Integral Equation Method for ARL on New Modified EWMA Chart in Change-Point Detection Problems

YUPAPORN AREEPONG*, SAOWANIT SUKPARUNGSEE
Department of Applied Statistics, Faculty of Applied Statistics,
King Mongkut's University of Technology North Bangkok,
Bangkok 10800,
THAILAND

*Corresponding Author

Abstract: - This research proposes an explicit formula for the Average Run Length (ARL) on a new modified EWMA (new MEWMA) control chart. This study proposes a mathematical algorithm for determining the ARL of a new MEWMA control chart for detecting autocorrelated processes for zero-state. The integral equation method is called Fredholm Integral Equations of the second kind can be effectively employed to calculate ARL. Banach's fixed point theorem is utilized to demonstrate the existence and uniqueness of the ARL solution. A process for constructing one-sided and two-sided new MEWMA control charts is presented, and the results were compared to the accuracy with numerical integral equations relying on various quadrature rules. This algorithm will utilize the autoregressive with the exogenous variables model (ARX(p,r)) and apply the algorithm to examine empirical data in the economic area. The effectiveness of control charts can be further evaluated using the expected average run length and the expected standard deviation of run length measures. Our analysis indicates that the new MEWMA control chart surpasses the MEWMA and EWMA control charts in performance. Comparisons are conducted for varying magnitudes of the process mean shift and varied levels of autocorrelation.

Key-Words: - Average Run Length, change-point detection, zero-state, Autoregressive with exogenous variable model, new Modified Exponentially Weighted Moving Average, control charts.

Received: October 5, 2024. Revised: January 19, 2025. Accepted: February 21, 2025. Published: April 10, 2025.

1 Introduction

The industrial sector extensively employs statistical process control (SPC) techniques to enhance quality and monitor processes. Conventional SPC charts rely on the fundamental assumption of normally distributed and statistically independent process data under control, [1], [2].

Early on, the study [3] invented the first control chart, called the Shewhart control chart. Later, many researchers developed various control charts for tracking changes in the process. Later, the cumulative sum control chart (CUSUM) proposed by [4] and the exponential weighted moving average (EWMA) control chart proposed by [5] were two important techniques for detecting small changes in the process. In many processes, the main assumption of observations being independently and identically distributed does not always hold. This assumption is wrong when the process data points are autocorrelated. More recently, [6] introduced the modified exponential weighted moving average (MEWMA) control chart, which added an observation term to the EWMA control chart statistic, which was found to be effective in detecting small changes. It was found that MEWMA control charts have been used in various practical applications, such as data with high correlation, such as in chemical temperature measurement. Later, [7] developed a statistic of the MEWMA control chart by modifying the observation term invented by [6] by increasing the constant value from 1 to any constant value so that the process changes can be detected more quickly. Recently, a study [8] proposed a new MEWMA control chart using the concept of unequal weighting of constants to assign more weight to the current data compared to the past data. It was found that this control chart can detect changes faster than the MEWMA control chart and uses equal weighting of the current data to the past data. Nevertheless, the conventional SPC methods may not be suitable for the monitoring, control, and enhancement of process quality in practice, as process data are not always independent. For instance, the study [9] applied the Shewhart, EWMA, CUSUM, or GMA control charts on the uncorrelated residuals of the time series process. This approach is a primary method for addressing both the stationary first-order autoregressive model and the trend stationary first-order autoregressive model, also known as trend AR(1). In many fields, including finance, economics, and the stock market, time series are extensively employed. The ARIMA model is extensively used as an analysis model for time series data. Nevertheless, exogenous variables that impact the variable of interest are frequently overlooked in prior studies. Most prior research concentrates on only the variable of interest. This research investigates the effect of potential exogenous inputs. The autoregressive with exogenous variables (ARX) model is a model employed in this study to determine the relationship between multiple variables, [10], [11]. Additionally, it is necessary to use time series models correctly to fit the autocorrelation observations. This can be done by looking at the ACF and PACF functions from the time series data. By determining the appropriate time series model and estimating the parameters of the selected model, the control chart will be more effective. Selecting the appropriate control chart for the data under consideration is crucial. Typically, the residual yields white noise. Nonetheless, exponential white noise can appear in certain datasets, [12], [13]. In economics, exponential white noise can represent random fluctuations in time series data, such as financial variables, stock returns, and commodity prices, [14], [15]. Consequently, this research examines ARX(p,r) with an exponentially distributed residual.

In the literature, two varieties of ARL are addressed: in-control ARL (ARL_0) and out-of-control ARL (ARL_Δ). ARL_0 represents the anticipated number of samples until a control chart signals, assuming that the process is under control. One may interpret this as a false alarm signal. The process is under control, so ARL_0 should be as large as possible. ARL_Δ represents the expected number of samples until a control chart indicates a signal, presuming that the process is out of control due to a shift in the mean. ARL_Δ is intended to be as minimal as possible.

A variety of methodologies, such as Monte Carlo simulation, numerical integral equation (NIE), the Markov Chain approach (MCA), and explicit formulations, may be utilized to evaluate ARL. For instance, the study [16] aims to create a triple HWMA (THWMA) chart for effective monitoring of process mean conditions. The suggested chart is tested against HWMA, DHWMA, EWMA, and double EWMA control schemes using the ARL criterion and Monte Carlo simulations to see how well it works. Using the numerical integral equation (NIE) method, the study [17] gets a rough idea of the average run length (ARL) for a long-memory fractionally integrated moving-average process with an exogenous variable (FI-MAX). The research [18] presents an estimated average run length (ARL) that utilizes four quadrature rules: the composite midpoint, trapezoidal, Simpson's, and Gauss-Legendre rules, to identify shifts in the process mean on a modified EWMA control chart. The observations originate from gamma or Weibull distributions. The criteria for evaluation were the ARL_1 and CPU time. The results show that all four quadrature methods for approximating the ARL on a modified EWMA control chart were about as accurate as each other. The study [19] looks at how to use a Markov Chain to estimate the average run length (ARL) for a Poisson EWMA chart with linear drifts. The results indicate that the MCA method yields an accurate estimation of the ARL in comparison to the Monte Carlo simulation. Utilizing explicit formulas. Numerous researchers have investigated this technique. The study [20] analyses the Average Run Length (ARL) for long memory in detecting mean shifts in the Max process on the Exponentially Weighted Moving Average (EWMA) control chart.

The research [21] formulated an explicit equation by integral analysis of the ARL on the Cumulative Sum (CUSUM) chart for the Seasonal Autoregressive Integrated Moving Average with Exogenous Variables ($SARX(P,r)_L$) model. The correctness of the ARL obtained using numerical integral equations utilizing the midpoint rule was evaluated through comparison. The study [22] demonstrates the ARL of the DEWMA control chart for identifying minor changes. The trends and seasonality of an autoregressive model were examined on the DEWMA chart. The explicit ARL was developed for simulated data and contrasted with the numerical integral equation (NIE) method.

The previously mentioned study revealed that no researcher had validated the average run length calculation for the new MEWMA control chart under the ARX(p,r) model. Consequently, this research derived the explicit formula for the Average Run Length (ARL) of the new MEWMA control chart under the zero-state, evaluated its efficacy in detecting shifts in the process mean, and compared it to both EWMA and MEWMA control charts. Ultimately, this study implemented the novel MEWMA control chart on empirical economic data.

2 Materials and Methods

2.1 Change Point Detection

Let $\xi_1, \xi_2, ...,$ be sequentially observed independent random variables with a distribution $F(x,\alpha)$, where α is a parameter. The change-point model for the exponential distribution can be expressed as follows. It is reasonable to presume that:

$$\varepsilon_t \sim \begin{cases} Exp(\alpha_0), & t = 1,2,...,\theta-1 \\ Exp(\alpha_1), & t = \theta, \theta+1, \theta+2,... \end{cases},$$

where α_0 and α_1 are known parameters. Typically, it is considered that the in-control state is defined by the parameter α_0, while the out-of-control state is indicated by the parameter α_1. It can be presumed that the value α_0 is sustained until some unknown time $\theta-1$ and at the time θ the parameter changes to the new value $\alpha > \alpha_0$. The time θ is referred to as "the change-point time".

The common criterion for on choice of stopping times τ will be as follows:

$$E_\infty(\tau) = L,$$

where L is given (usually large), and $E_\infty(.)$ denotes that the expectation under distribution $F(x,\alpha_0)$, in the control process is that the change-point occurs at point θ (where $\theta \le \infty$). In quality control literature, this is referred to as the Average Run Length for an in-control process (ARL_0). Consequently, by definition, the conventional practical constraint is:

$$ARL_0 = E_\infty(\tau) = L.$$

Another common constraint consists of minimizing the quantity:

$$ARL_1 = E_\theta\left(\tau - \theta + 1 \middle| \tau \ge \theta\right),$$

where $E_\theta(.)$ is the expectation under distribution $F(x,\alpha_1)$, α_1 is the value of the parameter after the change point. In this research, the zero-state is usually studied as the special case $\theta = 1$. The quantity $E_1(\tau)$ is called the ARL for the out-of-control process (ARL_1) A sequential chart is likely to exhibit near-optimal performance when ARL_1 approaches its smallest value.

2.2 The ARX(p,r) Model

An autoregressive model with exogenous variables, referred to as the ARX(p,r) model, is defined as:

$$Y_t = \delta + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + ... + \phi_p Y_{t-p} + \sum_{i=1}^{r} \beta_i X_{it} + \varepsilon_t \quad ; t = 1,2,3,..., \tag{1}$$

where δ is a constant $(\delta \ge 0)$, ϕ_i is an autoregressive coefficient for $i = 1,2,...,p$, $|\phi_p| < 1$ and ε_t is i.i.d. sequence $(\varepsilon_t \sim Exp(\alpha))$. The initial value for the ARX(p,r) is $Y_{t-1}, Y_{t-2}, ..., Y_{t-p} = 1$.

2.3 Control Charts

This research examines three control charts: The EWMA, the MEWMA, and the new MEWMA control charts. The details of each control chart are outlined below:

2.3.1 The EWMA Control Chart

The EWMA control chart usually employed for identifying small variations in the process mean is delineated as

$$E_t = (1-\lambda)E_{t-1} + \lambda Y_t \quad ; t = 1,2,3,... \tag{2}$$

where E_t is the EWMA statistic, Y_t is the sequence of the ARX(p,r) process, and λ is an exponential smoothing parameter $(0 < \lambda \le 1)$. The stopping time is defined as the time when the initial detection of an out-of-control observation occurs, which is acceptable to conclude that the process is out of control.

The UCL and LCL of the EWMA control chart are determined as follows:

$$\text{UCL/LCL} = \mu_0 \pm L_1 \sigma \sqrt{\frac{\lambda}{(2-\lambda)}}, \tag{3}$$

where μ_0 is the target mean, σ is the process standard deviation, and L_1 is an appropriate control width limit $(L_1 > 0)$.

The stopping time τ_b for the EWMA control chart can be written as:

$$\tau_b = \inf\{t > 0; Z_t < a \text{ or } Z_t > b\},$$

where a and b are constant parameters known as the lower control limit (LCL) and the upper control limit (UCL).

2.3.2 The MEWMA Control Chart

The study, [7], created a MEWMA control chart adapted from the MEWMA statistic proposed by [6], which incorporates both historical and present observations of the process. The fundamental concept is to adjust the weight of the observation term to a constant value. The MEWMA control chart is characterized as:

$$ME_t = (1-\lambda)ME_{t-1} + \lambda Y_t + d(Y_t - Y_{t-1}); t = 1,2,3,..., \quad (4)$$

where ME_t is the MEWMA statistic, Y_t is the sequence of the ARX(p,r) process, and d is a constant $(d>0)$.

