}^{cc}\right)=pq+2p+2$, when $p \neq q$ and $gcd(p,q+1)>1$.
Now, for chromatic number:

The largest complete subgraph is $K_{pq+2p+2}$ and thus a minimum of $pq+2p+2$ colors will be required to color $\Gamma_{SL(2,q)\times G}^{cc}$.
Number of vertices left to be colored = number of vertices in $v_1 = 2p-4$.

v_1 is not adjacent to v_{11}, v_5 and v_6 and hence it can be colored by choosing a color assigned to v_{11}, v_5 or v_6. Number of vertices in v_{11}, v_5 and $v_6 = q+1$.

Since $q>2p-5$, the $2p-4=2p-5+1<q+1$ vertices in v_1 can be colored by the $q+1$ colors in v_{11}, v_5 and v_6.
Hence, $\chi\left(\Gamma_{SL(2,q)\times G}^{cc}\right)=pq+2p+2$ when $p \neq q$ and $gcd(p,q+1)>1$.

Theorem 3.1.8: $\chi\left(\Gamma_{SL(2,q)\times G}^{cc}\right)=\omega\left(\Gamma_{SL(2,q)\times G}^{cc}\right)=pq+2p+2$ when $p \neq q$ and $gcd(p,q-1)>1$.

Proof: From Figure 2, the two largest complete subgraphs that can be formed are the subgraphs obtained by joining the vertices of $v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_{10}$ and another by joining $v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}$. The former complete subgraph gives the complete subgraph $K_{pq+4p-q-1}$ and the latter complete subgraph gives the complete subgraph $K_{pq+2p+2}$. Thus the clique number will be the greater one between $pq+2p+2$ and $pq+4p-q-1$. Now, take $A=pq+2p+2$ and $B=pq+4p-q-1$.

$$A-B=-2p+q+3$$

Thus, $A>B$ if $q>2p-3$ and $A<B$ if $q<2p-3$.

But, we have taken $gcd(p,q-1)>1$. Thus, $p|q-1 \Rightarrow q-1=kp$, where k is an integer. Since p and q are odd primes, $q-1$ is even and thus k must be even.

We can now conclude that, $q-1=kp \geq 2p \Rightarrow q \geq 2p+1 > 2p-3$. So, $A<B$ is not possible.
Hence, $A>B$ and $\omega\left(\Gamma_{SL(2,q)\times G}^{cc}\right)=pq+2p+2$, when $p \neq q$ and $gcd(p,q-1)>1$.

Now, for chromatic number:

The largest complete subgraph is $K_{pq+2p+2}$ and thus a minimum of $pq+2p+2$ vertices will be required to color $\Gamma_{SL(2,q)\times G}^{cc}$.

Number of vertices left to be colored = number of vertices in $v_1 = 2p-4$.
v_1 is not adjacent to v_{11}, v_5 and v_6 and hence it can be colored by choosing a color assigned to v_{11}, v_5 and v_6. Number of vertices in v_{11}, v_5 and $v_6 = q+1$.
Since $q>2p-3$, the $2p-4=2p-3-1<q-1$ vertices in v_1 can be colored by the $q+1$ colors in v_{11}, v_5 and v_6.

Hence, $\chi\left(\Gamma_{SL(2,q)\times G}^{cc}\right)=pq+2p+2$ when $p \neq q$ and $gcd(p,q-1)>1$.

Theorem 3.1.9: $\chi\left(\Gamma_{SL(2,q)\times G}^{cc}\right)=\omega\left(\Gamma_{SL(2,q)\times G}^{cc}\right)=\begin{cases} pq+2p+2 & \text{if } q>p-5 \\ pq+4p-2q-8 & \text{if } q<p-5 \\ p^2-3p+2 & \text{if } q=p-5 \end{cases}$ when $p \neq q$ and $gcd(p,q+1)=gcd(p,q-1)=1$.

Proof: From Figure 3, the two largest complete subgraphs that can be formed are the subgraphs obtained by joining the vertices of v_1, v_4, v_7, v_{10} and another by joining $v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}$. The former complete subgraph gives the complete subgraph $K_{pq+4p-2q-8}$ and the latter complete subgraph gives the complete subgraph $K_{pq+2p+2}$. Thus the clique number will be the greater one between $pq+2p+2$ and $pq+4p-2q-8$. Now, take $A=pq+2p+2$ and $B=pq+4p-2q-8$.

$$A-B=-2p+2q+10$$

Thus, $A>B$ if $q>p-5$, $A<B$ if $q<p-5$, $A=B$ if $q=p-5$ and,
$\omega\left(\Gamma_{SL(2,q)\times G}^{cc}\right)=\begin{cases} pq+2p+2 & \text{if } q>p-5 \\ pq+4p-2q-8 & \text{if } q<p-5 \\ p^2-3p+2 & \text{if } q=p-5 \end{cases}$ when $p \neq q$ and $gcd(p,q+1)=gcd(p,q-1)=1$.

