A Novel Interpolation-Based Order Six Method and its Convergence Analysis

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Abstract: This paper introduces an interpolation-based order six method for solving non-linear equations. Our method offers significant improvements in accuracy, stability, and efficiency, making it valuable for computational and applied mathematics. The main goal is to address various non-linear problems in fields such as physics, engineering, and finance. The paper explains the theoretical foundation and key principles of the method, which uses interpolation points instead of derivatives to approximate solutions. This approach enhances convergence behavior and numerical precision. We also conduct a detailed local and semilocal convergence analysis to evaluate the method's performance. This analysis provides insights into the convergence region, radii, and error boundaries. It also assesses the method's effectiveness in scenarios where accurate initial guesses are hard to obtain. Extensive numerical experiments on diverse test problems demonstrate the method's superior convergence rates and error estimates, confirming its effectiveness and reliability.

Key-Words: Non-linear equations, Fréchet derivative, Interpolation-based method, Banach Space, Convergence analysis, Convergence radii, Lipschitz continuity, Ball convergence.

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1 Introduction

In the realm of computational and applied mathematics, the quest for efficient and accurate numerical methods for solving non-linear equations remains a paramount pursuit. Non-linear equations pervade numerous scientific and engineering disciplines, presenting formidable challenges and complexities that necessitate innovative and sophisticated techniques for their solution, [1], [2], [3], [4]. Numerous researchers have explored and developed a rich array of iterative methods, [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], designed to address the challenges of solving non-linear equations of the form:

$$F(x) = 0 \tag{1}$$

where $F : \Delta \subset Q \rightarrow Q$. Here, the operator F is a non-linear mapping, continuously Fréchet differentiable, and operates on an open convex subset Δ within the Banach space Q.

Analytical methods for solving complex equations, particularly non-linear ones, are scarce and often infeasible. As a result, numerical approaches based on iterative procedures have become indispensable in obtaining approximate solutions. With the remarkable advancement of computing technology, the significance of numerical methods for solving non-linear equations has grown exponentially.

Throughout history, celebrated mathematicians such as Cauchy, Chebyshev, Euler, Fourier, Gauss, Lagrange, Laguerre, and Newton, [1], [2], [3], [4], [16], have made invaluable contributions to the field of equation-solving. Their pioneering work laid the foundation for modern numerical methods, inspiring researchers to explore innovative techniques for tackling intricate problems in various scientific and engineering disciplines.

The Newton-Raphson method, [1], [2], [3], [4], [10], [11], [12], [13], [14], [16], stands as one of the most prevalent algorithms for locating simple roots in various mathematical and engineering applications. This iterative technique commences with an initial guess x_0 that lies in proximity to the desired root and proceeds to generate a sequence of successive iterates $\{x_m\}_{m=0}^{\infty}$ that converge quadratically to the simple root. The method's recursive formula is given by:

$$x_{m+1} = x_m - \frac{F(x_m)}{F'(x_m)}, \quad m = 0, 1, 2, 3, \ldots \tag{2}$$

Numerous applications in diverse fields, such as transportation, electron theory, geometric theory of relativistic strings, chemical speciation, chemical engineering, and queuing models, give rise to a multitude of equations, [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16]. In most cases, these equations resist analytical solutions, necessitating numerical approximation through iterative methods. As such, researchers emphasize the development of higher-order iterative techniques to solve equations of the form (1) as they offer more efficient approximations and increased accuracy in finding solutions, [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15].

Higher-order iterative methods carry great significance in scenarios where faster convergence is essential. However, achieving an equilibrium between convergence rate and operational cost remains crucial. To this end, Newton's method has undergone modifications, involving additional function and derivative evaluations, as well as alterations in iteration points, to enhance its efficiency index and order of convergence. In the quest for enhancing the convergence order of the Newton-Raphson method, researchers have proposed and analyzed various higher-order multi-step methods, [3], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [17], [18]. A comprehensive survey of the literature addressing these improved order methods can be found in [2], [3], [4], [16], [19], [20], [21], [22], [23], [24], [25], and their respective references.

Below we investigate and compare a few sixth-order approaches. The study, [5], introduced a highly efficient sixth-order iterative method (NTM), which exhibits remarkable convergence properties. The NTM method, given for $m = 0, 1, 2, \ldots$ is defined by the following recursive formula:

$$\begin{aligned} w_m &= x_m - \frac{F(x_m)}{F'(x_m)}, \\ z_m &= w_m - \left[\frac{F(x_m) + 2F(w_m)}{F(x_m)}\right] \frac{F(w_m)}{F'(x_m)}, \\ x_{m+1} &= z_m - \left[\frac{F(x_m) - F(w_m) + F(z_m)}{F(x_m) - 3F(w_m) + F(z_m)}\right] \frac{F(z_m)}{F'(x_m)}. \end{aligned} \tag{3}$$

The NTM method requires only three evaluations of the function F and one evaluation of its first derivative F' per iteration.

The authors in study, [7], have devised a sixth-order variant of the Jarratt method (LKM) that exhibits enhanced convergence properties. The LKM method, given for $m = 0, 1, 2, \ldots$, is defined by the following recursive formula:

$$\begin{aligned} y_m &= x_m - \frac{2}{3} \frac{F(x_m)}{F'(x_m)}, \\ z_m &= x_m - J_f \frac{F(x_m)}{F'(x_m)}, \\ J_f &= \frac{3F'(y_m) + F'(x_m)}{6F'(y_m) - 2F'(x_m)}, \\ x_{m+1} &= z_m - \frac{F(z_m)}{\frac{3}{2} J_f F'(y_m) + \left(1 - \frac{3}{2} J_f\right) F'(x_m)}. \end{aligned} \tag{4}$$

The LKM method demands evaluations of two functions F and two of its first derivatives F' per iteration.

The author, [6], has contributed to the advancement of numerical methods by introducing two sixth-order iterative techniques for solving non-linear equations, applicable for $m = 0, 1, 2, \ldots$. These methods are detailed below. The first method, known as Singh Method 1 (SGM1), is expressed by the following recursive relations:

$$\begin{aligned} y_m &= x_m - \frac{F(x_m)}{F'(x_m)}, \\ z_m &= y_m - \frac{1}{2} \left(\frac{F'(x_m) - F'(y_m)}{F(x_m) + F'(x_m)}\right) \frac{F(x_m)}{F'(x_m)}, \\ x_{m+1} &= z_m - \frac{2F(z_m)(F(x_m) + F'(x_m))}{2F(x_m)F'(y_m) + 4F'(x_m)F'(y_m) - (F'(x_m))^2 - (F'(y_m))^2}. \end{aligned} \tag{5}$$

The SGM1 method requires two evaluations of the function F and its first derivative F' per iteration.

The second sixth-order method proposed by [6], demands three evaluations of the function F and one evaluation of its first derivative F' in each iteration. The SGM2 method, applicable for $m = 0, 1, 2, \ldots$ is expressed through the following recursive equations:

$$\begin{aligned} y_m &= x_m + \frac{F(x_m)}{F'(x_m)}, \\ z_m &= x_m - \frac{F(y_m) - F(x_m)}{F'(x_m)}, \\ x_{m+1} &= z_m - \frac{F(z_m)}{F[z_m, y_m] + F[z_m, x_m, x_m](z_m - y_m)}. \end{aligned} \tag{6}$$

The authors in [8], presented a novel sixth-order iterative method termed as Sharma's Sixth-order Method (SHM) for approximating solutions of non-linear equations. The SHM method, applicable for $m = 0, 1, 2, \ldots$, is defined by the following iterative process:

$$\begin{aligned} y_m &= x_m - \frac{F(x_m)}{F'(x_m)}, \\ z_m &= x_m - \left(\frac{3}{2} - \frac{1}{2}\frac{F'(y_m)}{F'(x_m)}\right)\frac{F(x_m)}{F'(x_m)}, \\ x_{m+1} &= z_m - \left[\frac{7}{2} - \left(-4 + \frac{3}{2}\frac{F'(y_m)}{F'(x_m)}\right)\frac{F'(y_m)}{F'(x_m)}\right]\frac{F(z_m)}{F'(x_m)}. \end{aligned} \tag{7}$$

The SHM method requires a computational cost of two evaluations of the function F and two evaluations of its first-order derivative F' at each step.

Armed with the techniques of linear interpolation and divided differences, [9], have pioneered the development of two higher-order iterative methods for solving non-linear equations. The first sixth-order method (ADM1) is derived by introducing a third step to a two-step third order method, but its derivative is approximated using divided differences up to the second order, resulting in a sixth-order method as well. The iterative process for the first method (ADM1), applicable for $m = 0, 1, 2, \ldots$, is given as:

$$\begin{aligned} y_m &= x_m - \frac{F(x_m)}{F'(x_m)}, \\ z_m &= x_m - \left(\frac{3}{2} - \frac{1}{2}\frac{F'(y_m)}{F'(x_m)}\right)\frac{F(x_m)}{F'(x_m)} \\ x_{m+1} &= z_m - \frac{F(z_m)}{F[z_m, y_m] + F[z_m, x_m, x_m](z_m - y_m)}. \end{aligned} \tag{8}$$

Each step in this procedure involves three evaluations of the function F and two evaluations of the first order derivative F'.

Similarly, the second sixth-order method (ADM2) is derived by introducing a third step to a two-step third oder method and employing linear interpolation to approximate its derivative. The iterative process for ADM2, applicable for $m = 0, 1, 2, \ldots$, is given as:

$$\begin{aligned} y_m &= x_m - \frac{F(x_m)}{F'(x_m)} \\ z_m &= x_m - \frac{1}{2}\left(\frac{1}{F'(x_m)} + \frac{1}{F'(y_m)}\right)F(x_m) \\ x_{m+1} &= z_m - \frac{2F(z_m)F'(y_m)}{2F'(x_m)F'(y_m) + (F'(y_m))^2 - (F'(x_m))^2}. \end{aligned} \tag{9}$$

At each stage of this approach, two evaluations of the function F and two evaluations of the first order derivative F' are used.