For the control limit, the UCL and LCL of the MEWMA control chart can respectively be expressed as:

$$\text{UCL/LCL} = \mu_0 \pm L_2\sigma\sqrt{\frac{(\lambda+2\lambda d+2d^2)}{(2-\lambda)}} \quad (5)$$

where L_2 is an appropriate control width limit $(L_2>0)$.

The stopping time τ_h for the MEWMA control chart can be written as:

$\tau_h = \inf\{t>0;\ ME_t < g \text{ or } ME_t > h\}$,

where g is the LCL, h is the UCL.

2.3.3 The New MEWMA Control Chart

The new MEWMA control chart is an enhancement of the original MEWMA control chart provided by [7], incorporating an additional constant to prioritize current data over historical data, specifically, $d_1 > d_2$. It is important to observe that if d_1 equals d_2, the new MEWMA control chart will correspond to the MEWMA control chart introduced by [7]. The new MEWMA control chart can be written as:

$$NM_t = (1-\lambda)NM_{t-1} + \lambda Y_t + d_1Y_t - d_2Y_{t-1} \ ;t=1,2,3,... \quad (6)$$

where NM_t is the new MEWMA statistic, Y_t is the sequence of the ARX(p,r) process with exponential white noise, $NM_0 = \vartheta$ and d_1 and d_2 are constants $(d_1 > d_2 > 0)$.

Meanwhile, for the control limit, the UCL and LCL of the new MEWMA control chart can be described as:

$$\text{UCL/LCL} = (\lambda+d_1-d_2)\frac{\mu_0}{\lambda} \pm L_3\sigma\sqrt{\frac{(\lambda+d_1)^2+d_2^2-2\lambda d_2+2\lambda^2 d_2-2d_1d_2+2\lambda d_1 d_2}{\lambda(2-\lambda)}} \quad (7)$$

where L_3 is an appropriate control width limit $(L_3>0)$.

The stopping time τ_q for the new MEWMA control chart can be written as:

$\tau_q = \inf\{t>0;\ NM_t < l \text{ or } NM_t > q\}$,

where l is the LCL, q is the UCL.

3 Performance Evaluation Measures

3.1 Explicit Formulas for the ARL of an ARX(p,r) Process on the New MEWMA Control Chart

In the analysis of time series data, stationarity can be assessed by unit root tests, as well as consider the autocorrelation function (ACF) and partial autocorrelation function (PACF) graphs to inform model selection. The ARX model can be expressed as follows:

$$Y_t = \delta + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + ... + \phi_p Y_{t-p} + \sum_{i=1}^{r}\beta_i X_{it} + \varepsilon_t \quad ;t=1,2,3,..., \quad (8)$$

3.1.1 The Explicit Formulas

The explicit formulas for the ARL of the new MEWMA control chart for an ARX(p,r) process are derived as follows:

$$NM_t = (1-\lambda)NM_{t-1} + (\lambda+d_1)\delta + (\lambda+d_1)\phi_1 Y_{t-1} + ... + (\lambda+d_1)\phi_p Y_{t-p} + (\lambda+d_1)\sum_{i=1}^{r}\beta_i X_{it} + (\lambda+d_1)\varepsilon_t - d_2 Y_{t-1}.$$

If Y_1 signals the out-of-control state NM_1, $NM_0 = \vartheta$ then:

$$NM_1 = (1-\lambda)\vartheta + (\lambda+d_1)\delta + (\lambda+d_1)\phi_1 Y_{t-1} + ... + (\lambda+d_1)\phi_p Y_{t-p} + (\lambda+d_1)\sum_{i=1}^{r}\beta_i X_{it} + (\lambda+d_1)\varepsilon_1 - d_2 v$$

If ε_1 is the in-control limit for NM_1, then $l \le NM_1 \le q$. Consider function $J(\vartheta)$

$$J(\vartheta) = 1 + \int J(NM_1) f(\varepsilon_1) d(\varepsilon_1). \quad (9)$$

Eq. (9) is a Fredholm integral equation of the second kind [23], and thus $J(\vartheta)$ can be rewritten as By modifying the integral variable, we derive the subsequent integral equation:

$$J(\vartheta)=1+\frac{1}{\lambda+d_1}*$$
$$\int_l^q J(\vartheta) f\left\{\frac{w-(1-\lambda)\vartheta}{(\lambda+d_1)}+\frac{d_2Y_{t-1}}{(\lambda+d_1)}-\delta-\phi_1Y_{t-1}-...-\phi_pY_{t-p}-\sum_{i=1}^{r}\beta_iX_{it}\right\}dw. \tag{10}$$

If $Y_t \sim Exp(\alpha)$ the $f(y)=\frac{1}{\alpha}e^{\frac{-y}{\alpha}}$; $y\geq 0$, then

$$J(\vartheta)=1+\frac{1}{\lambda+d_1}*$$
$$\int_l^q J(w)\frac{1}{\alpha}e^{-\frac{1}{\alpha}\left\{\frac{w-(1-\lambda)\vartheta}{(\lambda+d_1)}+\frac{d_2Y_{t-1}}{(\lambda+d_1)}-\delta-\phi_1Y_{t-1}-...-\phi_pY_{t-p}-\sum_{i=1}^{r}\beta_iX_{it}\right\}}dw \tag{11}$$

Let function

$$O(\vartheta)=e^{\frac{(1-\lambda)\vartheta}{\alpha(\lambda+d_1)}-\frac{d_2Y_{t-1}}{\alpha(\lambda+d_1)}+\frac{\delta}{\alpha}+\frac{\phi_1Y_{t-1}+...+\phi_pY_{t-p}+\sum_{i=1}^{r}\beta_iX_{it}}{\alpha}},$$ then we have

$$J(\vartheta)=1+\frac{O(\vartheta)}{\alpha(\lambda+d_1)}\int_l^q J(w)e^{\frac{-w}{\alpha(\lambda+d_1)}}dw;\ l\leq\vartheta\leq q.$$

Let $P=\int_l^q J(w)e^{\frac{-w}{\alpha(\lambda+d_1)}}dw$, then

$$J(\vartheta)=1+\frac{O(\vartheta)}{\alpha(\lambda+d_1)}\cdot P.$$

As a result, we acquire

$$J(\vartheta)=1+\frac{1}{\alpha(\lambda+d_1)}e^{\frac{(1-\lambda)\vartheta}{\alpha(\lambda+d_1)}-\frac{d_2Y_{t-1}}{\alpha(\lambda+d_1)}+\frac{\delta}{\alpha}+\frac{\phi_1Y_{t-1}+...+\phi_pY_{t-p}+\sum_{i=1}^{r}\beta_iX_{it}}{\alpha}}\cdot P \tag{12}$$

By solving for constant P, we obtain

$$P=\int_l^q J(w)e^{\frac{-w}{\alpha(\lambda+d_1)}}dw$$
$$=\int_l^q\left[1+\frac{P}{\alpha(\lambda+d_1)}O(w)\right]e^{\frac{-w}{\alpha(\lambda+d_1)}}dw$$
$$=\int_l^q e^{\frac{-w}{\alpha(\lambda+d_1)}}dw+\int_l^q\frac{Pe^{\frac{(1-\lambda)w}{\alpha(\lambda+d_1)}-\frac{d_2Y_{t-1}}{\alpha(\lambda+d_1)}+\frac{\delta}{\alpha}+\frac{\phi_1Y_{t-1}+...+\phi_pY_{t-p}+\sum_{i=1}^{r}\beta_iX_{it}}{\alpha}}}{\alpha(\lambda+d_1)}\cdot e^{\frac{-w}{\alpha(\lambda+d_1)}}dw$$
$$=\frac{-\alpha(\lambda+d_1)\left(e^{\frac{-q}{\alpha(\lambda+d_1)}}-e^{\frac{-l}{\alpha(\lambda+d_1)}}\right)}{1+\frac{e^{\frac{-d_2Y_{t-1}}{\alpha(\lambda+d_1)}+\frac{\delta}{\alpha}+\frac{\phi_1Y_{t-1}+...+\phi_pY_{t-p}+\sum_{i=1}^{r}\beta_iX_{it}}{\alpha}}}{\lambda}\left(e^{\frac{-\lambda q}{\alpha(\lambda+d_1)}}-e^{\frac{-\lambda l}{\alpha(\lambda+d_1)}}\right)}$$

By substituting constant P into Eq. (12), we arrive at:

$$J(\vartheta)=1+\frac{e^{\frac{(1-\lambda)\vartheta}{\alpha(\lambda+d_1)}-\frac{d_2Y_{t-1}}{\alpha(\lambda+d_1)}+\frac{\delta}{\alpha}+\frac{\phi_1Y_{t-1}+...+\phi_pY_{t-p}+\sum_{i=1}^{r}\beta_iX_{it}}{\alpha}}}{\alpha(\lambda+d_1)}*$$
$$\left(\frac{-\alpha(\lambda+d_1)\left[e^{\frac{-q}{\alpha(\lambda+d_1)}}-e^{\frac{-l}{\alpha(\lambda+d_1)}}\right]}{1+\frac{e^{\frac{-d_2Y_{t-1}}{\alpha(\lambda+d_1)}+\frac{\delta}{\alpha}+\frac{\phi_1Y_{t-1}+...+\phi_pY_{t-p}+\sum_{i=1}^{r}\beta_iX_{it}}{\alpha}}}{\lambda}\left[e^{\frac{-\lambda q}{\alpha(\lambda+d_1)}}-e^{\frac{-\lambda l}{\alpha(\lambda+d_1)}}\right]}\right). \tag{13}$$

Thus, the corresponding explicit formulas for the Average Run Length (ARL) of an ARX(p,r) process operating on a new MEWMA control chart, utilizing the Fredholm integral equation of the second kind, can be articulated as

$$ARL_{2-sided}=1-\frac{\lambda e^{\frac{(1-\lambda)\vartheta}{\alpha(\lambda+d_1)}}\left[e^{\frac{-q}{\alpha(\lambda+d_1)}}-e^{\frac{-l}{\alpha(\lambda+d_1)}}\right]}{\lambda e^{\frac{d_2Y_{t-1}}{\alpha(\lambda+d_1)}-\frac{\delta}{\alpha}-\frac{\phi_1Y_{t-1}-...-\phi_pY_{t-p}-\sum_{i=1}^{r}\beta_iX_{it}}{\alpha}}+e^{\frac{-\lambda q}{\alpha(\lambda+d_1)}}-e^{\frac{-\lambda l}{\alpha(\lambda+d_1)}}} \tag{14}$$

For $l=0$, the one-sided explicit formulas for the ARL on the new MEWMA control chart are expressed:

$$ARL_{1-sided}=1-\frac{\lambda e^{\frac{(1-\lambda)\vartheta}{\alpha(\lambda+d_1)}}\left[e^{\frac{-q}{\alpha(\lambda+d_1)}}-1\right]}{\lambda e^{\frac{d_2Y_{t-1}}{\alpha(\lambda+d_1)}-\frac{\delta}{\alpha}-\frac{\phi_1Y_{t-1}-...-\phi_pY_{t-p}-\sum_{i=1}^{r}\beta_iX_{it}}{\alpha}}+e^{\frac{-\lambda q}{\alpha(\lambda+d_1)}}-1} \tag{15}$$

3.1.2 The Existence and Uniqueness of Explicit Formulas

We demonstrate the existence and uniqueness of the solution to the integral equation presented in Eq. (11). Initially, we delineate:

$$T(J(\vartheta)) = 1 + \frac{1}{\lambda + d_1} \int_l^q J(w) \frac{1}{\alpha} e^{-\frac{1}{\alpha}\left\{\frac{w-(1-\lambda)\vartheta}{(\lambda+d_1)} + \frac{d_2 Y_{t-1}}{(\lambda+d_1)} - \delta - \phi_1 Y_{t-1} - \ldots - \phi_p Y_{t-p} - \sum_{i=1}^{r} \beta_i X_{it}\right\}} dw \tag{16}$$

Theorem 1. (Banach's fixed-point theorem)
Let's represent the set that contains all of the continuous functions on complete metric (X,d), and presume that $T: X \to X$ is a contraction mapping with contraction constant $0 \le s < 1$; i.e.,

$$\|T(J_1) - T(J_2)\| \le s\|J_1 - J_2\| \ \forall J_1, J_2 \in X .$$

Subsequently, $J(.) \in X$ is unique $T(J(\vartheta)) = J(\vartheta)$; i.e., it has a unique fixed point in X.