Now, for chromatic number:

Case 1: When $q>p-5$.
The largest complete subgraph is $K_{pq+2p+2}$ and thus a minimum of $pq+2p+2$ vertices will be required to color $\Gamma_{SL(2,q)\times G}^{cc}$.
Number of vertices left to be colored = number of vertices in $v_1 = 2p-4$.

v_1 is not adjacent to $v_2, v_3, v_5, v_6, v_8, v_9$ and v_{11} and hence can be colored by choosing a color in one of these vertices. Number of vertices in $v_2, v_3, v_5, v_6, v_8, v_9$ and $v_{11} = 2q + 6$.
Since $q > p - 5$, the $2p - 4 = 2p - 5 + 1 < q + 1 < 2q + 6$ vertices in v_1 can be colored by the $2q + 6$ colors in $v_2, v_3, v_5, v_6, v_8, v_9$ and v_{11}.
Thus, $\chi(\Gamma^{cc}_{SL(2,q)\times G}) = pq + 2p + 2$ when $q > p - 5$.

Case 2: When $q < p - 5$.
The largest complete subgraph is $K_{pq+4p-2q-8}$ and thus a minimum of $pq + 4p - 2q - 8$ vertices will be required to color $\Gamma^{cc}_{SL(2,q)\times G}$.
Number of vertices left to be colored = number of vertices in $v_2, v_3, v_5, v_6, v_8, v_9$ and $v_{11} = 2q + 6$.
The vertices in $v_2, v_3, v_5, v_6, v_8, v_9$ and v_{11} are not adjacent to the vertices in v_1 and hence can be colored by the $2p - 4$ colors in v_1.
Since $q < p - 5$, $2q < 2p - 10$ and $2q + 6 < 2p - 4$. Thus the $2q + 6$ vertices in $v_2, v_3, v_5, v_6, v_8, v_9$ and v_{11} can be colored by the $2p - 4$ colors in v_1.
Thus, $\chi(\Gamma^{cc}_{SL(2,q)\times G}) = pq + 4p - 2q - 8$ when $q < p - 5$.
Case 3: When $q = p - 5$.
In this case $pq + 2p + 2$ and $pq + 4p - 2q - 8$ become equal and substituting $q = p - 5$ in either of them results in $p^2 - 3p + 2$ and thus $\chi(\Gamma^{cc}_{SL(2,q)\times G}) = p^2 - 3p + 2$ when $q > p - 5$.
Hence we get,

$$\chi(\Gamma^{cc}_{SL(2,q)\times G}) = \begin{cases} pq + 2p + 2 & \text{if } q > p - 5 \\ pq + 4p - 2q - 8 & \text{if } q < p - 5 \\ p^2 - 3p + 2 & \text{if } q = p - 5 \end{cases} \quad \text{when}$$

$p \neq q$ and $gcd(p, q + 1) = gcd(p, q - 1) = 1$.

Theorem 3.1.10: $\chi(\Gamma^{cc}_{SL(2,q)\times G}) = \omega(\Gamma^{cc}_{SL(2,q)\times G}) =$

$$\begin{cases} pq + 2p + 2 & \text{if } p < 7 \\ pq + 4p - 12 & \text{if } p > 7 \\ 65 & \text{if } p = 7 \end{cases} \quad \text{when } p = q.$$

Proof: From Figure 4, the two largest complete subgraphs that can be formed are the subgraphs obtained by joining the vertices of $v_1, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}$ and another by joining $v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}$. The former complete subgraph gives the complete subgraph $K_{pq+4p-12}$ and the latter complete subgraph gives the complete subgraph $K_{pq+2p+2}$. Thus the clique number will be the greater one between $pq + 2p + 2$ and $pq + 4p - 12$. Now, take $A = pq + 2p + 2$ and $B = pq + 4p - 12$.

$$A - B = -2p + 14$$

Thus, $A > B$ if $p < 7$, $A < B$ if $p > 7$, $A = B$ if $p = 7$ and,

$$\omega(\Gamma^{cc}_{SL(2,q)\times G}) = \begin{cases} pq + 2p + 2 & \text{if } p < 7 \\ pq + 4p - 12 & \text{if } p > 7 \\ 65 & \text{if } p = 7 \end{cases} \quad \text{when } p = q.$$

