Investigating New Inclusive Subclasses of Bi-Univalent Functions Linked to Gregory Numbers

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Abstract: - In this article, we introduce three inclusive subclasses $\mathcal{Y}_\Gamma(\kappa,\eta,\sigma)$, $\mathcal{W}_\Gamma(\alpha,\varphi)$ and $\mathcal{K}_\Gamma(\alpha,\varphi)$ of the class of bi-univalent functions utilizing Gregory numbers. For each of these subclasses of analytic functions, we examine the Fekete-Szegö functional as well as the estimations of the Taylor-Maclaurin coefficients, $|s_2|$ and $|s_3|$. Such these subclasses may be the subject of future study due to the novelty of their characterizations and the proofs.

Key-Words: Analytic function, Univalent and bi-univalent functions, Gregory numbers, Fekete-Szegö problem

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1 Introduction and Preliminaries

Numerical analysis and approximation theory employ a numerical set of coefficients known as Gregory coefficients, which named after the mathematician James Gregory. These Gregory coefficients are derived from the properties of a specific function.

Gregory coefficients play an important role in numerical approximation methods by helping to create polynomial or spline functions which are precisely describe input data or functions. They are indispensable in various fields, including data analysis, computer graphics, scientific computing, and engineering.

Gregory coefficients, Υ_m, are the numbers $\frac{1}{2}, \frac{-1}{12}, \frac{1}{24}, \frac{-19}{720}, \cdots$. They appear in the following expansion.

$$\frac{\varepsilon}{\log(\varepsilon+1)} = 1 + \frac{1}{2}\varepsilon - \frac{1}{12}\varepsilon^2 + \frac{1}{24}\varepsilon^3 - \frac{19}{720}\varepsilon^4 + \frac{3}{160}\varepsilon^5 + \cdots.$$

These numbers were first proposed by James Gregory in 1670 and were later revitalized by other mathematicians, appearing in the works of modern writers.

The generating function $G(\varepsilon)$ of the Gregory coefficients [1, 2], given by

$$G(\varepsilon) = \frac{\varepsilon}{\log(\varepsilon+1)} = \sum_{m=0}^{\infty} \Upsilon_m \varepsilon^m = 1 + \frac{1}{2}\varepsilon - \frac{1}{12}\varepsilon^2 + \frac{1}{24}\varepsilon^3 - \frac{19}{720}\varepsilon^4 + \frac{3}{160}\varepsilon^5 + \cdots, \quad |\varepsilon| < 1. \quad (1)$$

Clearly, the initial values of Υ_m, for $m \in \mathbb{N}$, are

$$\Upsilon_0 = 1, \Upsilon_1 = \frac{1}{2}, \Upsilon_2 = \frac{-1}{12}, \Upsilon_3 = \frac{1}{24},$$
$$\Upsilon_4 = \frac{-19}{720}, \ and \ \Upsilon_5 = \frac{3}{160}.$$

Let Ω denote the subclass of analytic functions h in the open unit disk $\Delta = \{\varepsilon \in \mathbb{C} : |\varepsilon| < 1\}$ normalized by $h(0) = h'(0) - 1 = 0$. The result of this, every function $h \in \Omega$ has the form:

$$h(\varepsilon) = \varepsilon + \sum_{m=2}^{\infty} s_m \varepsilon^m, \quad (\varepsilon \in \Delta). \tag{2}$$

Further, let $\mathcal{U}$ represent the subclass of all univalent functions in the subclass Ω [3, 4]. So, every function $h \in \mathcal{U}$ has an inverse h^{-1}, defined by

$$h^{-1}(h(\varepsilon)) = \varepsilon \qquad (\varepsilon \in \Delta)$$

and

$$h(h^{-1}(\varpi)) = \varpi \qquad (|\varpi| < r_0(h); r_0(h) \geq \frac{1}{4})$$

where

$$h^{-1}(\varpi) = \varpi - s_2\varpi^2 + (2s_2^2 - s_3)\varpi^3 - + \cdots. \tag{3}$$

Now, $h_1 \prec h_2$ or $h_1(\varepsilon) \prec h_2(\varepsilon)$ (denote the subordination of analytic functions h_1 and h_2) if for all $\varepsilon \in \Delta$ there exists a function Φ with $\Phi(0) = 0$ and $|\Phi(\varepsilon)| < 1$; such that:

$$h_1(\varepsilon) = h_2(\Phi(\varepsilon)).$$

Also, if h_2 is univalent in Δ, then we have the following equivalence relation [5, 6].