Proof: To show that T defined in Eq. (16) is a contraction mapping $J_1, J_2 \in C[l,q]$, we use the inequality $\|T(J_1) - T(J_2)\| \le s\|J_1 - J_2\|$ $\forall J_1, J_2 \in C(l,q)$ with $0 \le s < 1$. Consider Eq. (11) and (16), then

$$\|T(J_1) - T(J_2)\|_\infty = \sup_{\vartheta \in [l,q]} \left| \frac{O(\vartheta)}{\alpha(\lambda + d_1)} \int_l^q (J_1(w) - J_2(w)) e^{\frac{-w}{\alpha(\lambda+d_1)}} dw \right|$$

$$\le \sup_{\vartheta \in [l,q]} \left| \|J_1 - J_2\|_\infty O(\vartheta) \left(e^{\frac{-l}{\alpha(\lambda+d_1)}} - e^{\frac{-q}{\alpha(\lambda+d_1)}} \right) \right|$$

$$= \|J_1 - J_2\|_\infty \left| e^{\frac{-l}{\alpha(\lambda+d_1)}} - e^{\frac{-q}{\alpha(\lambda+d_1)}} \right| \sup_{\vartheta \in [l,q]} |O(\vartheta)|$$

$$\le s\|J_1 - J_2\|_\infty ,$$

where $s = \left| e^{\frac{-l}{\alpha(\lambda+d_1)}} - e^{\frac{-q}{\alpha(\lambda+d_1)}} \right| \sup_{\vartheta \in [l,q]} |O(\vartheta)|$ and

$$O(\vartheta) = e^{\frac{(1-\lambda)\vartheta}{\alpha(\lambda+d_1)} - \frac{d_2 Y_{t-1}}{\alpha(\lambda+d_1)} + \frac{\delta}{\alpha} + \frac{\phi_1 Y_{t-1} + \ldots + \phi_p Y_{t-p} + \sum_{i=1}^{r} \beta_i X_{it}}{\alpha}};$$

$0 \le s < 1$.

Consequently, as verified through the application of Banach's fixed-point theorem, the solution is both existent and unique.

3.2 The Numerical Integral Equation for the ARL of an ARX(p,r) Process on the new MEWMA Control Chart

The NIE methodology is extensively employed for assessing the ARL. It may rely on various quadrature rules (midpoint, trapezoidal, Simpson's rule, and Gauss-Legendre), all of which provide ARLs that are quite similar to the others, [14]. Consequently, this study employed the Midpoint rule to assess the ARL. The second-kind integral equation for the ARL on the new MEWMA control chart for the ARX(p,r) process, as presented in Eq. (14) and Eq. (15), can be approximated utilizing the quadrature formula. The midpoint rule is implemented in the following manner:

$$\text{Given } f(a_j) = f\left\{ \begin{array}{l} \frac{a_j - (1-\lambda)a_i}{(\lambda + d_1)} + \frac{d_2 Y_{t-1}}{(\lambda + d_1)} - \delta \\ -\phi_1 Y_{t-1} - \ldots - \phi_p Y_{t-p} - \sum_{i=1}^{r} \beta_i X_{it} \end{array} \right\}. \tag{17}$$

The approximation for the integral is in the form:

$$\int_l^q J(w) f(w) dw \approx \sum_{j=1}^{m} w_j f(a_j), \tag{18}$$

where $w_j = \frac{(q-l)}{m}$ and $a_j = \left(j - \frac{1}{2}\right) w_j$; $j = 1,2,...,m.$

Using the midpoint rule, numerical approximation $\tilde{J}(\vartheta)$ for the integral equation can be found as the solution for the following linear equations:

$$\tilde{J}(a_i) = 1 + \frac{1}{\lambda + d_1} * \sum_{j=1}^{m} w_j \tilde{J}(a_j) f\left\{ \begin{array}{l} \frac{a_j - (1-\lambda)a_i}{(\lambda + d_1)} + \frac{d_2 Y_{t-1}}{(\lambda + d_1)} - \delta \\ -\phi_1 Y_{t-1} - \ldots - \phi_p Y_{t-p} - \sum_{i=1}^{r} \beta_i X_{it} \end{array} \right\} ; i = 1,2,...,m.$$

Thus,

$$\tilde{J}(a_1) = 1 + \frac{1}{\lambda + d_1} \sum_{j=1}^{m} w_j \tilde{J}(a_j) f\left\{ \begin{array}{l} \frac{a_j - (1-\lambda)a_1}{(\lambda + d_1)} + \frac{d_2 Y_{t-1}}{(\lambda + d_1)} - \delta \\ -\phi_1 Y_{t-1} - \ldots - \phi_p Y_{t-p} - \sum_{i=1}^{r} \beta_i X_{it} \end{array} \right\}$$

$$\tilde{J}(a_2) = 1 + \frac{1}{\lambda + d_1} \sum_{j=1}^{m} w_j \tilde{J}(a_j) f\left\{ \begin{array}{l} \frac{a_j - (1-\lambda)a_2}{(\lambda + d_1)} + \frac{d_2 Y_{t-1}}{(\lambda + d_1)} - \delta \\ -\phi_1 Y_{t-1} - \ldots - \phi_p Y_{t-p} - \sum_{i=1}^{r} \beta_i X_{it} \end{array} \right\}$$

$$\vdots$$

$$\tilde{J}(a_m)=1+\frac{1}{\lambda+d_1}\sum_{j=1}^{m}w_j\tilde{J}(a_j)\,f\left\{\begin{matrix}\frac{a_j-(1-\lambda)a_m}{(\lambda+d_1)}+\frac{d_2Y_{t-1}}{(\lambda+d_1)}-\delta\\ -\phi_1Y_{t-1}-...-\phi_pY_{t-p}-\sum_{i=1}^{r}\beta_iX_{it}\end{matrix}\right\}$$

This set of m equations with m unknowns can be described in matrix form. The column vector is:

$$J_{m\times1}=\left(\tilde{J}(a_1),\tilde{J}(a_2),...,\tilde{J}(a_m)\right)'.$$

Since $1_{m\times1}=(1,1,...,1)'$ is a column vector of ones and $\mathrm{R}_{m\times m}$ is a matrix, we can define m to m^{th} as elements of the matrix R as follows:

$$\left[R_{ij}\right]\approx\frac{1}{\lambda+d_1}w_j\,f\left\{\begin{matrix}\frac{a_j-(1-\lambda)a_i}{(\lambda+d_1)}+\frac{d_2Y_{t-1}}{(\lambda+d_1)}-\delta\\ -\phi_1Y_{t-1}-...-\phi_pY_{t-p}-\sum_{i=1}^{r}\beta_iX_{it}\end{matrix}\right\},$$

and $\mathrm{I}_m=diag(1,1,...,1)$ as a unit matrix of order m. If $(\mathrm{I}-\mathrm{R})^{-1}$ exists, the numerical approximation for the integral equation in matrix form can be expressed as $\mathrm{J}_{m\times1}=(\mathrm{I}_m-\mathrm{R}_{m\times m})^{-1}1_{m\times1}$.

Finally, by substituting a_i within $C(a_i)$, the numerical integration equation for function $\tilde{J}(\vartheta)$ can be derived as:

$$\tilde{J}(\vartheta)=1+\frac{1}{\lambda+d_1}\sum_{j=1}^{m}w_j\tilde{J}(a_j)\,f\left\{\begin{matrix}\frac{a_j-(1-\lambda)\vartheta}{(\lambda+d_1)}+\frac{d_2Y_{t-1}}{(\lambda+d_1)}-\delta\\ -\phi_1Y_{t-1}-...-\phi_pY_{t-p}-\sum_{i=1}^{r}\beta_iX_{it}\end{matrix}\right\} \quad (19)$$

Equation (19) can be approximated using a numerical integral equation, which can be computed by many approaches. This study employs the composite midpoint rule, the trapezoidal rule, Simpson's rule, and the Gauss-Legendre quadrature.

A. Midpoint Rule

Given $f(A_j)=f\left\{\begin{matrix}\frac{a_j-(1-\lambda)\vartheta}{(\lambda+d_1)}+\frac{d_2Y_{t-1}}{(\lambda+d_1)}-\delta\\ -\phi_1Y_{t-1}-...-\phi_pY_{t-p}-\sum_{i=1}^{r}\beta_iX_{it}\end{matrix}\right\}.$

The Integral Equation (19) can be approximated by

$$\tilde{J}_M(\vartheta)\approx1+\frac{1}{1+\lambda d_1}\sum_{j=1}^{m}w_jJ(a_j)\,f(A_j) \quad (20)$$

where $w_j=\frac{(q-l)}{m}$ and $a_j=\left(j-\frac{1}{2}\right)w_j$; $j=1,2,...,m.$

B. Trapezoidal Rule

Similarly, it can be written as follows:

$$\tilde{J}_T(\vartheta)\approx1+\frac{1}{1+\lambda d_1}\sum_{j=1}^{m+1}w_jJ(a_j)\,f(A_j) \quad (21)$$

where $a_j=jw_j$ and $w_j=\frac{(q-l)}{m}; j=1,2,...,m-1,$

in other cases, $w_j=\frac{(q-l)}{2m}.$

C. Simpson's Rule

By Simpson's rule, ARL can be solved as follows

$$\tilde{J}_S(\vartheta)\approx1+\frac{1}{1+\lambda d_1}\sum_{j=1}^{2m+1}w_jJ(a_j)\,f(A_j) \quad (22)$$

where $a_j=jw_j$ and

$w_j=\frac{4}{3}\left(\frac{(q-l)}{2m}\right); j=1,3,...,2m-1,$

$w_j=\frac{2}{3}\left(\frac{(q-l)}{2m}\right); j=2,4,...,2m-2,$

in other cases, $w_j=\frac{1}{3}\left(\frac{(q-l)}{2m}\right).$

D. Gauss-Legendre quadrature

Given $f(A_j)=f\left\{\begin{matrix}\frac{a_j-(1-\lambda)\vartheta}{(\lambda+d_1)}+\frac{d_2Y_{t-1}}{(\lambda+d_1)}-\delta\\ -\phi_1Y_{t-1}-...-\phi_pY_{t-p}-\sum_{i=1}^{r}\beta_iX_{it}\end{matrix}\right\}.$

The approximation for the integral in [l, q] is in the form $\int_l^q J(w)\,f(w)dw\approx\sum_{j=1}^{m}w_jf(a_j)$ where $J(w)=1,\ -1<w<1.$

The Integral Equation can be approximated by

$$\tilde{J}_G(\vartheta)\approx1+\frac{1}{1+\lambda d_1}\sum_{j=1}^{m}w_jJ(a_j)\,f(A_j). \quad (23)$$

3.3 Overall Performance Measures

The accuracy of the ARL is quantified by the percentage of accuracy that has been derived from:

$$\%Accuracy=100-\left|\frac{J(\vartheta)-\tilde{J}(\vartheta)}{J(\vartheta)}\right|\times100\%.$$

The effectiveness of control charts is additionally examined by the Standard Deviation Run Length (SDRL), [24]. The SDRL for the control process is computed as follows.

$$ARL_0 = \frac{1}{\pi_0}, SDRL_0 = \sqrt{\frac{1-\pi_0}{\pi_0^{\,2}}}, \qquad (24)$$

where π_0 represents a type I error. In this study, ARL_0 was set at 370, which can be determined using $SDRL_0$ via Eq.(24). To calculate $SDRL_\Delta$ for an out-of-control situation, replace π_0 with π_1 where π_1 represents type II error. The control chart that performs the best at detecting changes in the process mean will have the lowest ARL_Δ and $SDRL_\Delta$ values.