In light of the ongoing research in this area, our study aims to contribute to the advancement of higher-order iterative methods for solving non-linear equations. Inspired by the technique of linear interpolation, we have developed and thoroughly analyzed a novel sixth-order iterative method. Our primary focus lies in achieving higher-order accuracy by incorporating additional steps in the iterative process. The method is designed by introducing a third step and employing linear interpolation to approximate its derivative. This strategic inclusion of extra steps elevates the method to higher-order schemes, facilitating rapid convergence and greater precision in approximating solutions to complex nonlinear equations.

To ascertain the validity and effectiveness of the proposed method, we establish comprehensive convergence analyses. Local and semi-local convergence properties, [2], [11], [12], [13], [14], are rigorously investigated, providing valuable insights into the behavior of the iterative technique. The local convergence analysis investigates the convergence behavior of the method in the proximity of the true solution, providing crucial insights into its reliability and efficiency for nearby initial approximations. Furthermore, the semi-local convergence analysis examines the method's behavior when starting from initial approximations further away from the actual solution. This investigation aims to assess the method's performance in realistic scenarios where accurate initial guesses may be challenging to obtain. Additionally, we determine the convergence radii and error bounds, which shed light on the region of convergence and the accuracy of the numerical approximations. Furthermore, we explore the uniqueness of solution obtained through this method. By thoroughly assessing the uniqueness of the solution, we gain a deeper understanding of the reliability and stability of the proposed iterative technique. This comprehensive examination of their behavior provides valuable information for researchers and practitioners, enabling them to make informed decisions regarding the method's application in various contexts. The proposed method exhibits higher efficiency than Newton's method, leading to an improvement in the efficiency index from 1.414 to 1.56508. We evaluate the performance of our method by applying it to various examples, and our comparison results demonstrate the superior performance of the presented scheme over existing ones, [5], [6], [7], [8], [9].

There are limitations in the applicability of the methods we considered above (3,4,5,6,7,8,9) which constitute the motivation for this paper.

Motivation : The local convergence order of the methods is determined using the Taylor series expansion technique and by assuming that the operator F has higher order derivatives (which may not exist) and is bounded. Moreover, isolation of the solution ζ or a priori bounds on the distances $\|x_n - \zeta\|$ are not developed. These limitations restrict the applicability of the methods. The same limitations exist with other studies utilizing the Taylor series expansion approach on other methods, [1], [3], [4], [5], [6], [7], [8], [9]. Therefore, there is a need to work on the convergence conditions by relying only on the operators on the method.

Novelty : The convergence analysis developed in our study relies only on the operators involved in the method. This way isolation of the solution and computable a priori estimates on $\|x_n - \zeta\|$ become possible. Moreover, the more challenging and interesting semi-local convergence analysis is developed based on majorizing sequences. Although, we extend the applicability of method (16), our technique can be used to do the same on other methods along the same lines, [1], [3], [4], [5], [6], [7], [8], [9].

The paper's contents can be summarized into several key sections. In Section 2, we provide the necessary preliminaries, definitions, and auxiliary results to lay the foundation for the subsequent developments. Section 3 is dedicated to the introduction of the sixth-order method, wherein we present the iterative scheme and conduct a comprehensive convergence analysis utilizing Taylor series approach. Section 4 of the paper focuses on the local analysis of the proposed method, where we delve into the convergence behavior in the vicinity of the solutions. Moving on to Section 5, we shift our attention to the semi-local analysis. Here, we investigate the convergence properties over a broader region, considering the influence of initial guesses that might be farther away from the true solutions. In Section 6, we present a series of numerical examples to validate and verify the theoretical principles established earlier. Through these examples, we compare the performance of the proposed method against existing approaches, assessing their effectiveness in practical applications. Finally, Section 8 provides concluding remarks summarizing the contributions and implications of our work.

2 Preliminary Concepts and Definitions

Definition 2.1. *Consider a sequence $\{s_m\}$ converging to a parameter ζ. We classify the convergence as follows:*

1. *Linear convergence: If there exists a parameter v and a natural number m_0 such that for each $m \geq m_0$, we have $|s_{m+1} - \zeta| \leq v|s_m - \zeta|$.*
2. *Convergence of order p, where $p \geq 2$: If there exist a parameter v, with $v > 0$, and a natural number m_0 such that for each $m \geq m_0$, we have $|s_{m+1} - \zeta| \leq v|s_m - \zeta|^p$.*

Definition 2.2. *Let's use ζ to represent the root of the function F. Suppose that s_{m-1}, s_m, s_{m+1}, and s_{m+2} are successive iterations in proximity to ζ. In this context, the convergence order (computational), denoted as γ, is established using the following formula:*

$$\gamma \approx \frac{\ln\left(\left|\frac{s_{m+1}-\zeta}{s_m-\zeta}\right|\right)}{\ln\left(\left|\frac{s_m-\zeta}{s_{m-1}-\zeta}\right|\right)}. \tag{10}$$

This computation is valid when the value of ζ is known.

Alternatively, a second form of convergence order, termed the Approximate Computational convergence order (γ'), is defined using the subsequent formula:

$$\gamma' \approx \frac{\ln\left(\left|\frac{s_{m+2}-s_{m+1}}{s_{m+1}-s_m}\right|\right)}{\ln\left(\left|\frac{s_{m+1}-s_m}{s_m-s_{m-1}}\right|\right)}. \tag{11}$$

This definition is applicable when the value of ζ is unknown.

The computation of various methods often employs the **efficiency index $\mathbf{p}^{1/\eta}$**, where p represents the convergence order, and η signifies the total number of new function evaluations per iteration.

We proceed to state the Taylor's expansion formula in the context of real functions.

LEMMA 2.3. *Consider a function $F : \mathbb{R} \to \mathbb{R}$ that is q-times differentiable within an interval Q. For any*

$u, \alpha \in Q$, the following expression is valid:

$$F(u+\alpha) = F(u) + F'(u)\alpha + \frac{1}{2!}F''(u)\alpha^2 + \frac{1}{3!}F'''(u)\alpha^3 + \dots + \frac{1}{(p-1)!}F^{(p-1)}(u)\alpha^{p-1} + \delta_p,$$

where $|\delta_p| \leq \frac{1}{p!}\sup|F^{(p)}(u+\lambda\alpha)|$, and this holds for each $\lambda \in [0,1]$.

3 Methodology Formulation

Within this section, we present a pioneering new three-step iterative technique designed to address nonlinear equations in the form (1). This method extends the fourth-order approach introduced by [15], offering enhanced capabilities for solving such equations more effectively.

The method of fourth order as presented in [15], is expressed as follows for $x_0 \in \Delta$ and $\forall\ m = 0, 1, 2, \cdots$

$$\begin{aligned} y_m &= x_m - \frac{2}{3}F'(x_m)^{-1}F(x_m) \\ x_{m+1} &= x_m - \mathcal{Z}_m F'(y_m)^{-1}F(x_m), \end{aligned} \tag{12}$$

where $\mathcal{Z}_m = 1 + \frac{1}{4}(\mathcal{B}_m - 1) + \frac{3}{8}(\mathcal{B}_m - 1)^2$,

$\mathcal{B}_m = F'(x_m)^{-1}F'(y_m)$.

Extending the fourth-order approach (12) to achieve a sixth-order iterative method involves incorporating a step reminiscent of Newton's method. Here, for $m = 0, 1, 2, \ldots$, and with x_0 representing an initial approximation in proximity to the root, the new method is stated below:

$$\begin{aligned} y_m &= x_m - \frac{2}{3}F'(x_m)^{-1}F(x_m) \\ z_m &= x_m - \mathcal{Z}_m F'(y_m)^{-1}F(x_m), \\ x_{m+1} &= z_m - F'(z_m)^{-1}F(z_m). \end{aligned} \tag{13}$$

The primary objective of our investigation is to formulate an innovative higher-order iterative method characterized by an elevated efficiency index. To accomplish this, we strive to curtail the number of evaluations through the utilization of a linear interpolation formula. This formula is applied to the points $(x_m, F'(x_m))$ and $(y_m, F'(y_m))$ to approximate $F'(z_m)$ as depicted below:

$$F'(z_m) \simeq \frac{z_m - x_m}{y_m - x_m}F'(y_m) + \frac{z_m - y_m}{x_m - y_m}F'(x_m). \tag{14}$$

This simplification leads to the expression for $F'(z_m)$ as follows:

$$F'(z_m) \simeq F'(x_m)\left(1 + \frac{3}{2}\mathcal{Z}_m\left[1 - \frac{F'(x_m)}{F'(y_m)}\right]\right). \tag{15}$$

By substituting equation (15) into equation (13), the resulting formulation of the new three-step sixth-order method is as follows:

$$\begin{aligned} y_m &= x_m - \frac{2}{3}F'(x_m)^{-1}F(x_m), \\ z_m &= x_m - \mathcal{Z}_m F'(y_m)^{-1}F(x_m), \\ x_{m+1} &= z_m - 2\Psi_m^{-1}F'(y_m)F'(x_m)^{-1}F(z_m), \end{aligned} \tag{16}$$

where $\Psi_m = [2 + 3\mathcal{Z}_m]F'(y_m) - 3\mathcal{Z}_m F'(x_m)$, $\mathcal{Z}_m = \left[1 + \frac{1}{4}(\mathcal{B}_m - 1) + \frac{3}{8}(\mathcal{B}_m - 1)^2\right]$, $\mathcal{B}_m = F'(x_m)^{-1}F'(y_m)$.

This approach employs two function evaluations of F and two evaluations of its first-order derivative F' at each iteration. The convergence analysis for the sixth-order method (16) is subsequently established in the following theorem.

THEOREM 3.1. *Consider $F : \Delta \subset \mathbb{R} \to \mathbb{R}$, a sufficiently differentiable function within an open interval Δ, and let x_0 be a closely approximated value to its simple root $\zeta \in \Delta$. Under these conditions, the iterative method (16) adheres to the subsequent error equation:*

$$\varepsilon_{m+1} = -\frac{1}{9}\left(d_3(21d_2^3 - 9d_2d_3 + d_4)\right)\varepsilon_m^6 + O(\varepsilon_m^7), \tag{17}$$

where $d_m = \frac{F^{(m)}(\zeta)}{m!F'(\zeta)!}$ for $m = 2, 3, \ldots$.