$$\begin{aligned} h_1(0) &= h_2(0) \ and \\ h_1(\Delta) \subset h_2(\Delta) &\Leftrightarrow h_1(\varepsilon) \prec h_2(\varepsilon). \end{aligned}$$

A function h given by Eqn (2) will be in the subclass Γ (Γ is the subclass of all bi-univalent functions lie in Δ) if both $h(\varepsilon)$ and $h^{-1}(\varepsilon)$ are univalent in Δ. For details of the subclass Γ, we refere to [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17].

Inspected the subclass Γ, The author, [18], found that $|s_2| < 1.51$. The author, [19], showed that $\max s_2 = \frac{4}{3}$, and, [20], conjectured that $|s_2| < \sqrt{2}$. The problem of approximating the coefficient s_m of $m \geq 3, m \in \mathbb{N}$, is still an open problem.

The Fekete-Szegö inequality is one of the well-known issues. The first to do so was, [21], if $h \in \Gamma$, who stated that

$$|s_3 - \varrho s_2^2| \leq 1 + 2e^{-2\varrho/(1-\varrho)}, \ \varrho \in \mathbb{R},$$

this bound is sharp.

Several scholars such as [22], [23], [24], [25], [26], have recently begun studying bi-univalent functions related with orthogonal polynomials.

In the current work, we define new subclasses of Γ involving the Gregory coefficients that are denoted by $\mathcal{Y}_\Gamma(\kappa, \eta, \sigma)$, $\mathcal{W}_\Gamma(\alpha, \varphi)$ and $\mathcal{K}_\Gamma(\alpha, \varphi)$, and we will estimate the upper bounds for the coefficients $|s_2|$, $|s_3|$ and $|s_3 - \varrho s_2^2|$ for the above subclasses. Additionally, some of novel results have been proven.

2 Coefficient Bounds of the Subclasses $\mathcal{Y}_\Gamma(\kappa, \eta, \sigma), \mathcal{W}_\Gamma(\alpha, \varphi)$ and $\mathcal{K}_\Gamma(\alpha, \varphi)$

This section begins with definitions for the new comprehensive subclasses $\mathcal{Y}_\Gamma(\kappa, \eta, \sigma)$, $\mathcal{W}_\Gamma(\alpha, \varphi)$ and $\mathcal{K}_\Gamma(\alpha, \varphi)$ related to Gregory coefficients.

Definition 2.1 *Let $\kappa \geq 1$, $\eta, \sigma \geq 0$, $\varepsilon, \varpi \in \Delta$ and the function $G(\varepsilon)$ is given by (1). A function $h \in \Gamma$ given by (2) is said to be in the subclass $\mathcal{Y}_\Gamma(\kappa, \eta, \sigma)$ if the next subordinations are satisfied:*

$$(1-\kappa)\left(\frac{h(\varepsilon)}{\varepsilon}\right)^{\eta} + \kappa\left(h'(\varepsilon)\right)^{1-\eta} + \sigma\varepsilon h''(\varepsilon) \prec G(\varepsilon) \tag{4}$$

and

$$(1-\kappa)\left(\frac{d(\varpi)}{\varpi}\right)^{\eta} + \kappa\left(d'(\varpi)\right)^{1-\eta} + \sigma\varpi d''(\varpi) \prec G(\varpi), \tag{5}$$

where $d(\varpi) = h^{-1}(\varpi)$ is given by (3).

Definition 2.2 *Let $-\pi < \varphi \leq \pi$, $\alpha \geq 1$, $\varepsilon, \varpi \in \Delta$ and the function $G(\varepsilon)$ is given by Eqn (1). A function $h \in \Gamma$ given by (2) is said to be in the subclass $\mathcal{W}_\Gamma(\alpha, \varphi)$ if the following subordinations are satisfied:*

$$\left(\frac{h(\varepsilon)}{\varepsilon}\right)^{\alpha} + \frac{1+e^{i\varphi}}{2}\left(\frac{\varepsilon h''(\varepsilon)}{h'(\varepsilon)}\right) \prec G(\varepsilon) \tag{6}$$

and

$$\left(\frac{d(\varpi)}{\varpi}\right)^{\alpha} + \frac{1+e^{i\varphi}}{2}\left(\frac{\varpi d''(\varpi)}{d'(\varpi)}\right) \prec G(\varpi), \tag{7}$$

where $d(\varpi) = h^{-1}(\varpi)$ is given by (3).