We employ a few performance measures to evaluate the efficacy of control charts. This study provides the expected average run length (EARL) and the expected standard deviation run length (ESDRL) as follows [25]:

The EARL can be expressed mathematically by:

$$EARL = \frac{1}{\Delta}\sum_{\delta=\delta_{\min}}^{\delta_{\max}} ARL(\delta)$$

The ESDRL is described by

$$ESDRL = \frac{1}{\Delta}\sum_{\delta=\delta_{\min}}^{\delta_{\max}} SDRL(\delta)$$

where the $\delta_{\min}$ and represents the lower and upper bounds of shift parameter (δ), $ARL(\delta)$ is the ARL Δ value for a specific shift and Δ denotes the total number of increments between $\delta_{\min}$ and $\delta_{\max}$. $SDRL(\delta)$ can be computed in a manner comparable to $ARL(\delta)$.

3.4 The ARL Procedure for the New MEWMA Control Chart

The steps for determining the ARL value using the NIE approach and the explicit formula will be described in this section. When the process is in control, α_0 is given to the exponential white noise parameter. Additionally, $\alpha_1 = (1+\delta)\alpha_0$ is set when the process is out of control. The following are the ARL computations used to compare the ARL values from the two techniques:

Step 1: Determining the parameters of the control chart and ARX(p,r) process:

- The autoregressive coefficients (ϕ_i), the coefficient exogenous variables (β_j), the constant (δ), and the exogenous variables (X_{jt}) in the ARX(p,r) model
- Set the initial values for the ARX(p,r) process and the new MEWMA statistic.
- The smoothing constant (λ) and the initial value of the new MEWMA control chart $(E_0 = \nu)$
- The exponential white noise parameter for the in-control state, α_0.
- Determine acceptable $ARL_0 = 370$ for the in-control state and the shift sizes (δ).

Step 2: Calculate the UCL (q) that yields the desired ARL for the control process using Eq. (14) or Eq.(15).

Step 3: Evaluating of ARL:

- Calculate ARL via the explicit formula Eq. (14) or Eq. (15).
- Approximate ARL using the NIE approach by using Eq. (20), Eq. (21), Eq. (22) and Eq. (23).

Step 4: Examination of ARL:

- Compare the ARL values obtained using the explicit formula and NIE methods in Step 3.

Step 5: Comparison of the performance of new MEWMA with MEWMA and EWMA control charts.

4 Numerical Results

In Table 1 (Appendix), the ARL from explicit formula against NIE method using four quadrature rules for the new MEWMA control chart on ARX(1,2), ARX(2,1) and ARX(3,1) models given $a=0$, $\mu=1$, $\lambda_1 = 0.05$, $\lambda_2 = 0.025$ and ARL_0=370 are presented. The study's results showed that the ARL values found using the explicit formula method were the same as those found using the four numerical integral equation methods: midpoint, trapezoidal, Simpson, and Gauss-Legendre quadrature across all shift levels from 0.005 to 2.00. The ARL values obtained from the four numerical integral equation approaches were closely comparable, with the midway method often requiring the least processing time. Consequently, the next investigation will employ the midpoint approach for comparing the ARL values against the results obtained via the explicit formula method. In Table 2 (Appendix), a one-sided comparison of the ARL derived using explicit formulas and the NIE method is presented. The ARX(3,2) processes on the new MEWMA control chart wit $\delta = 1, \phi_1 = 0.1,$

$\phi_2 = 0.2,\ \beta_1 = 0.5, \beta_2 = 1.5,\ d_1 = 3,\ \ d_2 = 2,$ $\phi_3 = \pm 0.3, \lambda = 0.05, 0.10\ ARL_0 = 370$ are given. The results show that the ARL values obtained from the explicit formula method align with the results obtained from the numerical integration method across all change levels from 0.001 to 0.30. The explicit formula technique requires about 0.01 seconds, whereas the numerical integration method takes approximately 2.5 seconds to process. In Table 3 (Appendix), a two-sided comparison of the Average Run Length (ARL) calculated using explicit formulas and the NIE technique for ARX(2,1) processes on the new MEWMA control chart is shown. The given parameters are l=0.10, $\delta = 2, \phi_1 = 0.1,\ \phi_2 = \pm 0.2,\ \lambda = 0.05, 0.10,\ \beta_1 = 0.5,$ $d_1 = 2,\ \ d_2 = 1,\ ARL_0 = 370.$ After confirming the accuracy of the ARL values obtained from the explicit formula, we proceed with analyzing the performance of the control charts displayed in Appendix in Table 4 and Table 5. Table 4 (Appendix) presents a comparative analysis of the performance of one-sided EWMA, MEWMA, and the new MEWMA control charts for ARX(2,2) processes. The parameters given for the simulation are the comparative analysis of control chart efficiency indicating that at λ of 0.05 and 0.10, the findings aligned consistently, demonstrating that the novel control chart exhibited superior efficiency at d_2 equal to 0.5, yielding the lowest ARL and SDRL values across all shift levels from 0.0001 to 0.10. Upon evaluating the EARL and ESDRL criteria, it was determined that they yielded the lowest values. A comparison of the ARL for the ARX(2,3) process on two-sided EWMA, MEWMA and new MEWMA control charts is presented in Table 5 (Appendix). The specific parameters $2.5, 3.0,\ \lambda = 0.05, 0.10$ and $ARL_0 = 370.$ The comparative analysis of the efficiency of the control charts revealed a consistent direction with the evaluation of one-way control charts. Specifically, at λ of 0.05 and 0.10, the new MEWMA control chart demonstrated superior efficiency at d equal to 0.5, yielding the lowest ARL and SDRL values across all shift levels from 0.0001 to 0.10. Upon evaluating the EARL and ESDRL criteria, it was determined that they exhibited the lowest values. Consequently, the simulation results indicated that the revised control chart exhibited superior efficiency in identifying alterations in the process mean.

4.1 Application

This section presents economic data, specifically gold futures, which are significant for investment planning. The monthly data will be collected from January 2, 2024, to May 31, 2024. Gold futures are influenced by the following factors: The United States 5-Year Bond Yield is designated as model 1, and the EUR/USD currency is assigned as model 2. After fitting the model with the data, the ARX(1,1) model is produced, with estimated parameter values shown in Appendix in Table 6 and Table 7. Note that, for data obtained from data collection, it is necessary to study and select appropriate sampling methods and estimation techniques, [26], [27]. The equations are as follows:

Model 1: $\hat{Y}_t = 0.912Y_{t-1} + 503.304X_t$

Model 2: $\hat{Y}_t = 0.994Y_{t-1} + 1999.361X_t$

After obtaining the parameter estimates for the two models, the ARL values obtained from the explicit formula on the new MEWMA control chart were compared with the performance of the MEWMA and EWMA control charts, as shown in Appendix in Table 8 and Table 9. The results were summarized by the results presented in Appendix in Table 4 and Table 5. The new MEWMA control chart had the lowest ARL and SDRL values at all levels at d_2 equal to 0.5 and also had the lowest EARL and ESDRL values. In conclusion, the explicit formula method is the best method for practical applications in detecting changes in the process mean using the new MEWMA control chart. When the statistics of three control charts were plotted, it was found that for the data from Model 1, the new MEWMA control chart could detect the fastest, i.e., it could detect from the first observation. The MEWMA control chart records the 2nd observation as the first out of its control limits, while the EWMA control chart records the 64th observation as the first out of its control limits, as illustrated in Figure 1 (Appendix). For the second model, it was found that the new MEWMA control chart was able to detect the fastest, finding that the first observation outside the control limits was the 2nd observation, while the first observation outside the control limits of the MEWMA control chart was the 9th observation, and the first observation outside the control limits of the EWMA control chart was the 8th observation, as shown in Figure 2 (Appendix).

5 Conclusion

This research proved the explicit Average Run Length (ARL) formula for the ARX(p,r) model on both one-sided and two-sided new MEWMA control charts. When comparing the ARL values derived from the explicit formula against the values obtained by the four numerical integral equation methods: midpoint, trapezoidal, Simpson, and Gauss-Legendre quadrature rules. The results were not different, with the percentage of accuracy equal to 100. When considering the processing time, the recommended explicit formula for ARL takes the minimum time, as shown in the results. The findings indicate that the new MEWMA control chart is superior at detecting process changes compared to the MEWMA and EWMA control charts, evidenced by its lowest EARL and ESDRL values. This research employed the novel MEWMA control chart to analyze the economic data. Future studies may formulate ARL values for further novel control charts, attractive models, and applications in other fields. This formula yields the precise value and significantly reduces computing time. Nevertheless, if the explicit formula remains unprovable, the numerical integral equation approach may be employed to approximate the ARL instead.

Acknowledgement:

This research was located by the National Science, Research and Innovation Fund (NSRF) and King Mongkut's University of Technology North Bangkok with Contract no. KMUTNB-FF-68-B-08.

Declaration of Generative AI and AI-assisted Technologies in the Writing Process

During the preparation of this work, the authors used QuillBot to refine certain phrases in the introduction section in order to enhance the academic quality. After using this tool/service, the authors reviewed and edited the content as needed and take full responsibility for the content of the publication.

References:

[1] M.B.C. Khoo, V.H. Wong, A double moving average control chart, *Communication in Statistics- Simulation and Computation*, Vol. 37, 2008, pp.1696–1708. DOI: 10.1080/03610910701832459.

[2] N. Khan, M. Aslam, C.H. Jun, A EWMA control chart for exponentially distributed quality based on moving average, *Quality and Reliability Engineering International*, Vol.32, 2016, pp. 1179–1190. https://doi.org/10.1002/qre.1825.

[3] W. A. Shewhart, *Economic control of the quality of manufactured product*, D. Van Nostrand Company, Inc., New York, 1931.

[4] E. S. Page, Continuous inspection schemes, *Biometrika*, Vol 41, No. 1/2, 1954, pp.100-115. https://doi.org/10.2307/2333009.

[5] S. W. Roberts, Control chart tests based on geometric moving average, *Technometrics*, Vol 1, No. 3, 1959, pp.239-250. https://doi.org/10.2307/1266443.

[6] N. Khan, M. Alsam, and CH.Jun, Design of a control chart using MEWMA statistics, *Quality and Reliability Engineering International*, Vol. 33, NO. 5, 2017, pp.1095-1104. https://doi.org/10.1002/qre.2102

[7] A. K. Patel, J. Divecha, Modified exponentially weighted moving average (EWMA) control chart for an analytical process data, *Journal of Chemical Engineering and Materials Science*, Vol 2, No. 1, 2011, pp.12-20.

[8] K. Silpakob, Y. Areepong, S. Sukparungsee, R. Sunthornwat, A New Modified EWMA Control Chart for Monitoring Processes Involving Autocorrelated Data. *Intelligent Automation & Soft Computing*, Vol.36, No.1, 2023, pp. 281-298. https://doi.org/10.32604/iasc.2023.032487.

[9] AD. Karaoglan, GM. Bayhan, Performance comparison of residual control charts for trend stationary first order autoregressive processes. *Gazi University Journal of Science*, Vol.24, No. 2, 2011, pp. 329-339.

[10] R. Somyanonthanakul, K. Warin, W. Amasiri, K. Mairiang, C. Mingmalairak,W. Panichkitkosolkul, K. Silanun, T. Theeramunkong, S. Nitikraipot, S. Suebnukarn, Forecasting COVID-19 cases using time series modeling and association rule mining, *BMC Medical Research Methodology*, Vol. 22, 2022, Article number: 281. https://doi.org/10.1186/s12874-022-01755-x.