Now, for chromatic number:
Case 1: When $p < 7$.
The largest complete subgraph is $K_{pq+2p+2}$ and thus a minimum of $pq + 2p + 2$ vertices will be required to color $\Gamma^{cc}_{SL(2,q)\times G}$.
Number of vertices left to be colored = number of vertices in $v_1 = 2p - 4$.
The vertices of v_1 is not adjacent to the vertices in v_2, v_3, v_{11} and hence can be colored by the colors in these vertices. Number of vertices in $v_2, v_3, v_{11} = 10$.
When $p < 7$, $2p - 4 < 10$. Hence the $2p - 4$ vertices in v_1 can be colored by the 10 vertices in v_2, v_3, and v_{11}.
Thus, $\chi(\Gamma^{cc}_{SL(2,q)\times G}) = pq + 2p + 2$ when $p < 7$.

Case 2: When $p > 7$.
The largest complete subgraph is $K_{pq+4p-12}$ and thus a minimum of $pq + 4p - 12$ vertices will be required to color $\Gamma^{cc}_{SL(2,q)\times G}$.
Number of vertices left to be colored = number of vertices in v_2, v_3 and $v_{11} = 10$.
The vertices in v_2, v_3, v_{11} are not adjacent to the vertices in v_1 and hence can be colored by the colors in these vertices. A number of vertices in $v_1 = 2p - 4$.
When $p > 7$, $2p - 4 > 10$. So, the 10 vertices in v_2, v_3, v_{11} can be colored by the $2p - 4$ vertices in v_1. Hence, $\chi(\Gamma^{cc}_{SL(2,q)\times G}) = pq + 4p - 12$ when $p > 7$.
Case 3: When $p = 7$.
In this case, $pq + 2p + 2$ and $pq + 4p - 12$ become equal and equal to 65. Hence,

$$\chi(\Gamma^{cc}_{SL(2,q)\times G}) = \begin{cases} pq + 2p + 2 & \text{if } p < 7 \\ pq + 4p - 12 & \text{if } p > 7 \\ 65 & \text{if } p = 7 \end{cases} \quad \text{when } p = q.$$

3.2 Complement Graph of Conjugacy Class Graph

Theorem 3.2.1: The complement graph of $\Gamma^{cc}_{SL(2,q)\times G}$ is given as:

$$\overline{\Gamma^{cc}_{SL(2,q)\times G}} = \begin{cases} \overline{K_{pq+2p-q+1}} \cup K_{2(p-2),q+1}, & \text{if } p \neq q \text{ and } gcd(p,q+1) > 1 \\ \overline{K_{pq+2p-q+3}} \cup K_{2(p-2),q-1}, & \text{if } p \neq q \text{ and } gcd(p,q-1) > 1 \\ \overline{K_{pq+2p-2q-4}} \cup K_{2(p-2),2(q+3)}, & \text{if } gcd(p,q+1) = \gcd(p,q-1) = 1 \\ \overline{K_{pq+2p-8}} \cup K_{2(p-2),10}, & \text{if } p = q \end{cases}$$

Proof: Four cases arise for finding $\overline{\Gamma^{cc}_{SL(2,q)\times G}}$:

Case 1: When $p \neq q$ and $gcd(p,q+1) > 1$. From Figure 1, the vertices in $v_2, v_3, v_4, v_7, v_8, v_9, v_{10}$ are adjacent to all other vertices in $\Gamma^{cc}_{SL(2,q)\times G}$ and thus they will be isolated vertices in its complement.

Number of vertices in $v_2, v_3, v_4, v_7, v_8, v_9, v_{10} = pq + 2p - q + 1$. Thus, there will be $pq + 2p - q + 1$ isolated vertices in $\overline{\Gamma^{cc}_{SL(2,q)\times G}}$. Also, all vertices of v_1 will be adjacent to all vertices of v_5, v_6 and v_{11} in $\overline{\Gamma^{cc}_{SL(2,q)\times G}}$ but the vertices within a particular v_i will not be adjacent.

A number of vertices in $v_1 = 2(p-2)$ and number of vertices in v_5, v_6 and $v_{11} = q + 1$. Thus, we get the complete bipartite graph $K_{2(p-2),q+1}$.

Case 2: When $p \neq q$ and $gcd(p,q-1) > 1$. From Figure 2, the vertices in $v_2, v_3, v_4, v_5, v_6, v_7, v_{10}$ are adjacent to all other vertices in $\Gamma^{cc}_{SL(2,q)\times G}$ and thus they will be isolated vertices in its complement.