Definition 2.3 *Let $-\pi < \varphi \leq \pi$, $\alpha \geq 1$, $\varepsilon, \varpi \in \Delta$ and the function $G(\varepsilon)$ is given by Eqn (1). A function $h \in \Gamma$ given by (2) is said to be in the subclass $\mathcal{K}_\Gamma(\alpha, \varphi)$ if the following subordinations are satisfied:*

$$\left(h'(\varepsilon)\right)^{\alpha} + \frac{1+e^{i\varphi}}{2}\left(\varepsilon h''(\varepsilon)\right) \prec G(\varepsilon) \tag{8}$$

and

$$\left(d'(\varpi)\right)^{\alpha}+\frac{1+e^{i\varphi}}{2}\left(\varpi d''(\varpi)\right)\prec G(\varpi), \qquad (9)$$

where $d(\varpi)=h^{-1}(\varpi)$ is given by (3).

Remark 2.4 *For specific values of parameters κ, η,σ in Definition 2.1, and φ, α in Definition 2.2 and Definition 2.3, we obtain several subclasses of Γ studied by several authors.*

In this study, the following two lemmas are employed.

Lemma 2.5 ([27]) *Let the analytic function $\psi(\varepsilon)=1+s_1\varepsilon+s_2\varepsilon^2+\cdots$ with positive real parts in Δ, then $|s_m|\le 2$, for $m\ge 1$.*

Lemma 2.6 ([28]) *Let $g_1,g_2\in\mathbb{R}$ and $\chi_1,\chi_2\in\mathbb{C}$. If $|\chi_1|<\hbar$ and $|\chi_2|<\hbar$, then*

$$|(g_1+g_2)\chi_1+(g_1-g_2)\chi_2|\le \begin{cases} 2\,|g_1|\,\hbar\ for\ |g_1|\ge|g_2|, \\ 2\,|g_2|\,\hbar\ for\ |g_1|\le|g_2|. \end{cases}$$

Theorem 2.7 *Let $h\in\Gamma$ given by Eqn (2) belongs to the subclass $\mathcal{Y}_{\Gamma}(\kappa,\eta,\sigma)$. Then*

$$|s_2|\le\min\left\{\frac{1}{2|\eta(1-3\kappa)+2(\sigma+\kappa)|},\ \frac{1}{\sqrt{\left|\eta^2(3\kappa+1)+\eta(1-11\kappa)+6(2\sigma+\kappa)+\frac{14}{3}[\eta(1-3\kappa)+2(\sigma+\kappa)]^2\right|}}\right\},$$

$$|s_3|\le\min\left\{\frac{1}{4|\eta(1-3\kappa)+2(\sigma+\kappa)|^2}+\frac{1}{|2\eta(1-4\kappa)+6(2\sigma+\kappa)|},\ \frac{1}{\left|\eta^2(3\kappa+1)+\eta(1-11\kappa)+6(2\sigma+\kappa)+\frac{14}{3}[\eta(1-3\kappa)+2(\sigma+\kappa)]^2\right|}+\frac{1}{|2\eta(1-4\kappa)+6(2\sigma+\kappa)|}\right\}$$

and

$$\left|s_3-\Xi s_2^2\right|\le \begin{cases} \frac{1}{2|\eta(1-4\kappa)+3(2\sigma+\kappa)|} & |\Theta(\Xi)|<\frac{1}{8|\eta(1-4\kappa)+3(2\sigma+\kappa)|}, \\ 4\,|\Theta(\Xi)| & |\Theta(\Xi)|\ge\frac{1}{8|\eta(1-4\kappa)+3(2\sigma+\kappa)|}. \end{cases}$$

where

$$\Theta(\Xi)=\frac{1-\Xi}{4\left[\eta^2(3\kappa+1)+\eta(1-11\kappa)+6(2\sigma+\kappa)+\frac{14}{3}[\eta(1-3\kappa)+2(\sigma+\kappa)]^2\right]}.$$