[11] A.S.AI. Shammre and B. Chidmi, Oil Price Forecasting Using FRED Data: A Comparison between Some Alternative Models, *Energies*, Vol. 16, No. 11, 2023, 4451. https://doi.org/10.3390/en16114451.

[12] M. Ibazizen, H. Fellag, Bayesian estimation of an AR (1) process with exponential white

noise, *Statistics*, Vol. 37, No. 5, 2003, pp.365–372. https://doi.org/10.1080/0233188031000078042.

[13] G. N. Farah, B. Lindner, Exponentially distributed noise—its correlation function and its effect on nonlinear dynamic, *Journal of Physics A: Mathematical and Theoretical*, Vol.54, 2021, 035003 (14pp). https://doi.org/10.1088/1751-8121/abd2fd.

[14] A. A. Agboluaje, S.B. Ismail, C.Y. Yip, Evaluating Combine White Noise with US and UK GDP Quarterly Data, *Gazi University Journal of Science*, Vol. 29, No. 2, 2016, pp.365-372.

[15] I.M.S. Pereira and M.A. Amaral-Turkman, Bayesian prediction in threshold autoregressive models with exponential white noise, *Test*, Vol. 13, No. 1, 2004, pp. 45-64. DOI: 10.1007/BF02603000.

[16] M. Neveed, M, Azam, N. Khan, M. Aslam, Design a control chart using extended EWMA statistic. *Technologies,* Vol 6, No 4, 2018, pp.108–122. https://doi.org/10.3390/technologies6040108

[17] P. Wilasinee, Gauss–Legendre Numerical Integrations for Average Run Length Running on EWMA Control Chart with Fractionally Integrated MAX Process, *WSEAS Transactions on Mathematics*, Vol 23, 2024, pp. 579 – 590. https://doi.org/10.37394/23206.2024.23.61.

[18] P. Paichit, W. Peerajit, The average run length for continuous distribution process mean shift detection on a MEWMA control chart, *Asia-Pacific Journal of Science and Technology*, Vol. 27, No. 6, 2022, APST-27-06-21. https://doi.org/10.14456/apst.2022.103.

[19] H. Zhao, H. Tang, C. Pang, H. Jiang, A Markov Chain Model for Approximating the Run Length Distributions of Poisson EWMA Charts under Linear Drifts, *Mathematics*, Vol. 10, No. 24, 2022, 4786. https://doi.org/10.3390/math10244786.

[20] W. Peerajit, Accurate Average Run Length Analysis for Detecting Changes in a Long-Memory Fractionally Integrated MAX Process Running on EWMA Control Chart, *WSEAS Transactions on Mathematics*, Vol. 22, 2023, pp. 514 – 530. https://doi.org/10.37394/23206.2023.22.58.

[21] S. Phanyaem, Explicit formulas and numerical integral equation of ARL for SARX $(P,r)_L$ model based on CUSUM chart, *Mathematics and Statistics*, Vol 10, No. 1, 2022, pp. 88-99. DOI: 10.13189/ms.2022.100107.

[22] Y. Areepong, and K. Karoon, Detection capability of the dewma chart using explicit run length solutions: A case study on data of gross domestic product, *Engineering Letters*, Vol. 32, No. 7, 2024, pp.1300-1312.

[23] A. Turab, W. Sintunavarat, On the Existence and Uniqueness of the Solution of a Probabilistic Functional Equation Approached by the Banach Fixed Point Theorem, *Thai Journal of Mathematics*, Vol. 18, No. 4, 2020, pp. 1983-1994.

[24] A. Fonseca, PH. Ferreira, DC. Nascimento, R. Fiaccone, CU. Correa, AG. Piña, F. Louzada, Water Particles Monitoring in the Atacama Desert: SPC approach Based on proportional data, *Axioms*, Vol.10, No.3, 2021, pp. 154. https://doi.org/10.3390/axioms10030154.

[25] JC Malela-Majika, S.C. Shongwe, O.A. Adeoti, A hybrid homogeneously weighted moving average control chart for process monitoring: Discussion, *Quality and Reliability Engineering International*, Vol. 37, 2021, pp.3314-3322. https://doi.org/10.1002/qre.2911.

[26] N. Lawson and N. Thongsak, A class of population mean estimators in stratified random sampling: a case study on fine particulate matter in the north of Thailand, *WSEAS Transactions on Mathematics*, Vol.23, 2024, pp. 160-166. https://doi.org/10.37394/23206.2024.23.19.

[27] C. Ponkaew and N. Lawson, New Product Estimators for Population Mean Under Unequal Probability Sampling with Missing Data: A Case Study on the Number of New COVID-19 Patients, *Thailand Statistician*, Vol. 22, No.3, 2024, pp. 634-656.

APPENDIX

Table 1. The ARL from explicit formula against NIE method using four quadrature rules for the new MEWMA control chart on ARX(p,r) model given $\delta = 1, \lambda = 0.05$ and ARL$_0$=370

ARX ϕ_i, β_j, b	δ	**Explicit (CPU Time)**	**NIE (CPU Time in seconds)**			
			Midpoint	**Trapezoidal**	**Simpson's**	**Gauss- Legendre**
ARX(1,2) $\phi_1 = 0.1$ $\beta_1 = 0.5$ $\beta_2 = 0.8$ d_1=3, d_2=2.5 $q = 0.3811$	0.000	370.50095 (<0.001)	370.50095 (2.328)	370.50095 (2.328)	370.50095 (9.234)	370.50095 (26.577)
	0.005	54.71463 (<0.001)	54.71463 (2.344)	54.71463 (2.328)	54.71463 (9.125)	54.71463 (26.501)
	0.010	29.87347 (<0.001)	29.87347 (2.297)	29.87347 (2.344)	29.87347 (9.187)	29.87347 (26.610)
	0.025	13.00216 (<0.001)	13.00216 (2.344)	13.00216 (2.359)	13.00216 (9.157)	13.00216 (26.375)
	0.050	7.00453 (<0.001)	7.00453 (2.329)	7.00453 (2.360)	7.00453 (9.187)	7.00453 (26.453)
	0.100	3.93626 (<0.001)	3.93626 (2.313)	3.93626 (2.313)	3.93626 (9.250)	3.93626 (26.188)
	1.000	1.21138 (<0.001)	1.21138 (2.312)	1.21138 (2.328)	1.21138 (9.282)	1.21138 (26.531)
ARX(2,1) $\phi_1 = 0.1$ $\phi_2 = 0.2$ $\beta_1 = 0.5$ d_1=3, d_2=2.5 $q = 0.6962885$	0.000	370.58524 (<0.001)	370.58524 (2.375)	370.58524 (2.390)	370.58524 (9.500)	370.58524 (26.625)
	0.005	63.10727 (<0.001)	63.10727 (2.375)	63.10727 (2.390)	63.10727 (9.531)	63.10727 (26.265)
	0.010	34.84638 (<0.001)	34.84638 (2.390)	34.84638 (2.359)	34.84638 (9.515)	34.84638 (26.422)
	0.025	15.25197 (<0.001)	15.25197 (2.406)	15.25197 (2.390)	15.25197 (9.469)	15.25197 (26.406)
	0.050	8.20429 (<0.001)	8.20429 (2.359)	8.20429 (2.407)	8.20429 (9.453)	8.20429 (26.313)
	0.100	4.57837 (<0.001)	4.57837 (2.390)	4.57837 (2.390)	4.57837 (9.406)	4.57837 (26.484)
	1.000	1.30165 (<0.001)	1.30165 (2.422)	1.30165 (2.360)	1.30165 (9.469)	1.30165 (26.484)
ARX(3,1) $\phi_1 = 0.1$ $\phi_2 = 0.2$ $\phi_3 = 0.3$ $\beta_1 = 0.5$ d_1=3, d_2=2.5 $q = 0.5150228$	0.000	370.52344 (<0.001)	370.52344 (2.360)	370.52344 (2.313)	370.52344 (9.375)	370.52344 (26.407)
	0.005	58.66220 (<0.001)	58.66220 (2.360)	58.66220 (2.391)	58.66220 (9.468)	58.66220 (26.296)
	0.010	32.19711 (<0.001)	32.19711 (2.406)	32.19711 (2.375)	32.19711 (9.469)	32.19711 (26.266)
	0.025	14.04814 (<0.001)	14.04814 (2.406)	14.04814 (2.406)	14.04814 (9.610)	14.04814 (26.406)
	0.050	7.56089 (<0.001)	7.56089 (2.391)	7.56089 (2.406)	7.56089 (9.453)	7.56089 (26.314)
	0.100	4.23320 (<0.001)	4.23320 (2.390)	4.23320 (2.406)	4.23320 (9.532)	4.23320 (26.297)
	1.000	1.25199 (<0.001)	1.25199 (2.391)	1.25199 (2.406)	1.25199 (9.500)	1.25199 (26.360)

Table 2. ARL comparison using explicit formulas and the NIE method for an ARX(3,2) process on the one-sided new MEWMA control chart with $\delta = 1, \phi_1 = 0.1\ \ \phi_2 = 0.2\ \ \beta_1 = 0.5, \beta_2 = 1.5,\ d_1 = 3,\ d_2 = 2,$ and $ARL_0 = 370$.

λ	ϕ_3	*q*	Shift	Explicit	NIE	Time[a]	%*Accuracy*
0.05	0.3	0.160329	0.00	370.39640	370.39640	2.359	100.00
			0.001	151.59420	151.59420	2.406	100.00
			0.003	69.80943	69.80943	2.421	100.00
			0.005	45.52333	45.52333	2.375	100.00
			0.01	24.56818	24.56818	2.390	100.00
			0.03	9.03358	9.03358	2.422	100.00
			0.05	5.76221	5.76221	2.390	100.00
			0.10	3.27882	3.27882	2.375	100.00
			0.20	2.04161	2.04161	2.422	100.00
			0.30	1.63968	1.63968	2.407	100.00
	-0.3	0.292472	0.00	370.93590	370.93590	2.453	100.00
			0.001	164.82820	164.82820	2.500	100.00
			0.003	78.38761	78.38761	2.453	100.00
			0.005	51.60251	51.60251	2.484	100.00
			0.01	28.05832	28.05832	2.515	100.00
			0.03	10.33744	10.33744	2.485	100.00
			0.05	6.57466	6.57466	2.500	100.00
			0.10	3.70783	3.70783	2.547	100.00
			0.20	2.27092	2.27092	2.547	100.00
			0.30	1.79922	1.79922	2.531	100.00
0.10	0.3	0.161451	0.00	370.26140	370.26140	2.515	100.00
			0.001	151.30950	151.30950	2.532	100.00
			0.003	69.63982	69.63982	2.500	100.00
			0.005	45.40576	45.40576	2.500	100.00
			0.01	24.50208	24.50208	2.562	100.00
			0.03	9.00959	9.00959	2.531	100.00
			0.05	5.74753	5.74753	2.578	100.00
			0.10	3.27134	3.27134	2.563	100.00
			0.20	2.03783	2.03783	2.531	100.00
			0.30	1.63716	1.63716	2.562	100.00
	-0.3	0.294831	0.00	370.34150	370.34150	2.500	100.00
			0.001	164.54360	164.54360	2.563	100.00
			0.003	78.24844	78.24844	2.532	100.00
			0.005	51.51024	51.51024	2.484	100.00
			0.01	28.00801	28.00801	2.546	100.00
			0.03	10.31918	10.31918	2.547	100.00
			0.05	6.56328	6.56328	2.531	100.00
			0.10	3.70176	3.70176	2.563	100.00
			0.20	2.26759	2.26759	2.547	100.00
			0.30	1.79686	1.79686	2.578	100.00

The computations for the NIE method were carried out on a Windows 10 Professional 64-bit with RAM of 8 GB and an AMD RYZEN 7 CPU.