Proof. Let $\varepsilon_m = x_m - \zeta$ represent the error in the m^{th} iteration. By employing the Taylor expansion of $F(x_m)$ and $F'(x_m)$ around ζ, we derive the expressions:

$$F(x_m) = F'(\zeta)\left(\varepsilon_m + d_2\varepsilon_m^2 + d_3\varepsilon_m^3 + d_4\varepsilon_m^4 + d_5\varepsilon_m^5 + d_6\varepsilon_m^6\right) + O(\varepsilon_m^7), \tag{18}$$

$$F'(x_m) = F'(\zeta)\left(1 + 2d_2\varepsilon_m + 3d_3\varepsilon_m^2 + 4d_4\varepsilon_m^3 + 5d_5\varepsilon_m^4 + 6d_6\varepsilon_m^5\right) + O(\varepsilon_m^6). \tag{19}$$

Upon substituting equations (18) and (19) into the first sub-step of equation (16), we acquire

$$\begin{aligned} y_m = {} & \zeta + \frac{\varepsilon_m}{3} + \frac{2d_2}{3}\varepsilon_m^2 - \frac{4}{3}(d_2^2 - d_3)\varepsilon_m^3 + \frac{2}{3}(4d_2^3 - 7d_2d_3 + 3d_4)\varepsilon_m^4 \\ & - \frac{4}{3}(4d_2^4 - 10d_2^2d_3 + 3d_3^2 + 5d_2d_4 - 2d_5)\varepsilon_m^5 + \frac{2}{3}(16d_2^5 - 52d_2^3d_3 + 33d_2d_3^2 \\ & + 28d_2^2d_4 - 17d_3d_4 - 13d_2d_5 + 5d_6)\varepsilon_m^6 + O(\varepsilon_m^7). \end{aligned} \tag{20}$$

Subsequently, employing the Taylor expansion centered around ζ yields

$$F(y_m) = F'(\zeta)\left[\frac{\varepsilon_m}{3} + \frac{7d_2}{9}\varepsilon_m^2 + \left(-\frac{8d_2^2}{9} + \frac{37d_3}{27}\right)\varepsilon_m^3 + \frac{1}{81}(180d_2^3 - 288d_2d_3 + 163d_4)\varepsilon_m^4 + \left(-\frac{16d_2^4}{3} + 12d_2^2d_3 - \frac{32d_3^2}{9} - \frac{424d_2d_4}{81} + \frac{649d_5}{243}\right)\varepsilon_m^5 + \frac{1}{729}(9072d_2^5 - 26352d_2^3d_3 + 12384d_2^2d_4 - 7632d_3d_4 + 192d_2(81d_3^2 - 26d_5) + 2431d_6)\varepsilon_m^6\right] + O(\varepsilon_m^7) \tag{21}$$

and

$$F'(y_m) = F'(\zeta)\left[1 + \frac{2d_2}{3}\varepsilon_m + \frac{1}{3}(4d_2^2 + d_3)\varepsilon_m^2 + \left(-\frac{8d_2^3}{3} + 4d_2d_3 + \frac{4d_4}{27}\right)\varepsilon_m^3 + \frac{1}{81}(432d_2^4 - 864d_2^2d_3 + 216d_3^2 + 396d_2d_4 + 5d_5)\varepsilon_m^4 - \frac{2}{81}(432d_2^5 - 1080d_2^3d_3 + 486d_2d_3^2 + 540d_2^2d_4 - 234d_3d_4 - 236d_2d_5 - d_6)\varepsilon_m^5 + \left(\frac{64d_2^6}{3} - 64d_2^4d_3 - \frac{8d_3^3}{3} + \frac{944d_2^3d_4}{27} + \frac{8d_4^2}{3} + \frac{124}{81}d_2^2(27d_3^2 - 11d_5) + \frac{512d_3d_5}{81} - \frac{4}{81}d_2(549d_3d_4 - 140d_6) + \frac{7d_7}{729}\right)\varepsilon_m^6\right] + O(\varepsilon_m^7) \tag{22}$$

By substituting equations (18), (19) and (22) into the second sub-step of equation (16), we get

$$z_m = \zeta + \frac{1}{9}(21d_2^3 - 9d_2d_3 + d_4)\varepsilon_m^4 - \frac{2}{27}(174d_2^4 - 216d_2^2d_3 + 27d_3^2 + 30d_2d_4 - 4d_5)\varepsilon_m^5 + \frac{2}{27}(623d_2^5 - 1257d_2^3d_3 + 321d_2^2d_4 - 99d_3d_4 + 9d_2(51d_3^2 - 5d_5) + 7d_6)\varepsilon_m^6 + O(\varepsilon_m^7). \tag{23}$$

Utilizing Taylor expansion for $F(z_k)$ around ζ, we obtain

$$F(z_m) = F'(\zeta)\left[\frac{1}{9}(21d_2^3 - 9d_2d_3 + d_4)\varepsilon_m^4 - \frac{2}{27}(174d_2^4 - 216d_2^2d_3 + 27d_3^2 + 30d_2d_4 - 4d_5)\varepsilon_m^5 + \frac{2}{27}(623d_2^5 - 1257d_2^3d_3 + 321d_2^2d_4 - 99d_3d_4 + 9d_2(51d_3^2 - 5d_5) + 7d_6)\varepsilon_m^6\right] + O(\varepsilon_m^7). \tag{24}$$

With reference to equation (15), we acquire

$$F'(z_m) = F'(\zeta)\Big[1 - d_3\varepsilon_m^2 - \frac{2}{9}(9d_2d_3 + 8d_4)\varepsilon_m^3 + \left(\tfrac{14d_2^4}{3} + 2d_2^2d_3 - 4d_3^2 - \tfrac{38d_2d_4}{9} - \tfrac{65d_5}{27}\right)\varepsilon_m^4 - \frac{2}{27}(348d_2^5 - 450d_2^3d_3 - 81d_2d_3^2 - 36d_2^2d_4 + 195d_3d_4 + 82d_2d_5 + 40d_6)\varepsilon_m^5 + \frac{1}{243}\big(22428d_2^6 - 52758d_2^4d_3 + 972d_3^3 + 12240d_2^3d_4 - 3084d_4^2 - 4896d_3d_5 + 270d_2^2(81d_3^2 + 2d_5) + 54d_2(67d_3d_4 - 34d_6) - 847d_7\big)\varepsilon_m^6\Big] + O(\varepsilon_m^7). \tag{25}$$

By incorporating equations (24) and (25) into the final sub-step of equation (16), we attain

$$\varepsilon_{m+1} = x_{m+1} - \zeta = -\frac{1}{9}\left(d_3(21d_2^3 - 9d_2d_3 + d_4)\right)\varepsilon_m^6 + O(\varepsilon_m^7). \tag{26}$$

□

Hence, it is shown that the method (16) have convergence order six.

The method (16) demonstrates an improved efficiency index of $6^{1/4} = 1.565085$, surpassing the efficiency of method (13). Furthermore, the superiority of method (16) over (13) becomes evident due to its need for one less derivative evaluation in each iteration.

4 Convergence Type I: Local

The assumptions for convergence order six necessitate the existence of at least the seventh derivative of $F(x)$. This limitation confines the method's applicability to scenarios where higher order derivatives are non-existent.

For illustrative purposes, let's consider a motivating example where F is defined on the interval $\Delta = [-0.5, 1.5]$ as follows:

$$F(x) = \begin{cases} \frac{1}{3}x^3\ln(x) + 8x^5 - 8x^4, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}. \tag{27}$$

Evidently, we can discern that the solution $\zeta = 1 \in \Delta$ and the third derivative is expressed as:

$$F'''(x) = \frac{11}{3} - 192x + 480x^2 + 2\ln(x). \tag{28}$$

It's noticeable that F''' is not bounded within Δ. Therefore, relying on the observations made in

Section 3, it is apparent that convergence cannot be consistently assured. To expand the method's scope of applicability, we introduce the notion of local convergence utilizing only the operators within the method. This concept is extended to the broader context of a Banach space.

4.1 Analysis

The introduction of several functions and constants contributes significantly to the analysis concerning the local convergence of the algorithm (16). We establish the set $\mathcal{W} = [0, +\infty)$. Assume:

(P_1) A root ζ within the domain Δ satisfies the equation $F(x) = 0$.

(P_2) The existence of a continuous and non-decreasing function $\kappa_0 : \mathcal{W} \to \mathcal{W}$ is such that the equation $\kappa_0(t) - 1 = 0$ has the smallest positive solution $\mathcal{R}_0$, leading to the interval $\mathcal{W}_0 = [0, \mathcal{R}_0)$.

(P_3) An operator $\mathcal{T} \in \mathscr{L}(Q, Q)$ exists, along with its inverse $\mathcal{T}^{-1} \in \mathscr{L}(Q, Q)$, and for all $x \in \Delta$, $\|\mathcal{T}^{-1}(F'(x) - \mathcal{T})\| \leq \kappa_0(\|x - \zeta\|)$. We define $\Delta_0 = U(\zeta, \mathcal{R}_0) \cap \Delta$.

(P_4) The smallest solution $\mathcal{R}_1 \in \mathcal{W}_0 - \{0\}$ of the equation $\kappa_0(q_1(t)t) - 1 = 0$ defines $\mathcal{W}_1 = [0, \mathcal{R}_1)$.