Number of vertices in $v_2, v_3, v_4, v_5, v_6, v_7, v_{10} = pq + 2p - q + 3$. Thus, there will be $pq + 2p - q + 3$ isolated vertices in $\overline{\Gamma^{cc}_{SL(2,q)\times G}}$. Also, all vertices of v_1 will be adjacent to all vertices in v_8, v_9 and v_{11} but the vertices within a particular v_i will not be adjacent to each other.

Number of vertices in $v_1 = 2(p-2)$ and number of vertices in v_8, v_9 and $v_{11} = q - 1$. Thus, we get the complete bipartite graph $K_{2(p-2),q-1}$.

Case 3: When $p \neq q$ and $gcd(p,q+1) = gcd(p,q-1) = 1$. From Figure 3, the vertices in v_4, v_7, v_{10} are adjacent to all other vertices in $\Gamma^{cc}_{SL(2,q)\times G}$ and thus they will be isolated vertices in its complement.

Number of vertices in $v_4, v_7, v_{10} = pq + 2p - 2q - 4$. Thus, there will be $pq + 2p - 2q - 4$ isolated vertices in $\overline{\Gamma^{cc}_{SL(2,q)\times G}}$. Also, all vertices of v_1 will be adjacent to all vertices in $v_2, v_3, v_5, v_6, v_8, v_9$ and v_{11} but the vertices within these will not be adjacent to each other. Also, the vertices of a particular v_i will not be adjacent to any other vertex in v_i itself.

Number of vertices in $v_1 = 2(p-2)$ and number of vertices in $v_2, v_3, v_5, v_6, v_8, v_9$ and $v_{11} = 2(q+3)$. Thus, we get the complete bipartite graph $K_{2(p-2),2(q+3)}$.

Case 4: When $p = q$. From Figure 4, the vertices in $v_4, v_5, v_6, v_7, v_8, v_9, v_{10}$ are adjacent to all other vertices in $\Gamma^{cc}_{SL(2,q)\times G}$ and thus they will be isolated vertices in its complement.

Number of vertices in $v_4, v_5, v_6, v_7, v_8, v_9, v_{10} = pq + 2p - 8$. Thus, there will be $pq + 2p - 8$ isolated vertices in $\overline{\Gamma^{cc}_{SL(2,q)\times G}}$. Also, all vertices of v_1 will be adjacent to all vertices in v_2, v_3 and v_{11} but the vertices within these will not be adjacent. Also, the vertices of a particular v_i will not be adjacent to any other vertex in v_i itself.

Number of vertices in $v_1 = 2(p-2)$ and number of vertices in v_2, v_3 and $v_{11} = 10$. Thus, we get the complete bipartite graph $K_{2(p-2),10}$. Hence the theorem.

Theorem 3.2.2: $\Gamma^{cc}_{SL(2,q)\times G}$ is planar when $p = 3$ and non-planar when $p > 3$.

Proof: We know that a graph is non-planar if and only if it contains a sub-division of K_5 or $K_{3,3}$[refer to Result 2.4].

Case1. When $p = 3$,

$2(p-2) = 2$. Thus, $\overline{\Gamma^{cc}_{SL(2,q)\times G}}$ becomes the complete bipartite graph $K_{2,x}$ where x represents one of the values $q+1, q-1, 2(q+3)$, 10 depending on the values of p and q and ignoring the isolated vertices. In all the cases, we see that the graph is planar as $K_{2,x}$ does not contain a subdivision of $K_{3,3}$ and hence $\overline{\Gamma^{cc}_{SL(2,q)\times G}}$ is planar for all q.

Case 2. When $p > 3$, two sub-cases arise:

Case(i) When $q > 3$: Let the bipartite graph $K_{m,n}$ represent the cases of $\overline{\Gamma^{cc}_{SL(2,q)\times G}}$ where $m = 2(p-2)$ and $n = q+1$ or $q-1$ or $2(q+3)$ or 10. Since $p > 3 \Rightarrow m \geq 6$ and since $q > 3 \Rightarrow n \geq 4$. Hence it contains a subdivision of $K_{3,3}$ and is non-planar.

Case(ii) When $q = 3$: $gcd(p,q+1) = \gcd(p,q-1) = 1$ and thus $\overline{\Gamma^{cc}_{SL(2,q)\times G}} = \overline{K_{pq+2p-2q-4}} \cup K_{2(p-2),12}$, which also contains a subdivision of $K_{3,3}$. Hence, $\overline{\Gamma^{cc}_{SL(2,q)\times G}}$ is non-planar when $p > 3$.

Theorem 3.2.3: $\chi\left(\overline{\Gamma^{cc}_{SL(2,q)\times G}}\right) = \omega\left(\overline{\Gamma^{cc}_{SL(2,q)\times G}}\right) = 2$.