Table 3. ARL comparison using explicit formulas and the NIE method for an ARX(2,1) process on a two-sided new MEWMA control chart with l=0.10 $\delta = 2, \phi_1 = 0.1$ $\beta_1 = 0.5$ $d_1 = 2,$ $d_2 = 1,$ and $ARL_0 = 370$.

λ	ϕ_2	q	Shift	Explicit	NIE	Time[a]	%*Accuracy*
0.05	0.2	0.303515	0.00	370.40012	370.40012	2.235	100.00
			0.001	162.70491	162.70491	2.250	100.00
			0.003	77.02293	77.02293	2.250	100.00
			0.005	50.64063	50.64063	2.234	100.00
			0.01	27.51549	27.51549	2.250	100.00
			0.03	10.14967	10.14967	2.266	100.00
			0.05	6.46637	6.46637	2.266	100.00
			0.10	3.66053	3.66053	2.281	100.00
			0.20	2.25342	2.25342	2.253	100.00
			0.30	1.79083	1.79083	2.234	100.00
	-0.2	0.403998	0.00	370.28842	370.28842	2.250	100.00
			0.001	172.44963	172.44963	2.234	100.00
			0.003	83.69628	83.69628	2.250	100.00
			0.005	55.44806	55.44806	2.250	100.00
			0.01	30.31624	30.31624	2.250	100.00
			0.03	11.20988	11.20988	2.266	100.00
			0.05	7.13032	7.13032	2.219	100.00
			0.10	4.01419	4.01419	2.250	100.00
			0.20	2.44527	2.44527	2.250	100.00
			0.30	1.92616	1.92616	2.265	100.00
0.10	0.2	0.307055	0.00	370.53150	370.53150	2.265	100.00
			0.001	162.79331	162.79331	2.250	100.00
			0.003	77.06977	77.06977	2.235	100.00
			0.005	50.67188	50.67188	2.265	100.00
			0.01	27.53199	27.53199	2.266	100.00
			0.03	10.15446	10.15446	2.265	100.00
			0.05	6.46865	6.46865	2.235	100.00
			0.10	3.66094	3.66094	2.234	100.00
			0.20	2.25302	2.25302	2.219	100.00
			0.30	1.79024	1.79024	2.234	100.00
	-0.2	0.4096598	0.00	370.08386	370.08386	2.234	100.00
			0.001	172.59884	172.59884	2.250	100.00
			0.003	83.82035	83.82035	2.250	100.00
			0.005	55.54019	55.54019	2.250	100.00
			0.01	30.37025	30.37025	2.266	100.00
			0.03	11.22881	11.22881	2.250	100.00
			0.05	7.14112	7.14112	2.281	100.00
			0.10	4.01869	4.01869	2.234	100.00
			0.20	2.44669	2.44669	2.250	100.00
			0.30	1.92666	1.92666	2.250	100.00

The computations for the NIE method were carried out on a Windows 10 Professional 64-bit with RAM of 8 GB and an AMD RYZEN 7 CPU.

Table 4. Efficiency comparison of one-sided EWMA, MEWMA and new MEWMA control charts for the ARX(2,2) process with $\delta = 2$, $\phi_1 = 0.1, \phi_2 = 0.3$ $\beta_1 = 2.5, \beta_2 = 1.5$ and $ARL_0 = 370$.

λ	Shift	Control Chart	New MEWMA $(d_1 = 2.5)$ $d_2 = 2.0$	$d_2 = 1.5$	$d_2 = 1.0$	$d_2 = 0.5$	MEWMA $(d = 2.5)$	EWMA $a = 0$
			$q = 0.00925768$	$q = 0.0076092$	$q = 0.006254275$	$q = 0.00514063$	$h = 0.01126333$	$b = 0.000081606$
0.05	0.0000	ARL_0	370.58950	370.59146	370.53976	370.58461	370.55000	370.55434
		$SDRL_0$	370.08916	370.09112	370.03942	370.08427	370.04966	370.05400
	0.0001	ARL_Δ	323.15776	321.21400	319.25251	**317.38458**	325.09494	360.56058
		$SDRL_\Delta$	322.65737	320.71361	318.75212	**316.88419**	324.59455	360.06023
	0.0005	ARL_Δ	213.83407	209.64248	205.59345	**201.72462**	218.18097	325.40885
		$SDRL_\Delta$	213.33348	209.14188	205.09284	**201.22400**	217.68040	324.90847
	0.0007	ARL_Δ	182.94091	178.66627	174.57294	**170.68175**	187.41290	310.26066
		$SDRL_\Delta$	182.44022	178.16557	174.07222	**170.18102**	186.91223	309.76026
	0.0010	ARL_Δ	150.39405	146.28943	142.39332	**138.71146**	154.72682	289.98493
		$SDRL_\Delta$	149.89322	145.78857	141.89244	**138.21056**	154.22601	289.48450
	0.0030	ARL_Δ	69.03301	66.48873	64.12356	**61.92350**	71.77790	201.61144
		$SDRL_\Delta$	68.53119	65.98684	63.62160	**61.42147**	71.27615	201.11082
	0.0050	ARL_Δ	44.97052	43.19325	41.55090	**40.03048**	46.90014	154.14110
		$SDRL_\Delta$	44.46771	42.69032	41.04786	**39.52732**	46.39745	153.64029
	0.0070	ARL_Δ	33.44119	32.08018	30.82602	**29.66763**	34.92336	124.51776
		$SDRL_\Delta$	32.93740	31.57622	30.32190	**29.16334**	34.41973	124.01675
	0.0100	ARL_Δ	24.24425	23.23782	22.33125	**21.45936**	25.34289	96.37813
		$SDRL_\Delta$	23.73899	22.73232	21.82552	**20.95340**	24.83786	95.87683
	0.1000	ARL_Δ	3.22471	3.10586	2.99719	**2.89747**	3.35521	10.42500
		$SDRL_\Delta$	2.67844	2.55744	2.44662	**2.34475**	2.81109	9.91240
	EARL		116.13783	113.76867	111.51568	**109.38676**	118.63501	208.14316
	ESDRL		115.63089	113.26142	111.00812	**108.87889**	118.12838	207.64117
			$q = 0.00929878$	$q = 0.00767174$	$q = 0.006329425$	$q = 0.005221997$	$h = 0.011270965$	$b = 0.0001650715$
0.1	0.0000	ARL_0	370.52270	370.52319	370.53643	370.53960	370.54569	370.51782
		$SDRL_0$	370.02236	370.02285	370.03609	370.03926	370.04535	370.01748
	0.0001	ARL_Δ	322.85209	320.94230	319.06339	**317.19806**	324.80160	344.97218
		$SDRL_\Delta$	322.35170	320.44191	318.56300	**316.69767**	324.30121	344.47182
	0.0005	ARL_Δ	213.25614	209.15308	205.20699	**201.40250**	217.52914	270.38746
		$SDRL_\Delta$	212.75555	208.65248	204.70638	**200.90188**	217.02856	269.88700
	0.0007	ARL_Δ	182.35618	178.17577	174.18383	**170.36328**	186.74116	244.00318
		$SDRL_\Delta$	181.85549	177.67507	173.68311	**169.86254**	186.24049	243.50267
	0.0010	ARL_Δ	149.83549	145.82530	142.02461	**138.41444**	154.07457	212.84277
		$SDRL_\Delta$	149.33465	145.32444	141.52373	**137.91353**	153.57376	212.34218
	0.0030	ARL_Δ	68.68913	66.20939	63.90276	**61.75123**	71.36245	114.91388
		$SDRL_\Delta$	68.18730	65.70749	63.40079	**61.24919**	70.86069	114.41279
	0.0050	ARL_Δ	44.73090	42.99994	41.39861	**39.91271**	46.60796	78.66277
		$SDRL_\Delta$	44.22807	42.49700	40.89555	**39.40954**	46.10525	78.16117
	0.0070	ARL_Δ	33.25812	31.93300	30.71032	**29.57857**	34.69911	59.77694
		$SDRL_\Delta$	32.75430	31.42902	30.20618	**29.07427**	34.19546	59.27483
	0.0100	ARL_Δ	24.10934	23.12972	22.22769	**21.39438**	25.17700	43.92935
		$SDRL_\Delta$	23.60405	22.62420	21.72194	**20.88840**	24.67193	43.42647
	0.1000	ARL_Δ	3.21014	3.09452	2.98886	**2.89124**	3.33685	4.91501
		$SDRL_\Delta$	2.66362	2.54589	2.43812	**2.33838**	2.79244	4.38661
	EARL		115.81084	113.49589	111.30078	**109.21182**	118.25887	152.71150
	ESDRL		115.30386	112.98861	110.79320	**108.70393**	117.75220	152.20728

Table 5. Efficiency comparison of two-sided EWMA, MEWMA and new MEWMA control charts for the ARX(2,3) process $\delta=1.5$, $l=0.1$, $\phi_1=0.1, \phi_2=0.2$, $\beta_1=1.5, \beta_2=2, \beta_3=2.5$ and $ARL_0=370$.

λ	Shift	Control Chart	New MEWMA $(d_1=2.5)$				MEWMA	EWMA
			$d_2=3.0$	$d_2=2.5$	$d_2=1.5$	$d_2=0.5$	$d=4.0, g=0.1$	$a=0.1$
			$q=0.14235217$	$q=0.1374322$	$q=0.12924078$	$q=0.1228422$	$h=0.15421785$	$b=0.100319656$
0.05	0.0000	ARL_0	370.51539	370.56248	370.52588	370.59241	370.51340	370.75207
		$SDRL_0$	370.01505	370.06214	370.02554	370.09207	370.01306	370.25173
	0.0001	ARL_Δ	322.00812	320.79056	318.28218	**315.88272**	324.53699	317.23804
		$SDRL_\Delta$	321.50773	320.29017	317.78179	**315.38232**	324.03660	316.73765
	0.0005	ARL_Δ	211.44306	208.78620	203.62026	**198.72418**	216.98604	201.23711
		$SDRL_\Delta$	210.94247	208.28560	203.11964	**198.22355**	216.48546	200.73649
	0.0007	ARL_Δ	180.50780	177.80199	172.59032	**167.68760**	186.18689	170.18211
		$SDRL_\Delta$	180.00711	177.30128	172.08959	**167.18685**	185.68622	169.68137
	0.0010	ARL_Δ	148.06193	145.46869	140.51944	**135.90166**	153.54189	138.23434
		$SDRL_\Delta$	147.56108	144.96783	140.01855	**135.40074**	153.04107	137.73343
	0.0050	ARL_Δ	43.97727	42.86280	40.79203	**38.91177**	46.38845	39.84616
		$SDRL_\Delta$	43.47440	42.35985	40.28893	**38.40852**	45.88573	39.34298
	0.0070	ARL_Δ	32.68714	31.83436	30.25417	**28.82361**	34.53705	29.53352
		$SDRL_\Delta$	32.18326	31.33037	29.74997	**28.31920**	34.03338	29.02921
	0.0100	ARL_Δ	23.69397	23.06370	21.89837	**20.84585**	25.06408	21.36759
		$SDRL_\Delta$	23.18858	22.55816	21.39253	**20.33971**	24.55899	20.86160
	0.1000	ARL_Δ	3.18113	3.10638	2.96895	**2.84562**	3.34451	2.90659
		$SDRL_\Delta$	2.63410	2.55797	2.41779	**2.29171**	2.80022	2.35408
	EARL		114.79550	113.30173	110.43772	**107.76752**	109.13229	117.95821
	ESDRL		114.28844	112.79446	109.93007	**107.25947**	108.62444	117.45154
			$r=0.1425516$	$r=0.13766403$	$r=0.12950917$	$r=0.12312048$	$h=0.1543132$	$b=0.100639413$
0.1	0.0000	ARL_0	370.51735	370.56185	370.53613	370.53182	370.59051	370.55045
		$SDRL_0$	370.01701	370.06151	370.03579	370.03148	370.09017	370.05011
	0.0001	ARL_Δ	321.94345	320.73831	318.26652	**315.84368**	324.50103	317.09935
		$SDRL_\Delta$	321.44306	320.23792	317.76613	**315.34328**	324.00064	316.59896
	0.0005	ARL_Δ	211.30142	208.67658	203.57593	**198.71659**	216.80043	201.19562
		$SDRL_\Delta$	210.80083	208.17598	203.07531	**198.21596**	216.29985	200.69500
	0.0007	ARL_Δ	180.36325	177.69087	172.54489	**167.68498**	185.98803	170.15754
		$SDRL_\Delta$	179.86256	177.19016	172.04416	**167.18423**	185.48736	169.65680
	0.0010	ARL_Δ	147.92306	145.36263	140.47584	**135.90266**	153.34334	138.22315
		$SDRL_\Delta$	147.42221	144.86177	139.97495	**135.40174**	152.84252	137.72224
	0.0050	ARL_Δ	43.91703	42.81765	40.77349	**38.91471**	46.29475	39.85061
		$SDRL_\Delta$	43.41415	42.31470	40.27039	**38.41146**	45.79202	39.34743
	0.0070	ARL_Δ	32.64097	31.79981	30.23999	**28.82596**	34.46472	29.53737
		$SDRL_\Delta$	32.13708	31.29582	29.73579	**28.32155**	33.96104	29.03307
	0.0100	ARL_Δ	23.65978	23.03815	21.88788	**20.84763**	25.01023	21.37066
		$SDRL_\Delta$	23.15438	22.53260	21.38204	**20.34149**	24.50513	20.86467
	0.1000	ARL_Δ	3.17693	3.10323	2.96762	**2.84578**	3.33785	2.90694
		$SDRL_\Delta$	2.62982	2.554762	2.41643	**2.29187**	2.79346	2.35443
	EARL		114.71543	113.24036	110.41326	**107.76338**	109.11008	117.84961
	ESDRL		114.20835	112.73309	109.90559	**107.25532**	108.60223	117.34292