(P_5) A continuous and non-decreasing function $\kappa : \mathcal{W}_0 \to \mathcal{W}$ exists, satisfying the equations $q_1(t) - 1 = 0$ and $q_2(t) - 1 = 0$, where the functions q_1 and q_2 are defined on $\mathcal{W}_0$ and $\mathcal{W}_1$ respectively as follows:

$$q_1(t) = \frac{\int_0^1 \kappa((1-\beta)t)d\beta + \frac{1}{3}\left(1 + \int_0^1 \kappa_0(\beta t)d\beta\right)}{1 - \kappa_0(t)},$$

$$\begin{aligned} q_2(t) = \Bigg[& \frac{\int_0^1 \kappa((1-\beta)t)d\beta}{1-\kappa_0(t)} + \frac{\bar{\kappa}(t)(1 + \int_0^1 \kappa_0(\beta t)d\beta)}{(1-\kappa_0(t))(1-\kappa_0(q_1(t)t))} \\ & + \left(\frac{1}{4}\frac{\bar{\kappa}(t)}{1-\kappa_0(t)} + \frac{3}{8}\left(\frac{\bar{\kappa}(t)}{1-\kappa_0(t)}\right)^2\right) \\ & \times \frac{(1 + \int_0^1 \kappa_0(\beta t)d\beta)}{1 - \kappa_0(q_1(t)t)} \Bigg], \end{aligned} \tag{29}$$

where

$$\bar{\kappa}(t) = \min\{\kappa_0(q_1(t)t) + \kappa_0(t), \quad \kappa((1 + q_1(t))t)\}$$

and they possess the smallest solutions $\delta_1 \in \mathcal{W}_0 - \{0\}$ and $\delta_2 \in \mathcal{W}_1 - \{0\}$, respectively.

(P_6) $\|\mathcal{T}^{-1}(F'(u) - F'(v))\| \leq \kappa(\|u - v\|)$, for each $u, v \in \Delta_0$.

(P_7) The smallest solution $\mathcal{R}_2 \in \mathcal{W}_0 - \{0\}$ of the equation $\lambda(t) - 1 = 0$ defines $\mathcal{W}_2 = [0, \mathcal{R})$, where

$$\lambda(t) = \frac{1}{2}\left[(2 + 3\alpha(t))\kappa_0(q_1(t)t) + 3\alpha(t)\kappa_0(t)\right],$$

$$\alpha(t) = 1 + \frac{1}{4}\left(\frac{\bar{\kappa}(t)}{1-\kappa_0(t)}\right) + \frac{3}{8}\left(\frac{\bar{\kappa}(t)}{1-\kappa_0(t)}\right)^2$$

and $\mathcal{R} = \min\{\mathcal{R}_0, \mathcal{R}_1, \mathcal{R}_2\}$.

(P_8) The equation $q_3(t) - 1 = 0$ has the smallest solution $\delta_3 \in \mathcal{W}_2 - \{0\}$, with the function q_3 defined for the domain $\mathcal{W}_2$ as:

$$\begin{aligned} q_3 = \Bigg[& 1 + \frac{1 + \int_0^1 \kappa_0(\beta q_2(t)t)d\beta}{1-\kappa_0(t)} + \frac{3\alpha(t)\bar{\kappa}(t)}{2(1-\lambda(t))} \\ & \times \left(\frac{1 + \int_0^1 \kappa_0(\beta q_2(t)t)d\beta}{1-\kappa_0(t)}\right)\Bigg] q_2(t). \end{aligned} \tag{30}$$

The radius of convergence for the method (16) is defined by $\delta = \min\{\delta_k : k = 1, 2, 3\}$.

(P_9) The set $U[\zeta, \delta] \subset \Delta$.

It's worth noting that selecting $\mathcal{T} = F'(\zeta)$ is one possible option. In such a scenario, ζ stands as a simple solution to the equation $F(x) = 0$. Nevertheless, it's important to emphasize that $\mathcal{T}$ has the flexibility to be any alternate linear operator that meets the prescribed conditions. Consequently, the solution ζ is not necessarily required to be simple in such cases.

Let's consider the interval $\mathcal{W}_3 = [0, \delta)$. Consequently, for any t belonging to $\mathcal{W}_3$, it can be deduced that:

$$\begin{aligned} & 0 \leq \kappa_0(t) < 1 \\ & 0 \leq \kappa_0(q_1(t)t) < 1 \\ & 0 \leq \lambda(t) < 1 \\ & 0 \leq q_m(t) < 1. \end{aligned} \tag{31}$$

The rationale behind introducing these real functions becomes evident through a sequence of calculations, commencing from the initial sub-step of the method:

$$
\begin{aligned}
y_m-\zeta &= x_m-\zeta-F'(x_m)^{-1}F(x_m)\\
&\quad+\frac{1}{3}F'(x_m)^{-1}F(x_m)\\
&= -\left(F'(x_m)^{-1}F'(\zeta)\right)\int_0^1 F'(\zeta)^{-1}\left[F'(\zeta+\beta(x_m-\zeta))\right.\\
&\quad\left.-F'(x_m)\right]d\beta(x_m-\zeta)\\
&\quad+\frac{1}{3}F'(x_m)^{-1}F'(\zeta)\int_0^1 F'(\zeta)^{-1}(F'(\zeta+\beta(x_m-\zeta)))d\beta\\
&\quad\times(x_m-\beta),
\end{aligned}
\tag{32}
$$

$$
\begin{aligned}
\|y_m-\zeta\| &\le \frac{1}{1-\kappa_0(\|x_m-\zeta\|)}\left[\int_0^1\kappa((1-\beta)\|x_m-\zeta\|)d\beta\right.\\
&\quad\left.+|\frac{1}{3}|\left(1+\int_0^1\kappa_0(\beta\|x_m-\zeta\|)d\beta\right)\right]\|x_m-\zeta\|\\
&\le q_1(\|x_m-\zeta\|)\|x_m-\zeta\|\\
&\le \|x_m-\zeta\|\le\delta,
\end{aligned}
$$

$$
\begin{aligned}
z_m-\zeta &= x_m-\zeta-F'(x_m)^{-1}F(x_m)\\
&\quad+F'(x_m)^{-1}(F'(y_m)-F'(x_m))F'(y_m)^{-1}F(x_m)\\
&\quad-\left[\frac{1}{4}(\mathcal{B}_m-1)+\frac{3}{8}(\mathcal{B}_m-1)^2\right]F'(y_m)^{-1}F(x_m),
\end{aligned}
$$

$$
\begin{aligned}
\|z_m-\zeta\| &\le \left[\frac{\int_0^1\kappa((1-\beta)\|x_m-\zeta\|)d\beta}{1-\kappa_0(\|x_m-\zeta\|)}\right.\\
&\quad+\frac{\bar\kappa(\|x_m-\zeta\|)}{(1-\kappa_0(\|x_m-\zeta\|))(1-\kappa_0(\|y_m-\zeta\|))}\\
&\quad\times\left(1+\int_0^1\kappa_0(\beta\|x_m-\zeta\|)d\beta\right)\\
&\quad+\left[\frac{1}{4}\left(\frac{\bar\kappa(\|x_m-\zeta\|)}{1-\kappa_0(\|x_m-\zeta\|)}\right)+\frac{3}{8}\left(\frac{\bar\kappa(\|x_m-\zeta\|)}{1-\kappa_0(\|x_m-\zeta\|)}\right)^2\right]\\
&\quad\left.\times\frac{1+\int_0^1\kappa_0(\beta\|x_m-\zeta\|)d\beta}{1-\kappa_0(\|y_m-\zeta\|)}\right]\|x_m-\zeta\|\\
&\le q_2(\|x_m-\zeta\|)\|x_m-\zeta\|\\
&\le \|x_m-\zeta\|,
\end{aligned}
$$

where,

$$
\bar\kappa(\|x_m-\zeta\|)=\begin{cases}\kappa(\|x_m-\zeta\|+q_1(\|x_m-\zeta\|)\|x_m-\zeta\|)\\ \text{or}\\ \kappa_0(\|x_m-\zeta\|)+\kappa_0(q_1(\|x_m-\zeta\|)\|x_m-\zeta\|),\end{cases}
$$

$$
\begin{aligned}
x_{m+1}-\zeta &= z_m-\zeta-F'(x_m)^{-1}F(z_m)\\
&\quad+F'(x_m)^{-1}\left(\Psi_m-2F'(y_m)\right)\Psi_m^{-1}F(z_m),
\end{aligned}
$$

$$
\begin{aligned}
\|x_{m+1}-\zeta\| &\le \|z_m-\zeta\|\\
&\quad+\frac{\left(1+\int_0^1\kappa_0(\beta\|z_m-\zeta\|)d\beta\right)\|z_m-\zeta\|}{1-\kappa_0(\|x_0-\zeta\|)}\\
&\quad+\frac{3\alpha(\|x_m-\zeta\|)\bar\kappa(\|x_m-\zeta\|)}{2(1-\lambda(\|x_m-\zeta\|))}\\
&\quad\times\left(\frac{1+\int_0^1\kappa_0(\beta\|z_m-\zeta\|)d\beta}{1-\kappa_0(\|x_m-\zeta\|)}\right)\|z_m-\zeta\|\\
&\le q_3(\|x_m-\zeta\|)\|x_m-\zeta\|\\
&\le \|x_m-\zeta\|,
\end{aligned}
$$

and

$$
\begin{aligned}
\|2F'(\zeta)^{-1}(\Psi_m-2F'(\zeta))\| &\le \frac{1}{2}\left[(2+3\alpha(\|x_m-\zeta\|))\right.\\
&\quad\times\|F'(\zeta)^{-1}(F'(y_m)-F'(\zeta))\|\\
&\quad+3\alpha(\|x_m-\zeta\|)\\
&\quad\left.\times\|F'(\zeta)^{-1}(F'(x_m)-F'(\zeta))\|\right]\\
&\le \lambda(\|x_m-\zeta\|)\\
&\le 1,
\end{aligned}
$$

$$
\|\Psi_m^{-1}F'(\zeta)\|\le\frac{1}{2(1-\lambda(\|x_m-\zeta\|))},
$$

$$
\begin{aligned}
\bar\alpha(\|x_m-\zeta\|) &= \left[1+\frac{1}{4}\left(\frac{\bar\kappa(\|x_m-\zeta\|)}{1-\kappa_0(\|x_m-\zeta\|)}\right)\right.\\
&\quad\left.+\frac{3}{8}\left(\frac{\bar\kappa(\|x_m-\zeta\|)}{1-\kappa_0(\|x_m-\zeta\|)}\right)^2\right]\\
&\le \alpha(\|x_m-\zeta\|),
\end{aligned}
$$

$$
\begin{aligned}
\|F'(\zeta)^{-1}(\Psi_m-2F'(y_m))\| &\le 3\alpha(\|x_m-\zeta\|)\\
&\quad\times\|F'(\zeta)^{-1}(F'(x_m)-F'(y_m))\|\\
&\le 3\alpha(\|x_m-\zeta\|)\bar\kappa(\|x_m-\zeta\|).
\end{aligned}
$$

Since,

$$
\begin{aligned}
F(z_m) &= F(z_m)-F(\zeta)\\
&= \int_0^1 F'(\zeta+\beta(z_m-\zeta))d\beta\\
&\quad\times(z_m-\beta),
\end{aligned}
$$

we get,

$$
\begin{aligned}
\|F'(\zeta)^{-1}F(z_m)\| &\le \left(1+\int_0^1\kappa_0(\beta\|z_m-\zeta\|)d\beta\right)\\
&\quad\times\|z_m-\zeta\|.
\end{aligned}
\tag{33}
$$

Consequently, subject to the conditions (P_1) - (P_9), the sequence of iterates $\{x_m\}$ remains within the domain $U(\zeta,\delta)$, and as m tends to infinity, $\lim_{m\to+\infty}x_m=\zeta$. Hence, we established through an inductive reasoning:

THEOREM 4.1. *Given the assumptions (P_1) - (P_9), it can be inferred that the iterates $\{x_m\}$ lie within the region $U(\zeta,\delta)$ and converge to ζ as m approaches infinity. This holds true when the initial value x_0 is selected from the set $U(\zeta,\delta)-\{\zeta\}$.*

We now proceed to establish a result that confirms the uniqueness of the solution within the context of local convergence.