Proof: Since $\overline{\Gamma^{cc}_{SL(2,q)\times G}}$ is a union of a complete bipartite graph and isolated vertices, we require at least 2 colors to color the complete bipartite graph. The rest of the isolated vertices can then be colored by one of those two vertices since they are not adjacent to any other vertex in the graph. Also, the clique number of a complete bipartite graph is always 2. Hence, $\chi\left(\overline{\Gamma^{cc}_{SL(2,q)\times G}}\right) = \omega\left(\overline{\Gamma^{cc}_{SL(2,q)\times G}}\right) = 2$.

Theorem 3.2.4: The independence number of $\overline{\Gamma^{cc}_{SL(2,q)\times G}}$ is given as, $\alpha\left(\overline{\Gamma^{cc}_{SL(2,q)\times G}}\right) = \begin{cases} pq + 2p - q + 1 + \max(2p-4, q+1), & \text{if } p \neq q \text{ and } gcd(p,q+1) > 1 \\ pq + 2p - q + 3 + \max(2p-4, q-1) & \text{if } p \neq q \text{ and } gcd(p,q-1) > 1 \\ pq + 2p - 2q - 4 + \max(2p-4, 2q+6), & \text{if } gcd(p,q+1) = \gcd(p,q-1) = 1 \\ pq + 2p - 8 + \max(2p-4, 10), & \text{if } p = q \end{cases}$

Proof: Recall that for a complete bipartite graph $K_{m,n}$, the independence number is the size of the larger partition i.e. $\max(m,n)$. Thus, in $\overline{\Gamma^{cc}_{SL(2,q)\times G}}$, the independence number will simply be the sum of the number of isolated vertices and the maximum value between the two partitions of the complete bipartite graph. Hence the theorem.

Theorem 3.2.5: The dominating number of $\overline{\Gamma^{cc}_{SL(2,q)\times G}}$ is given as, $\gamma\left(\overline{\Gamma^{cc}_{SL(2,q)\times G}}\right) = \begin{cases} pq + 2p - q + 1 + \min(2p-4, q+1), & \text{if } p \neq q \text{ and } gcd(p,q+1) > 1 \\ pq + 2p - q + 3 + \min(2p-4, q-1), & \text{if } p \neq q \text{ and } gcd(p,q-1) > 1 \\ pq + 2p - 2q - 4 + \min(2p-4, 2q+6), & \text{if } gcd(p,q+1) = \gcd(p,q-1) = 1 \\ pq + 2p - 8 + \min(2p-4, 10), & \text{if } p = q \end{cases}$

Proof: For a complete bipartite graph $K_{m,n}$, the dominating number is the size of the smaller partition i.e. $\min(m,n)$. Thus, in $\overline{\Gamma^{cc}_{SL(2,q)\times G}}$, the dominating number will simply be the sum of the number of isolated vertices and the minimum value between the two partitions of the complete bipartite graph. Hence the theorem.

Declaration of Generative AI and AI-assisted Technologies in the Writing Process

During the preparation of this work, the authors used ChatGpt and Grammarly solely to refine the language and improve readability. After using these tools, the authors reviewed and edited the content as needed and take full responsibility for the content of the publication.

4 Conclusion

In this paper, we have derived the conjugacy class graphs associated with $SL(2,q) \times SL(2,q')$, where q and q' are odd primes, $SL(2,q) \times G$, where G is the metacyclic group of order $p(p-1)$ and the complement graph. The graph for the first case is a complete graph of order $(q'+4)(q+4)-4$, while for the second case, the graph is found to be a connected graph with $p(q+4)-2$ vertices. We examine four scenarios: $p \neq q$ and $gcd(p,q+1) > 1$, $p \neq q$ and $gcd(p,q-1) > 1$, $p \neq q$ and $gcd(p,q+1) = gcd(p,q-1) = 1$, and lastly $p = q$. In each case, it is observed that the graph is a perfect graph, the independence number is found to be 2, and the dominating number is 1 for all values of p and q. The complement graph is a union of isolated vertices and a complete bipartite graph which is planar when $p = 3$ and non-planar when $p \neq 3$.

Beyond their theoretical significance, a visual framework is provided by the conjugacy class graphs to understand the structure of groups. This helps in the identification of patterns, symmetries, and relationships between group elements, offering insights that may not be apparent through purely algebraic approaches. By deriving and analyzing these graphs for the direct product of Special linear and K-metacyclic groups, this work provides new insights into the structural properties of conjugacy class graphs and their applications in algebraic graph theory.

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