Table 6. The ARX(p,r,) model parameters and the model fit for Gold Futures

Model	Variables	Coefficient	t	Sig	Model fit	
					MAPE	Normalized BIC
1 ARX(1,1)	AR(1) $(\hat{\phi}_1)$	0.912	20.128	0.000	**1.391**	**7.497**
	United States 5-Year Bond Yield $(\hat{\beta}_1)$	503.304	47.686	0.000		
2 ARX(1,1)	AR(1) $(\hat{\phi}_1)$	0.99	96.866	0.000	**0.658**	**6.266**
	EUR/USD currency $(\hat{\beta}_1)$	1999.361	14.802	0.041		

Table 7. Exponential white noise Testing

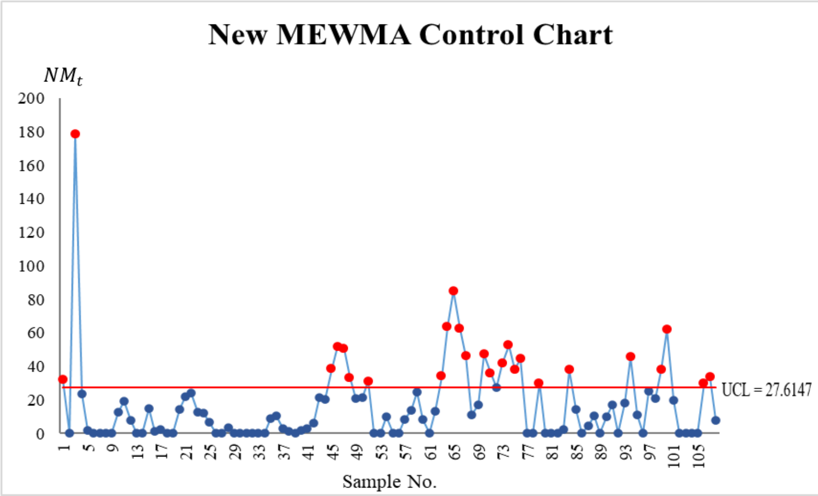
Exogenous variable	**Mean** (α_0)	**Kolmogorov-Smirnov Z**	**Sig.**
United States 5-Year Bond Yield	30.1757	0.928	**0.356**
EUR/USD currency	14.5520	1.005	**0.265**

Table 8. ARL comparison for the ARX(1,1) process on one and two-sided EWMA, MEWMA and new MEWMA control charts with $\lambda = 0.1$, $\phi_1 = 0.912$, $\beta_1 = 503.304$, $\alpha = 30.1757$, and $ARL_0 = 370$.

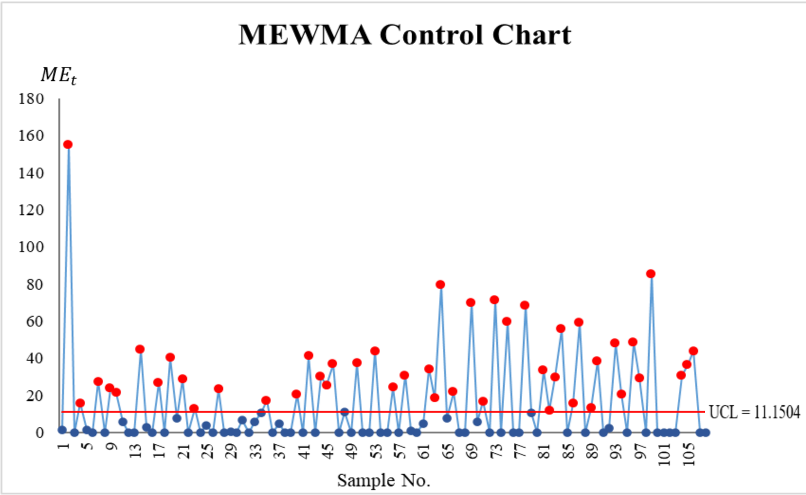
l	**Shift**	**Control Chart**	**New MEWMA** $(d_1 = 3)$ $d_2 = 2.5$	$d_2 = 2.0$	$d_2 = 1.5$	$d_2 = 0.5$	**MEWMA** $d = 3, g = 0$	**EWMA** $a = 0$
0			$q = 0.606244$	$q = 0.60301109$	$q = 0.5997955$	$q = 0.5934157$	$b = 0.609494$	$h = 0.019651$
	0.0000	ARL_0	370.41791	370.39766	370.39437	370.39076	370.38233	370.39948
		$SDRL_0$	369.91757	369.89732	369.89403	369.89042	369.88199	369.89914
	0.0001	ARL_Δ	365.91549	365.87210	365.84526	**365.79449**	365.90440	365.92151
		$SDRL_\Delta$	365.41515	365.37176	365.34492	**365.29415**	365.40406	365.42117
	0.0005	ARL_Δ	348.95228	348.82691	348.71666	**348.49899**	349.02817	349.04502
		$SDRL_\Delta$	348.45192	348.32655	348.21630	**347.99863**	348.52781	348.54466
	0.0007	ARL_Δ	341.04865	340.88789	340.74163	**340.45201**	341.16220	341.17886
		$SDRL_\Delta$	340.54828	340.38752	340.24126	**339.95164**	340.66183	340.67849
	0.0010	ARL_Δ	329.84425	329.63639	329.44221	**329.05687**	330.00807	330.02440
		$SDRL_\Delta$	329.34387	329.13601	328.94183	**328.55649**	329.50769	329.52402
	0.0030	ARL_Δ	270.61912	270.22181	269.83468	**269.06531**	270.98774	271.00101
		$SDRL_\Delta$	270.11866	269.72135	269.33422	**268.56484**	270.48728	270.50055
	0.0050	ARL_Δ	229.47134	229.00128	228.53961	**227.62299**	229.92185	229.93167
		$SDRL_\Delta$	228.97079	228.50073	228.03906	**227.12244**	229.42131	229.43113
	0.0070	ARL_Δ	199.22015	198.72734	198.24183	**197.27878**	199.69922	199.70583
		$SDRL_\Delta$	198.71952	198.22671	197.74120	**196.77814**	199.19859	199.20520
	0.0100	ARL_Δ	166.37505	165.88698	165.40516	**164.45049**	166.85472	166.85717
		$SDRL_\Delta$	165.87430	165.38622	164.90440	**163.94973**	166.35397	166.35642
	0.1000	ARL_Δ	28.69769	28.56204	28.42779	**28.16313**	28.83433	28.81251
		$SDRL_\Delta$	28.19326	28.05759	27.92331	**27.65861**	28.32992	28.30810
	EARL		253.34934	253.06919	252.79943	**252.26478**	253.60008	253.60866
	ESDRL		252.84842	252.56827	252.29850	**251.76385**	253.09916	253.10775
0.1			$q = 0.70631$	$q = 0.7030768$	$q = 0.6998609$	$q = 0.6934804$	$b = 0.7095605$	$h = 0.119161473$ $a = 0.1$
	0.0000	ARL_0	370.32642	370.32027	370.32623	370.32165	370.32368	370.33441
		$SDRL_0$	369.82608	369.81993	369.82589	369.82131	369.82334	369.83407
	0.0001	ARL_Δ	365.82243	365.79278	365.77496	**365.72322**	365.8434	365.73956
		$SDRL_\Delta$	365.32209	365.29244	365.27462	**365.22288**	365.34306	365.23922
	0.0005	ARL_Δ	348.8539	348.74099	348.63889	**348.42026**	348.95899	348.44929
		$SDRL_\Delta$	348.35354	348.24063	348.13853	**347.91990**	348.45863	347.94893
	0.0007	ARL_Δ	340.94811	340.79924	340.66076	**340.37018**	341.08958	340.40465
		$SDRL_\Delta$	340.44774	340.29887	340.16039	**339.86981**	340.58921	339.90428
	0.0010	ARL_Δ	329.74102	329.54426	329.35731	**328.97105**	329.93096	329.01273
		$SDRL_\Delta$	329.24064	329.04388	328.85693	**328.47067**	329.43058	328.51235
	0.0030	ARL_Δ	270.50845	270.11859	269.73629	**268.96624**	270.89469	269.03629
		$SDRL_\Delta$	270.00799	269.61813	269.23583	**268.46577**	270.39423	268.53582
	0.0050	ARL_Δ	229.36228	228.89761	228.43923	**227.52237**	229.82543	227.60262
		$SDRL_\Delta$	228.86173	228.39706	227.93868	**227.02182**	229.32488	227.10207
	0.0070	ARL_Δ	199.11582	198.62712	198.14428	**197.18098**	199.60438	197.26381
		$SDRL_\Delta$	198.61519	198.12649	197.64365	**196.68034**	199.10375	196.76317
	0.0100	ARL_Δ	166.27926	165.79411	165.31412	**164.35949**	166.76548	164.44047
		$SDRL_\Delta$	165.77851	165.29335	164.81336	**163.85873**	166.26473	163.93971
	0.1000	ARL_Δ	28.67616	28.54068	28.40656	**28.14207**	28.81291	28.16382
		$SDRL_\Delta$	28.17172	28.03622	27.90208	**27.63755**	28.30850	27.65930
	EARL		253.25638	252.98393	252.71916	**252.18398**	253.52509	252.23480
	ESDRL		252.75546	252.48301	252.21823	**251.68305**	253.02417	251.73387

Table 9. ARL comparison for the ARX(1,1) process on one and two-sided EWMA, MEWMA and new MEWMA with $\lambda = 0.1$, $\phi_1 = 0.994$, $\beta_1 = 1999.361$, $\alpha = 14.552$, and $ARL_0 = 370$.