PROPOSITION 4.2. *Suppose there exists a solution $\zeta^*\in U(\zeta,\mathcal{R}_3)$ for the equation $F(x)=0$, with $\mathcal{R}_3>0$.*

Moreover, if the condition in (P_3) holds within the ball $U(\zeta,\mathcal{R}_3)$, and there is a larger radius $\mathcal{R}_4\ge\mathcal{R}_3$ that satisfies

$$
\int_0^1\kappa_0(\beta\mathcal{R}_4)d\beta<1.
\tag{34}
$$

Define $U_1 = \Omega \cap U[\zeta, \mathcal{R}_4]$. Then, within the set U_1, ζ stands as the unique solution to the equation $F(x) = 0$.

Proof. Let's define the linear operator $\mathscr{S} = \int_0^1 F'(\zeta + \beta(\zeta^* - \zeta))d\beta$. By using the condition in (P_3) and (34), we can infer the following:

$$\begin{aligned} \|F'(\zeta)^{-1}(\mathscr{S} - F'(\zeta))\| &\leq \int_0^1 \kappa_0(\beta\|\zeta^* - \zeta\|)d\beta \\ &\leq \int_0^1 \kappa_0(\beta\mathcal{R}_4)d\beta \\ &< 1. \end{aligned} \tag{35}$$

As a result, we can conclude that $\mathscr{S}^{-1} \in \mathscr{L}(\mathcal{Q})$. Also, based on the approximation

$$\zeta^* - \zeta = \mathscr{S}^{-1}(F(\zeta^*) - F(\zeta)) = \mathscr{S}^{-1}(0) = 0,$$

we can firmly establish that $\zeta^* = \zeta$. □

5 Convergence Type II: Semi-Local

In a manner analogous to local convergence, let's assume:

(L_1) A continuous, non-decreasing function $\mu_0 : \mathcal{W} \to \mathcal{W}$ is such that the equation $\mu_0(t) - 1 = 0$ possesses a minimal positive solution denoted by χ_0.

(L_2) An operator $\mathcal{T} \in \mathscr{L}(Q)$ exists for which $\mathcal{T}^{-1} \in \mathscr{L}(Q)$, and for every $x \in \Delta$, it follows that $\|\mathcal{T}^{-1}(F'(x) - \mathcal{T})\| \leq \mu_0(\|x - x_0\|)$ for some x_0.

It's important to note that choosing $\mathcal{T} = F'(x_0)$ is a viable choice. Notably, the condition in (L_1) ensures that $\mu_0(\|x - x_0\|) < 1$, leading to $F'(x_0)^{-1} \in \mathscr{L}(Q)$ and the well-defined nature of the iterate y_0 through the initial sub-step of the method (16). By setting $\|F'(x_0)^{-1}F(x_0)\| \leq p_0$, we also define $\Delta_1 = U(x_0, \chi_0) \cap \Delta$.

(L_3) Within Δ_1, there exists a continuous, non-decreasing function μ with values in $\mathcal{W}$, ensuring that for each x and y, $\|\mathcal{T}^{-1}(F'(y) - F'(x))\| \leq \mu(\|y - x\|)$. This prompts the definition of the scalar sequence $\{k_m\}$ with $k_0 = 0$ and $p_0 \in [0, \chi_0)$ as follows:

$$\begin{aligned}
\bar{\mu}_m &= \begin{cases} \mu(p_m + k_m) \\ \text{or} \\ \mu_0(p_m) + \mu_0(k_m), \end{cases} \\
\xi_m &= 1 + \frac{1}{4}\left(\frac{\bar{\mu}_m}{1 - \mu_0(k_m)}\right) + \frac{3}{8}\left(\frac{\bar{\mu}_m}{1 - \mu_0(k_m)}\right)^2, \\
j_m &= p_m + \left(1 + \frac{3}{2}\xi_m\left[\frac{1 + \mu_0(k_m)}{1 - \mu_0(p_m)}\right]\right)(p_m - k_m), \\
\delta_m &= \left(1 + \int_0^1 \mu_0(k_m + \beta(j_m - k_m))d\beta\right)(j_m - k_m) \\
&\quad + \frac{3}{2}\left(1 + \mu_0(k_m)\right)(p_m - k_m), \\
\sigma_m &= \frac{1}{2}\left[(2 + 3\xi_m)\mu_0(p_m) + 3\xi_m\mu_0(k_m)\right], \\
k_{m+1} &= j_m + \left(1 + \frac{3\xi_m\bar{\mu}_m}{(1 - \sigma_m)}\right)\left(\frac{\delta_m}{1 - \mu_0(k_m)}\right), \\
\theta_{m+1} &= \left(1 + \int_0^1 \mu_0(k_m + \beta(k_{m+1} - k_m))d\beta\right)(k_{m+1} - k_m) \\
&\quad + \frac{3}{2}\left(1 + \mu_0(k_m)\right)(p_m - k_m), \\
p_{m+1} &= k_{m+1} + \frac{\theta_{m+1}}{1 - \mu_0(k_m)}.
\end{aligned}$$

There exists $\chi \in [0, \chi_0)$ such that for each $m = 0, 1, 2, \ldots$, the condition $\mu_0(k_m) < 1$ and $k_m \leq \chi$ holds. Consequently, the inequalities $0 \leq k_m \leq p_m \leq j_m \leq \chi^\star \leq \chi$ emerge, where $\chi^\star = \lim_{m \to +\infty} k_m$.

(L_4) $U[x_0, \chi^\star] \subset \Delta$ is satisfied.

The rationale behind these conditions is:

$$\begin{aligned}
z_m - y_m &= -(y_m - x_m) - \mathcal{Z}_m F'(y_m)^{-1}F(x_m), \\
\|z_m - y_m\| &\leq \left(1 + \frac{3}{2}\xi_m\left[\frac{1 + \mu_0(k_m)}{1 - \mu_0(p_m)}\right]\right)(p_m - k_m) \\
&\leq j_m - p_m,
\end{aligned}$$

where,

$$\begin{aligned}
\bar{\xi}_m &= 1 + \frac{1}{4}\left(\frac{\bar{\mu}_m}{1 - \mu_0(k_m)}\right) + \frac{3}{8}\left(\frac{\bar{\mu}_m}{1 - \mu_0(k_m)}\right)^2 \\
&\leq \xi_m, \\
\bar{\mu}_m &= \begin{cases} \mu(p_m + k_m) \\ \text{or} \\ \mu_0(p_m) + \mu_0(k_m), \end{cases} \\
F(x_m) &= -\frac{3}{2}F'(x_m)(y_m - x_m),
\end{aligned}$$

and hence

$$\begin{aligned}\|F'(x_0)^{-1}F(x_m)\| &\le \frac{3}{2}\left(1+\mu_0(\|x_m-x_0\|)\right)\|y_m-x_m\|,\\ x_{m+1}-z_m &= F'(x_m)^{-1}F(z_m)\\ &+ F'(x_m)^{-1}(\Psi_m-2F'(y_m))\Psi_m^{-1}F(z_m),\\ \|x_{m+1}-z_m\| &\le \left(1+\frac{3\xi_m\bar{\mu}_m}{(1-\sigma_m)}\right)\left(\frac{\delta_m}{1-\mu_0(k_m)}\right)\\ &\le k_{m+1}-j_m.\end{aligned}$$

Since from

$$\begin{aligned}F(z_m) &= F(z_m)+F(x_m)-F(x_m)\\ &= \int_0^1 F'(x_m+\beta(z_m-x_m))d\beta(z_m-x_m)\\ &- \frac{3}{2}F'(x_m)(y_m-x_m),\\ \|F'(x_0)^{-1}F(z_m)\| &\le \left(1+\int_0^1 \mu_0(\|x_m+\beta(z_m-x_m)-x_0\|)d\beta\right)\\ &\times \|z_m-x_m\|\\ &\quad + \frac{3}{2}\left(1+\mu_0(\|x_m-x_0\|)\right)\\ &\times \|y_m-x_m\|\\ &= \bar{\delta}_m \le \delta_m,\end{aligned}$$

$$\begin{aligned}\|2F'(x_0)^{-1}(\Psi_m-2F'(x_0))\| &\le \frac{1}{2}\left[(2+\xi_m)\right.\\ &\times\|F'(x_0)^{-1}(F'(y_m)-F'(x_0))\|\\ &\left.+3\xi_m\|F'(x_0)^{-1}(F'(x_m)-F'(x_0))\|\right]\\ &\le \frac{1}{2}\left[(2+3\xi_m)\right.\\ &\times\mu_0(\|y_m-x_0\|)\\ &\left.+3\xi_m\mu_0(\|x_m-x_0\|)\right]\\ &\le \sigma(\|x_m-x_0\|)\\ &\le 1,\\ \|\Psi_m^{-1}F'(x_0)\| &\le \frac{1}{1-\sigma(\|x_m-x_0\|)},\\ \|F'(x_0)^{-1}(\Psi_m-2F'(y_m))\| &\le 3\xi_m\\ &\times\|F'(x_0)^{-1}(F'(x_m)-F'(y_m))\|\\ &\le 3\xi_m\bar{\mu}_m,\end{aligned}$$

and

$$\begin{aligned}F(x_{m+1}) &= F(x_{m+1})-F(x_m)\\ &- \frac{3}{2}F'(x_m)(y_m-x_m)\\ &= \int_0^1 F'(x_m+\beta(x_{m+1}-x_m))\\ &\times (x_{m+1}-x_m)d\beta\\ &- \frac{3}{2}F'(x_m)(y_m-x_m),\\ \|F'(x_0)^{-1}F(x_{m+1})\| &\le \Big(1\\ &+\int_0^1 \mu_0(\|x_m+\beta(x_{m+1}-x_m)-x_0\|)d\beta\Big)\\ &\times \|x_{m+1}-x_m\|\\ &+ \frac{3}{2}\left(1+\mu_0(\|x_m-x_0\|)\right)\\ &\times \|y_m-x_m\|\\ &= \bar{\theta}_{m+1}\\ &\le \theta_{m+1}.\end{aligned}$$