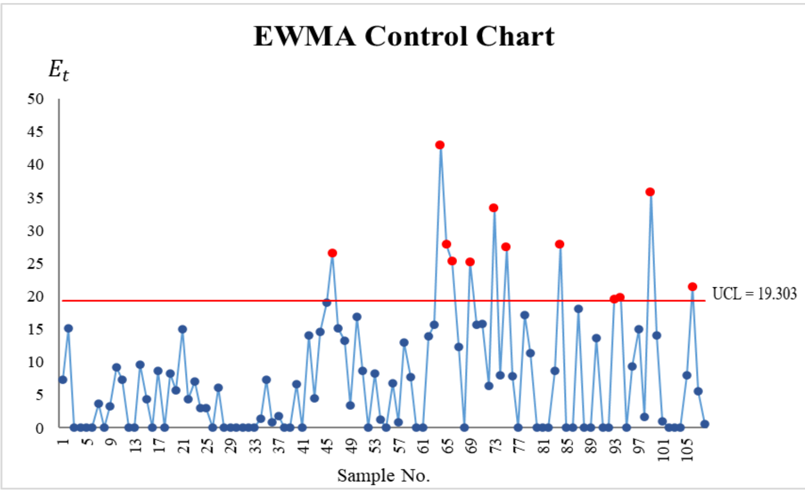
l	Shift	Control Chart	New MEWMA ($d_1 = 3$) $d_2 = 2.5$	$d_2 = 2.0$	$d_2 = 1.5$	$d_2 = 0.5$	MEWMA $d = 3, g = 0$	EWMA $a = 0$
0			$q = 6.055914$	$q = 5.988707$	$q = 5.922251$	$q = 5.791556$	$h = 6.12388$	$b = 0.1859969$
	0.0000	ARL_0	370.59493	370.58436	370.57734	370.52230	370.59482	370.60208
		$SDRL_0$	370.09459	370.08402	370.07700	370.02196	370.09448	370.10174
	0.0001	ARL_Δ	360.82995	360.72078	360.61511	**360.36537**	360.92914	360.97726
		$SDRL_\Delta$	360.32960	360.22043	360.11476	**359.86502**	360.42879	360.47691
	0.0005	ARL_Δ	326.43744	326.02410	325.61485	**324.76878**	326.84383	327.01782
		$SDRL_\Delta$	325.93706	325.52372	325.11447	**324.26839**	326.34345	326.51744
	0.0007	ARL_Δ	311.59448	311.07060	310.55134	**309.49050**	312.11296	312.33257
		$SDRL_\Delta$	311.09408	310.57020	310.05094	**308.99010**	311.61256	311.83217
	0.0010	ARL_Δ	291.70652	291.05405	290.40713	**289.09832**	292.35578	292.62823
		$SDRL_\Delta$	291.20609	290.55362	289.90670	**288.59789**	291.85535	292.12780
	0.0030	ARL_Δ	204.74131	203.78856	202.84639	**200.97794**	205.70047	206.09176
		$SDRL_\Delta$	204.24070	203.28795	202.34577	**200.47732**	205.19986	205.59115
	0.0050	ARL_Δ	157.83208	156.89366	155.96766	**154.14301**	158.78058	159.15935
		$SDRL_\Delta$	157.33129	156.39286	155.46686	**153.64220**	158.27979	158.65856
	0.0070	ARL_Δ	128.48465	127.61774	126.76353	**125.08617**	129.36282	129.70616
		$SDRL_\Delta$	127.98367	127.11676	126.26254	**124.58517**	128.86185	129.20519
	0.0100	ARL_Δ	100.54794	99.79370	99.05153	**97.59865**	101.31351	101.60326
		$SDRL_\Delta$	100.04669	99.29244	98.55026	**97.09736**	100.81227	101.10202
	0.1000	ARL_Δ	14.18152	14.04959	13.92024	**13.66909**	14.31613	14.31928
		$SDRL_\Delta$	13.67238	13.54036	13.41092	**13.15960**	13.80708	13.81023
	EARL		210.70621	210.11253	209.52642	**208.35531**	211.30169	211.53730
	ESDRL		210.20462	209.61093	209.02480	**207.85364**	210.80011	211.03572
0.1			$q = 6.1573$	$q = 6.090078$	$q = 6.023607$	$q = 5.892884$	$h = 6.225282$	$b = 0.2873186$
	0.0000	ARL_0	370.67083	370.67142	370.67155	370.66807	370.70281	370.60392
		$SDRL_0$	370.17049	370.17108	370.17121	370.16773	370.20247	370.10358
	0.0001	ARL_Δ	360.88500	360.78616	360.68702	**360.48559**	360.99246	360.44631
		$SDRL_\Delta$	360.38465	360.28581	360.18667	**359.98524**	360.49211	359.94596
	0.0005	ARL_Δ	326.42718	326.02169	325.61715	**324.80906**	326.84095	324.84665
		$SDRL_\Delta$	325.92680	325.52131	325.11677	**324.30867**	326.34057	324.34626
	0.0007	ARL_Δ	311.55998	311.04303	310.52785	**309.50108**	312.0854	309.56672
		$SDRL_\Delta$	311.05958	310.54263	310.02745	**309.00068**	311.58500	309.06632
	0.0010	ARL_Δ	291.64326	290.99664	290.35309	**289.07357**	292.29881	289.17206
		$SDRL_\Delta$	291.14283	290.49621	289.85266	**288.57314**	291.79838	288.67163
	0.0030	ARL_Δ	204.60230	203.65231	202.71168	**200.85715**	205.56467	201.03700
		$SDRL_\Delta$	204.10169	203.15169	202.21106	**200.35653**	205.06406	200.53638
	0.0050	ARL_Δ	157.68599	156.74944	155.82458	**154.00857**	158.63617	154.19166
		$SDRL_\Delta$	157.18519	156.24864	155.32378	**153.50776**	158.13538	153.69085
	0.0070	ARL_Δ	128.34616	127.48073	126.62750	**124.95629**	129.22521	125.12746
		$SDRL_\Delta$	127.84518	126.97975	126.12651	**124.45529**	128.72424	124.62646
	0.0100	ARL_Δ	100.42529	99.67221	98.93090	**97.48227**	101.19112	97.63233
		$SDRL_\Delta$	99.92404	99.17095	98.42963	**96.98098**	100.68988	97.13104
	0.1000	ARL_Δ	14.16069	14.02895	13.89981	**13.64902**	14.29513	13.67559
		$SDRL_\Delta$	13.65154	13.51971	13.39048	**13.13951**	13.78607	13.16610
	EARL		210.63732	210.04791	209.46440	**208.31362**	211.23666	208.41064
	ESDRL		210.13572	209.54630	208.96278	**207.81198**	210.73508	207.90900



(a)

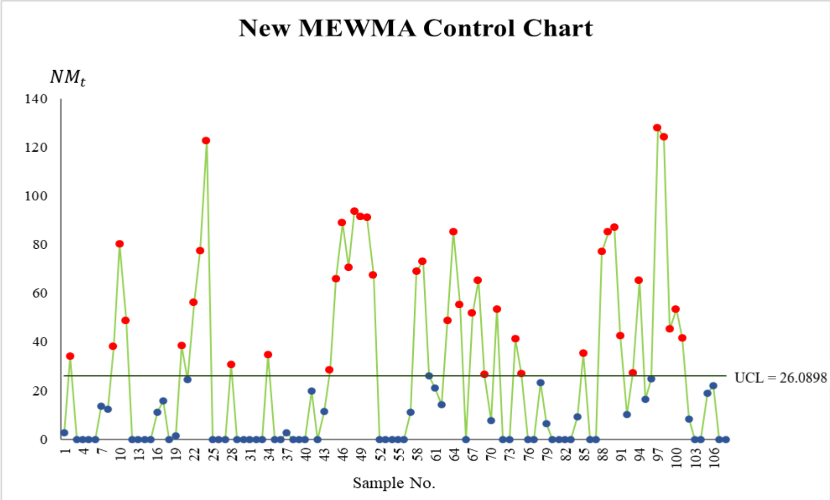


(b)

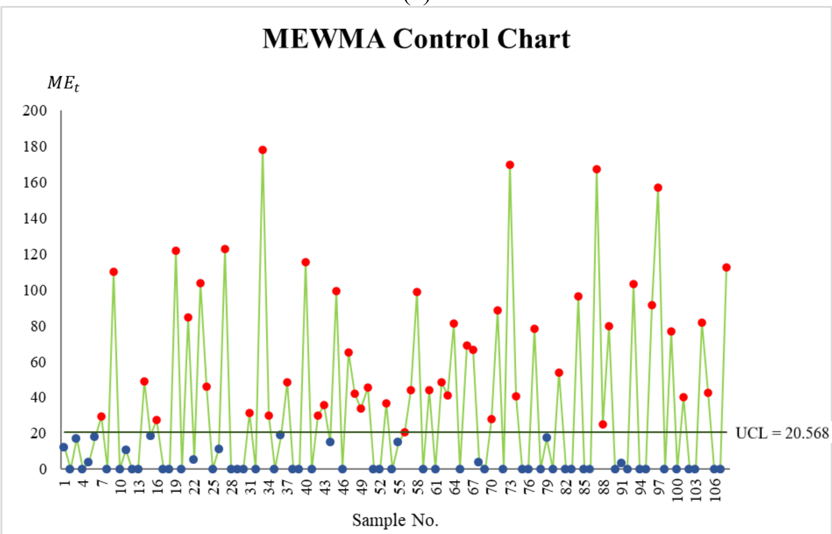


(c)

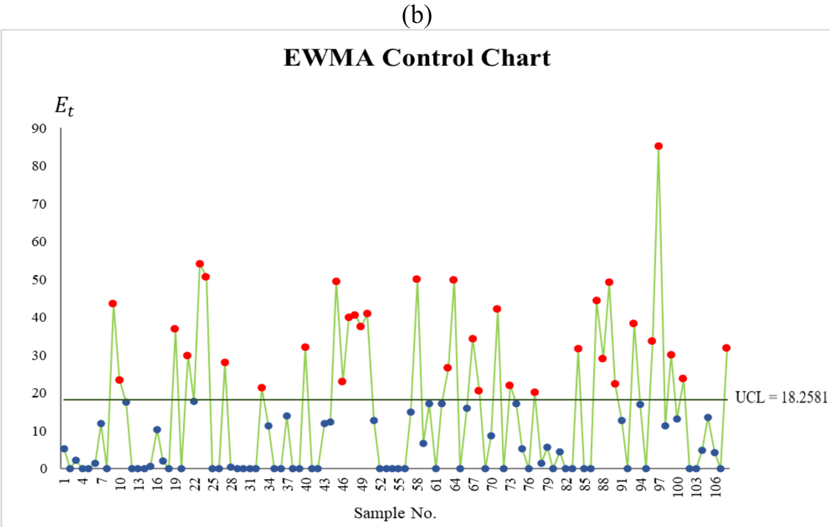
Fig. 1: The data plotted on $\lambda = 0.75$, $\phi_1 = 0.912$, $\beta_1 = 503.304$, $\alpha = 30.1757$ (a) New MEWMA control chart, (b) MEWMA control chart, and (c) EWMA control chart



(a)



(b)



(c)

Fig. 2: The data plotted on $\lambda = 0.75,\ \phi_1 = 0.994,\ \beta_1 = 1999.361,\ \alpha = 14.552$ (a) New MEWMA control chart, (b) MEWMA control chart, and (c) EWMA control chart

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

- Yupaporn Areepong has organized the conceptualization and simulation, writing-original draft.
- Saowanit Sukparungsee has implemented the methodology, software, and validation.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

This research was funded by National Science Research and Innovation Fund (NSRF), and King Mongkut's University of Technology North Bangkok with Contract no. KMUTNB-FF-68-B-08.

Conflicts of Interest

The authors declare no conflict of interest.