Therefore,

$$\begin{aligned}\|y_{m+1}-x_{m+1}\| &\le \|F'(x_m)^{-1}F'(x_0)\|\\ &\times \|F'(x_0)^{-1}F(x_{m+1})\|\\ &\le \frac{\theta_{m+1}}{1-\mu_0(k_m)}\\ &= p_{m+1}-k_{m+1}.\end{aligned}$$

The sequences of iterates $\{x_m\}$, $\{y_m\}$, and $\{z_m\}$ all belong to the set $U(x_0,\chi^\star)$. Notably, we can establish the following bounds:

$$\begin{aligned}\|y_m-x_0\| &\le p_m-k_0 < \chi^\star,\\ \|z_m-x_0\| &\le \|z_m-y_m\|+\|y_m-x_0\|\\ &\le j_m-p_m+p_m-k_0\\ &= j_m < \chi^\star,\\ \|x_{m+1}-x_0\| &\le \|x_{m+1}-z_m\|+\|z_m-x_0\|\\ &\le k_{m+1}-j_m+j_m-k_0\\ &= k_{m+1} < \chi^\star,\\ \|y_{m+1}-x_0\| &\le \|y_{m+1}-x_m\|+\|x_{m+1}-x_0\|\\ &\le p_{m+1}-k_{m+1}+k_{m+1}-k_0\\ &= p_{m+1} < \chi^\star.\end{aligned}$$

This establishes the existence of a point $\zeta \in U[x_0,\chi^\star]$ that solves the equation $F(x)=0$ and satisfies the error estimate

$$\|\zeta-x_m\| \le \chi^\star-k_m.$$

Therefore, we can conclude with the following result:

THEOREM 5.1. *Given the assumptions* (L_1)*-*(L_4)*, the sequence* $\{x_m\}$ *converges to a solution* $\zeta \in U[x_0,\chi^\star]$ *of the equation* $F(x)=0$.

We proceed to establish the uniqueness of the solution domain through the following proposition.

PROPOSITION 5.2. *Consider the following conditions:*

(i) A solution $\zeta^\star$ of the equation $F(x) = 0$ exists within $U(x_0, \mathcal{R}_5)$ for some $\mathcal{R}_5 > 0$.

(ii) Condition (L_2) is satisfied within $U(x_0, \mathcal{R}_5)$.

(iii) There exists $\mathcal{R}_6 > \mathcal{R}_5$ such that

$$\int_0^1 \mu_0((1-\beta)\mathcal{R}_5 + \beta\mathcal{R}_6)d\beta < 1.$$

Define $U_2 = \Delta \cap U[x_0, \mathcal{R}_6]$. Under these conditions, the only point in the domain U_2 that satisfies the equation $F(x) = 0$ is $\zeta^\star$.

Proof. Let's assume there exists $\zeta' \in U_2$ such that $F(x) = 0$. By applying conditions (ii) and (iii), we can derive the following inequality:

$$\begin{aligned}\|F'(x_0)^{-1}(P - F'(x_0))\| &\le \int_0^1 \mu_0((1-\beta)\|\zeta^\star - x_0\| \\ &\quad + \beta\|\zeta' - x_0\|)d\beta \\ &\le \textstyle\int_0^1 \mu_0((1-\beta)\mathcal{R}_5 + \beta\mathcal{R}_6)d\beta \\ &< 1,\end{aligned}$$

where $P = \int_0^1 F'(\zeta^\star + \beta(\zeta' - \zeta^\star))d\beta$. Therefore, it follows that $\zeta' = \zeta^\star$. □

REMARK 5.3. *(i) Within condition (L_4), the limit point χ^* can be interchanged with χ_0.*

(ii) Under the assumptions (L_1)-(L_4), consider $\zeta^\star = \zeta$ and $\mathcal{R}_5 = \chi^\star$ in Proposition 5.2.

6 Numericals

In this section, we showcase the practical application of the proposed technique (16), namely JJM, to a variety of nonlinear equations. This validation process serves to confirm the theoretical findings presented earlier. These nonlinear equations have implications across a range of scientific and engineering domains [5,6]. The obtained results are then compared against methods NTM, LKM, $SGM1$, $SGM2$, SHM, $ADM1$ and $ADM2$, represented by (3), (4), (5), (6), (7), (8) and (9) respectively.

The test functions used are detailed in Table 1(Appendix), along with the initial approximations, x_0 and roots accurate up to 15 decimal places. Table 2(Appendix) and Table 3(Appendix) demonstrate, respectively, comparisons of the number of iterations and the overall number of function evaluations. The term "div" indicates that the iteration diverges from the considered initial point.

The comparison results for $|x_{m+1} - x_m|$, $|f(x_m)|$ and computational time for all the considered examples are presented in Table 4(Appendix), Table 5(Appendix) and Table 6(Appendix), respectively, up to the third iteration. These computations were conducted using the Mathematica programming package version 11.3.0.0, employing 600 significant digits. The computations were carried out on an Intel(R) Core(TM) i5 - 8250U CPU @ 1.60 GHz 1.80 GHz with 8 GB of RAM, running on Windows 11 Home version 22H2. It's clear that in the majority of examples, the proposed method JJM exhibits higher accuracy in numerical approximations of the root and also requires less computational time compared to the existing methods. As a result, these numerical experiments effectively showcase the innovation and practicality of the current study.

Next, we examine examples that pertain to the convergence analysis discussed in Sections 4 and 5.

EXAMPLE 6.1. *Consider the settings with $\mathcal{Q} = C[0,1]$ using the maximum norm. Let $\Delta = U[0,1]$. Define the operator F on Δ as*

$$F(\Lambda)(x) = \Lambda(x) - 5\int_0^1 x\beta\Lambda(\beta)^3 d\beta.$$

With this definition, we have

$$F'(\Lambda(v))(x) = v(x) - 15\int_0^1 x\beta\Lambda(\beta)^2 v(\beta)d\beta,$$

for each $v \in \Delta$.

The solution is given by $\zeta = 0$. As a result, conditions (P_1)-(P_9) are met with $\kappa_0(t) = 7.5t$, $\kappa(t) = 15t$, $R_0 = 0.133333$, $R_1 = 0.068872$, $R_2 = 0.031716$ and hence $R = 0.031716$. Consequently, the radii of the convergence domain have been computed and are provided in Table 7(Appendix).

EXAMPLE 6.2. *Consider $\mathcal{Q} = \Delta = \mathbb{R}$. Let's examine the function F defined on Δ as $F(x) = \sin(x)$. The derivative of F is given by $F'(x) = \cos(x)$. The fixed point is $\zeta = 0$. To validate conditions $(P_1) - (P_9)$, we choose $\kappa_0(t) = \kappa(t) = t$, $R_0 = 1$, $\Delta_0 = \Delta \cap U(\zeta, R_0)$, $R_1 = 0.581139$, $R_2 = 0.25732$ and thus $R = 0.25732$. The radii of convergence are detailed in Table 7(Appendix).*

EXAMPLE 6.3. *Consider a system of differential equations described by:*

$$F_1'(x_1) = e^{x_1}, \quad F_2'(x_2) = (e-1)x_2 + 1, \quad F_3'(x_3) = 1$$

with initial conditions $F_1(0) = F_2(0) = F_3(0) = 0$. *Here,* $F = (F_1, F_2, F_3)$*, and we take* $\mathcal{Q} = \mathbb{R}^3$ *and* $\Delta = U[0,1]$. *The vector* $\zeta = (0,0,0)^T$ *is a solution of this system. The function* F *on* Ω *for any vector* $x = (x_1, x_2, x_3)^T$ *is defined as:*

$$F(x) = (e^{x_1} - 1, \frac{e-1}{2}x_2^2 + x_2, x_3)^T.$$

The derivative matrix of F is given by:

$$F'(x) = \begin{bmatrix} e^{x_1} & 0 & 0 \\ 0 & (e-1)x_2 + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notably, $F'(\zeta) = \mathbf{I}$. *To verify the local convergence criteria, the conditions* $(P_1) - (P_9)$ *need to be satisfied. By selecting* $\kappa_0(t) = (e-1)t$*,* $\kappa(t) = e^{\frac{1}{e-1}}t$*,* $R_0 = 0.581977$*,* $\Delta_0 = \Delta \cap U(\zeta, R_0)$*,* $R_1 = 0.336295$*,* $R_2 = 0.149209$*, and hence* $R = 0.149209$*, these conditions can be met. Refer to Table 7(Appendix) for the radii corresponding to method (16).*

EXAMPLE 6.4. *Consider the examination of a system consisting of twenty nonlinear equations:*

$$x_i - \cos\left(2x_i - \sum_{j\neq i, j=1}^{20} x_j\right) = 0, \quad 1 \le i \le 20.$$

The solution-seeking process commences with the initial approximation $x_0 = \{-0.89, -0.89, \ldots, -0.89\}^T$*, aiming to determine the solution:* $\zeta = \{-0.89707478633292, -0.89707478633292, \ldots, -0.89707478633292\}^T$. *The error estimates for the solution are detailed in Table 8(Appendix). Upon a thorough analysis of the system of equations, it is observed that convergence to* ζ *is achieved within a maximum of two iterations.*

6.1 Some Application Problems

This section uses several real-life applications to demonstrate the effectiveness of the innovative sixth-order iterative technique. The error estimates are detailed in Table (9)(Appendix).

EXAMPLE 6.5. *The following nonlinear equation determines the embedment depth of a sheet-pile wall:*

$$g_1(x) = \frac{1}{4.62}\left(x^3 + 2.87x^2 - 10.28\right) - x.$$

The root is approximated to be $\zeta = 2.0021187789538272$.

EXAMPLE 6.6. *Vertical stress is a fundamental stress experienced by finite underground structures and is expressed as*

$$g_2(x) = \frac{x + \cos x \sin x}{\pi} - \frac{1}{4}.$$

The solution to the nonlinear equation $g_2(x) = 0$ *is* 0.4160444988100767043.

EXAMPLE 6.7. *The motion of an electron in the region between two parallel plates is*

$$x(t) = x_0 + \left(v_0 + eE_0(mw)^{-1}\sin(wt_0 + \eta)\right)(t - t_0) + eE_0(mw^2)^{-1}\left(\cos(wt_0 + \eta) + \sin(wt_0 + \eta)\right),$$

where x_0 *represents the electron's position,* v_0 *denotes its velocity,* e *is the electron charge,* m *is its rest mass, and* $E_0 \sin(\omega t_0 + \eta)$ *is the RF electric field between the plates at time* t_0. *For specific values, this can be simplified to a polynomial form as*

$$g_3(x) = x - 0.5\cos x + \pi/4.$$

This function has a simple root approximately at $\zeta \approx -0.309466139208214$.

EXAMPLE 6.8. *The nonlinear equation describing the velocity of the parachutist is*

$$g_4(x) = \frac{gm}{x}\left(1 - e^{-\frac{x}{m}t}\right) - v.$$

Using the parameter values $g = 9.8\,m/s^2$*,* $m = 70\,kg$*,* $t = 10\,s$*, and* $v = 42\,m/s$*, the root of the nonlinear equation is approximately* 14.17851672262242.

7 Discussion

The test functions discussed in Table 1(Appendix), along with the corresponding results of computational time, iteration counts, function evaluations, and absolute differences, showcase that our method performs better than existing methods. Specifically, our method demonstrates higher accuracy and efficiency.

The analysis of the convergence radius further underscores the robustness of our method. Examples (6.1) -(6.4) provide local and semi-local results, including convergence radii and error estimates. By evaluating the convergence radius, we have been able to determine the extent to which our method can be applied effectively across different initial guesses.

Additionally, we have tested our method on several real-life problems to verify its practical usefulness. These applications demonstrate that our method is not only theoretically sound but also effective in solving real-world problems, making it a valuable tool in various scientific and engineering fields. Our findings suggest a broader applicability, making our method a versatile tool in practical applications.

8 Conclusion

The scope of this research encompasses the creation of a sixth-order algorithm designed for tackling nonlinear equations. This novel approach involves the incorporation of a Newton-like step while approximating the derivative using linear interpolation. The remarkable efficiency index of 1.56508 achieved by this iterative method underscores the significance and motivation driving this study. The detailed analyses of local and semi-local convergence contribute to the thorough understanding of its performance characteristics. The radii of the convergence domain, error estimates, and uniqueness of solution results collectively underline the robustness and reliability of the proposed sixth-order composition method. This study not only enriches the theoretical underpinnings but also lays the foundation for practical applications in a wide array of scientific and engineering domains. These encompass not only single-step but also multi-step methods, as exemplified by references, [4], [15], [17], [19], [22], [24], [25], [26], [27], [28], [29]. This approach is also applicable for the extension of methods found in [1], [2], [5], [6], [7], [8], [9], [10], [11], [12], [13], [30] where our future research plans will uncover them.

Declaration of Generative AI and AI-assisted technologies in the writing process

During the preparation of this work the authors used ChatGPT in order to improve readability and language. After using this tool/service, the authors reviewed and edited the content as needed and take full responsibility for the content of the publication

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Conflict of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.

Appendix

Table 1: Test Functions

$f(x)$	Root (ζ)	x_0
$f_1(x) = x - 0.9995\sin x - 0.01$	$\zeta = 0.389977774946362$	$x_0 = 2.99$
$f_2(x) = x^3 - x^2 - 1$	$\zeta = 1.465571231876768$	$x_0 = 2$
$f_3(x) = \exp(-x^2 + x - 2) - \cos(x+1) + x^3 + 1$	$\zeta = -1.000000000000000$	$x_0 = -2$
$f_4(x) = \sin^2(x) - x^2 - 1$	$\zeta = 1.404494648215341$	$x_0 = 3$
$f_5(x) = x\exp(x^2) - \sin^2(x) + 3\cos(x) + 5$	$\zeta = -1.207647827130919$	$x_0 = -1$
$f_6(x) = x^3 + 4x^2 - 10$	$\zeta = 1.365230013414097$	$x_0 = 4$
$f_7(x) = x^2\exp(x) - \sin x + x$	$\zeta = -1.499393096901409$	$x_0 = -2$
$f_8(x) = \log(x^2 + x + 2) - x + 1$	$\zeta = 4.15290736757158$	$x_0 = 3$
$f_9(x) = \exp(-x) + \cos x$	$\zeta = 1.365230013414097$	$x_0 = -0.5$
$f_{10}(x) = \arcsin(x^2 - 1) - \frac{x}{2} + 1$	$\zeta = 0.5948109683983692$	$x_0 = 1$

Table 2: Comparison of iteration counts.

$f(x)$	NTM	LKM	$SGM1$	$SGM2$	SHM	$ADM1$	$ADM2$	JJM
$f_1(x)$	3	3	3	3	div	3	3	3
$f_2(x)$	2	2	3	3	4	3	3	3
$f_3(x)$	3	3	4	4	6	3	3	3
$f_4(x)$	4	2	2	3	div	3	2	2
$f_5(x)$	2	2	2	2	7	3	2	2
$f_6(x)$	3	3	3	4	div	4	3	3
$f_7(x)$	2	2	2	3	5	3	2	2
$f_8(x)$	2	2	2	2	3	2	2	2
$f_9(x)$	2	3	3	div	5	3	2	2
$f_{10}(x)$	2	2	2	3	4	2	2	2

Table 3: Comparison of the number of function evaluations.

$f(x)$	NTM	LKM	$SGM1$	$SGM2$	SHM	$ADM1$	$ADM2$	JJM
$f_1(x)$	12	12	12	12	-	15	12	12
$f_2(x)$	8	8	12	12	16	15	12	12
$f_3(x)$	12	12	16	16	24	15	12	12
$f_4(x)$	16	8	8	12	-	15	8	8
$f_5(x)$	8	8	8	8	28	15	8	8
$f_6(x)$	12	12	12	16	-	20	12	12
$f_7(x)$	8	8	8	12	20	15	8	8
$f_8(x)$	8	8	8	8	12	10	8	8
$f_9(x)$	8	12	12	-	20	15	8	8
$f_{10}(x)$	8	8	8	12	16	10	8	8

TABLE 4: Comparing the Absolute Difference $|x_{m+1} - x_m|$

$f(x)$	m	NTM	LKM	$SGM1$	$SGM2$	SHM	$ADM1$	$ADM2$	JJM
	1	4.86e-002	5.94e-002	6.17e-002	9.42e-002	–	9.27e-002	9.88e-003	2.89e-002
f_1	2	4.60e-007	7.97e-007	8.37e-006	2.01e-005	–div–	1.13e-004	1.19e-011	3.26e-007
	3	8.88e-016	4.44e-016	3.89e-016	5.00e-016	–	9.99e-016	5.55e-016	5.55e-016
	1	2.10e-003	1.80e-003	7.30e-003	2.30e-002	1.74e-001	2.35e-002	8.50e-004	2.49e-003
f_2	2	8.14e-011	0.00e-000	8.78e-013	4.12e-009	1.36e-001	1.41e-008	2.22e-016	5.47e-007
	3	0.00e-000	-	2.22e-016	0.00e-000	1.30e-002	0.00e-000	0.00e-000	0.00e-000
	1	1.29e-001	4.22e-001	1.24e-000	4.39e-001	2.93e-001	4.39e-001	2.82e-001	1.63e-002
f_3	2	3.25e-005	3.94e-004	3.80e-001	7.44e-003	2.60e-001	1.27e-002	1.73e-004	8.48e-008
	3	1.11e-016	1.11e-016	1.28e-003	1.31e-013	5.74e-002	5.10e-010	1.11e-016	1.11e-016
	1	7.67e-002	1.64e-004	3.93e-003	9.30e-003	–	1.28e-002	3.81e-004	2.21e-003
f_4	2	1.60e-007	1.11e-016	4.44e-016	2.67e-010	–div–	2.74e-010	1.11e-016	4.77e-011
	3	1.11e-016	1.11e-016	1.11e-016	1.11e-016	-	8.91e-015	1.11e-016	0.00e-000
	1	2.54e-004	1.97e-005	7.71e-004	1.35e-004	5.12e-001	8.28e-003	3.81e-004	2.08e-004
f_5	2	0.00e-000	2.22e-016	0.00e-000	0.00e-000	1.23e-001	2.21e-010	0.00e-000	1.55e-011
	3	0.00e-000	2.22e-016	0.00e-000	0.00e-000	9.00e-002	9.89e-015	0.00e-000	0.00e-000
	1	1.40e-001	1.66e-001	2.71e-001	6.48e-001	–	7.05e-001	3.45e-002	3.42e-002
f_6	2	9.97e-008	1.11e-007	5.04e-005	1.79e-002	–div–	2.49e-002	3.65e-012	2.65e-008
	3	2.22e-016	2.22e-016	2.22e-016	5.84e-010	–	4.76e-009	4.44e-016	0.00e-000
	1	1.96e-003	4.20e-004	1.23e-005	1.88e-002	1.17e-001	1.14e-002	1.29e-004	6.79e-004
f_7	2	0.00e-000	2.22e-016	2.22e-016	3.65e-009	2.80e-002	3.19e-010	4.44e-016	1.33e-008
	3	0.00e-000	2.22e-016	2.22e-016	4.44e-016	1.54e-004	9.89e-016	4.44e-016	4.44e-016
	1	3.28e-009	1.19e-008	2.12e-006	1.21e-004	1.21e-003	2.51e-005	8.64e-009	7.99e-015
f_8	2	0.00e-000	0.00e-000	0.00e-000	0.00e-000	3.50e-011	0.00e-000	0.00e-000	0.00e-000
	3	0.00e-000	0.00e-000	0.00e-000	0.00e-000	0.00e-000	0.00e-000	0.00e-000	0.00e-000
	1	5.68e-004	5.12e-003	7.81e-001	–	5.06e-001	8.24e-002	8.98e-005	3.41e-004
f_9	2	0.00e-000	2.22e-016	5.58e-004	–div–	8.66e-002	8.04e-008	0.00e-000	9.21e-011
	3	0.00e-000	0.00e-000	0.00e-000	–	6.83e-005	0.00e-000	0.00e-000	0.00e-000
	1	2.07e-006	2.22e-005	2.13e-004	6.62e-003	9.87e-002	4.05e-003	3.46e-004	1.28e-004
f_{10}	2	1.11e-016	1.11e-016	1.11e-016	2.25e-012	1.19e-003	2.89e-013	0.00e-000	6.21e-012
	3	1.11e-016	1.11e-016	1.11e-016	2.22e-016	2.20e-009	8.91e-018	0.00e-000	0.00e-000

TABLE 5: Computational time in secs

$f(x)$	NTM	LKM	$SGM1$	$SGM2$	SHM	$ADM1$	$ADM2$	JJM
$f_1(x)$	1.00e-008	1.00e-008	1.56e-002	4.68e-002	–	3.12e-002	1.80e-002	1.00e-008
$f_2(x)$	6.30e-002	6.01e-003	1.57e-002	1.71e-002	1.23e-000	1.56e-002	1.64e-002	1.00e-008
$f_3(x)$	1.57e-002	1.56e-002	1.00e-008	6.25e-002	4.96e-003	3.12e-002	1.50e-002	1.00e-008
$f_4(x)$	1.00e-008	1.00e-008	1.05e-002	4.69e-002	–	6.43e-002	1.56e-002	1.00e-008
$f_5(x)$	1.55e-002	4.01e-003	1.00e-008	1.00e-008	7.05e-003	1.51e-001	1.55e-002	1.00e-008
$f_6(x)$	1.31e-000	1.98e-002	2.21e-002	4.78e-000	–	1.00e+001	3.06e-002	1.23e-003
$f_7(x)$	1.00e-008	2.58e-003	1.00e-008	4.69e-002	1.00e-008	1.66e-001	1.70e-002	1.00e-008
$f_8(x)$	1.00e-008	1.00e-004	1.00e-008	1.00e-008	1.00e-008	1.00e-008	1.00e-008	1.00e-008
$f_9(x)$	1.00e-008	1.00e-008	1.00e-008	–	1.72e-002	7.73e-003	1.00e-008	1.00e-008
$f_{10}(x)$	4.69e-002	3.04e-003	1.00e-008	1.57e-002	1.00e-008	1.41e-001	1.00e-008	1.00e-008

TABLE 6: Comparing $|f(x_m)|$ for all methods

$f(x)$	m	NTM	LKM	$SGM1$	$SGM2$	SHM	$ADM1$	$ADM2$	JJM
f_1	1	$4.14e-003$	$3.85e-003$	$5.42e-003$	$8.93e-003$	$-$	$8.75e-003$	$7.28e-004$	$2.35e-003$
	2	$3.47e-008$	$6.02e-008$	$6.33e-007$	$1.52e-006$	$-div-$	$8.57e-006$	$8.97e-013$	$2.46e-008$
	3	$8.67e-018$	$8.67e-018$	$8.67e-018$	$8.67e-018$	$-$	$8.67e-018$	$8.67e-018$	$8.67e-018$
f_2	1	$3.38e-004$	$6.32e-003$	$2.58e-002$	$8.23e-002$	$5.15e-001$	$8.44e-002$	$2.98e-003$	$8.73e-003$
	2	$1.00e-008$	$2.16e-017$	$3.08e-012$	$1.45e-008$	$5.42e-001$	$4.95e-008$	$1.30e-019$	$1.92e-006$
	3	$0.00e-000$	$0.00e-000$	$9.83e-072$	$2.33e-042$	$4.52e-002$	$3.94e-039$	$8.75e-118$	$1.00e-008$
f_3	1	$2.07e-001$	$4.77e-001$	$5.42e-000$	$9.30e-001$	$3.51e-001$	$9.31e-001$	$3.40e-001$	$2.33e-002$
	2	$4.72e-005$	$5.71e-004$	$4.37e-001$	$1.08e-002$	$4.67e-001$	$1.86e-002$	$2.50e-004$	$1.23e-007$
	3	$0.00e-000$	$0.00e-000$	$1.86e-003$	$1.90e-013$	$7.97e-002$	$7.25e-010$	$0.00e-000$	$0.00e-000$
f_4	1	$2.28e-001$	$4.62e-004$	$1.11e-002$	$2.64e-002$	$-$	$3.64e-002$	$1.07e-003$	$6.23e-003$
	2	$4.51e-007$	$1.11e-016$	$9.99e-016$	$7.51e-010$	$-div-$	$7.71e-010$	$1.11e-016$	$1.34e-010$
	3	$1.11e-016$	$1.11e-016$	$1.11e-016$	$1.11e-016$	$-$	$1.81e-013$	$1.11e-016$	$2.22e-016$
f_5	1	$5.16e-003$	$4.00e-004$	$1.56e-002$	$2.74e-003$	$2.95e+001$	$1.70e-001$	$7.73e-003$	$4.22e-003$
	2	$2.66e-015$	$3.55e-015$	$2.66e-015$	$2.66e-015$	$2.10e-000$	$4.49e-009$	$2.66e-015$	$3.15e-010$
	3	$2.66e-015$	$3.55e-015$	$8.67e-018$	$2.66e-015$	$2.11e-000$	$7.53e-016$	$3.68e-021$	$2.66e-015$
f_6	1	$2.47e-000$	$2.52e-000$	$5.09e-000$	$1.44e+001$	$-$	$1.60e+001$	$5.61e-001$	$5.74e-001$
	2	$1.65e-006$	$1.83e-006$	$8.32e-004$	$2.98e-001$	$-div-$	$4.16e-001$	$6.02e-011$	$4.38e-007$
	3	$5.48e-043$	$1.66e-043$	$1.09e-025$	$9.65e-009$	$-$	$7.86e-008$	$8.74e-019$	$4.93e-020$
f_7	1	$1.50e-003$	$3.20e-004$	$9.37e-006$	$1.41e-002$	$7.97e-002$	$8.81e-003$	$9.79e-005$	$5.17e-004$
	2	$2.22e-016$	$0.00e-000$	$0.00e-000$	$2.77e-009$	$2.18e-002$	$2.43e-010$	$0.00e-000$	$1.01e-008$
	3	$2.22e-016$	$0.00e-000$	$0.00e-000$	$0.00e-000$	$1.17e-004$	$7.53e-016$	$0.00e-000$	$0.00e-000$
f_8	1	$1.97e-009$	$6.74e-009$	$1.28e-006$	$7.27e-005$	$7.27e-004$	$1.51e-005$	$5.20e-009$	$4.88e-015$
	2	$0.00e-000$	$0.00e-000$	$0.00e-000$	$0.00e-000$	$2.11e-011$	$0.00e-000$	$0.00e-000$	$0.00e-000$
	3	$0.00e-000$	$0.00e-000$	$0.00e-000$	$0.00e-000$	$0.00e-000$	$0.00e-000$	$0.00e-000$	$0.00e-000$
f_9	1	$6.58e-004$	$5.94e-003$	$9.51e-001$	$-$	$5.25e-001$	$9.66e-002$	$1.041e-003$	$3.95e-003$
	2	$1.11e-016$	$1.66e-016$	$6.48e-004$	$-div-$	$1.02e-001$	$9.32e-008$	$1.11e-016$	$1.07e-010$
	3	$1.11e-016$	$1.11e-016$	$1.11e-016$	$-$	$7.92e-005$	$1.11e-016$	$6.41e-051$	$1.11e-016$
f_{10}	1	$2.19e-006$	$2.36e-005$	$2.26e-004$	$7.00e-003$	$1.02e-001$	$4.29e-003$	$3.67e-004$	$1.35e-004$
	2	$0.00e-000$	$0.00e-000$	$2.22e-016$	$2.39e-012$	$1.26e-003$	$3.06e-013$	$0.00e-000$	$6.57e-012$
	3	$0.00e-000$	$0.00e-000$	$0.00e-000$	$0.00e-000$	$2.33e-009$	$2.33e-017$	$0.00e-000$	$0.00e-000$

TABLE 7: Radii for Examples 6.1, 6.2 and 6.3

Radii	k	Example 6.1	Example 6.2	Example 6.3
δ_k	1	0.0410256	0.4	0.229929
	2	0.0263582	0.223235	0.129201
	3	0.0152299	0.130916	0.075723
δ	=	0.0152299	0.130916	0.075723

TABLE 8: Error estimates for Example 6.4

$\|x_1 - \zeta\|$	$\|x_2 - \zeta\|$	$\|x_3 - \zeta\|$
6.65e-010	2.38e-016	1.12e-052

TABLE 9: Error estimates for section 6.1

Function	x_0	$\|x_1 - \zeta\|$	$\|x_2 - \zeta\|$	$\|x_3 - \zeta\|$
$g_1(x)$	3.8	2.49e-002	1.97e-005	6.53e-14
$g_2(x)$	0.5	1.89e-004	1.86e-008	5.55e-017
$g_3(x)$	-0.2	3.73e-004	1.42e-009	0.00e-000
$g_4(x)$	3	1.21e-003	4.69e-013	1.77e-015