Proof. Let $h\in\mathcal{Y}_{\Gamma}(\kappa,\eta,\sigma)$. So, by using the subordinations in Eqns (4) and (5), there exist two analytic functions r,t such that $r(0)=t(0)=0$ and $|r(\varepsilon)|<1,|t(\varpi)|<1$, such that

$$(1-\kappa)\left(\frac{h(\varepsilon)}{\varepsilon}\right)^{\eta}+\kappa\left(h'(\varepsilon)\right)^{1-\eta}+\sigma\varepsilon h''(\varepsilon)=G(r(\varepsilon)), \qquad (10)$$

and

$$(1-\kappa)\left(\frac{d(\varpi)}{\varpi}\right)^{\eta}+\kappa\left(d'(\varpi)\right)^{1-\eta}+\sigma\varpi d''(\varpi)=G(t(\varpi)). \qquad (11)$$

So, the function

$$\beta(\varepsilon)=\frac{r(\varepsilon)+1}{1-r(\varepsilon)}=1+a_1\varepsilon+a_2\varepsilon^2+\cdots,$$

hence,

$r(\varepsilon)=\frac{a_1}{2}\varepsilon+\frac{1}{2}\left(a_2-\frac{a_1^2}{2}\right)\varepsilon^2+\frac{1}{2}\left(a_3-a_1a_2+\frac{a_1^3}{4}\right)\varepsilon^3+\cdots$

and

$G(r(\varepsilon))=1+\frac{a_1}{4}\varepsilon+\frac{1}{48}\left(12a_2-7a_1^2\right)\varepsilon^2+\frac{1}{192}\left(17a_1^3-56a_1a_2+48a_3\right)\varepsilon^3+\cdots.$

Also, the function

$\delta(\varpi)=\frac{t(\varpi)+1}{1-t(\varpi)}=1+b_1\varpi+b_2\varpi^2+\cdots,$

hence,

$t(\varpi)=\frac{b_1}{2}\varpi+\frac{1}{2}\left(b_2-\frac{b_1^2}{2}\right)\varpi^2+\frac{1}{2}\left(b_3-b_1b_2+\frac{b_1^3}{4}\right)\varpi^3+\cdots$ and $G(t(\varpi))=1+\frac{b_1}{4}\varpi+\frac{1}{48}\left(12b_2-7b_1^2\right)\varpi^2+\frac{1}{192}\left(17b_1^3-56b_1b_2+48b_3\right)\varpi^3+\cdots.$

Thus we have $(1-\kappa)\left(\frac{h(\varepsilon)}{\varepsilon}\right)^{\eta}+\kappa\left(h'(\varepsilon)\right)^{1-\eta}+\sigma\varepsilon h''(\varepsilon)$
$=1+\frac{a_1}{4}\varepsilon+\frac{1}{48}\left(12a_2-7a_1^2\right)\varepsilon^2+\frac{1}{192}\left(17a_1^3-56a_1a_2+48a_3\right)\varepsilon^3+\cdots.$ and $(1-\kappa)\left(\frac{d(\varpi)}{\varpi}\right)^{\eta}+\kappa\left(d'(\varpi)\right)^{1-\eta}+\sigma\varpi d''(\varpi)$
$=1+\frac{b_1}{4}\varpi+\frac{1}{48}\left(12b_2-7b_1^2\right)\varpi^2+\frac{1}{192}\left(17b_1^3-56b_1b_2+48b_3\right)\varpi^3+\cdots.$

When we contrast the coefficients in (2) and (2), we get

$$\left[\eta(1-3\kappa)+2\left(\sigma+\kappa\right)\right]s_2=\frac{a_1}{4}, \qquad (12)$$

$$\left[\frac{\eta\left(\eta-1\right)\left(3\kappa+1\right)}{2}\right]s_2^2+\left[\eta(1-4\kappa)+3\left(2\sigma+\kappa\right)\right]s_3=\frac{1}{48}\left(12a_2-7a_1^2\right), \qquad (13)$$

$$-\left[\eta(1-3\kappa)+2(\sigma+\kappa)\right]s_2=\frac{b_1}{4}, \quad (14)$$

and

$$\left[\frac{\eta^2(3\kappa+1)+\eta(3-19\kappa)+12(2\sigma+\kappa)}{2}\right]s_2^2 - \left[\eta(1-4\kappa)+3(2\sigma+\kappa)\right]s_3 = \frac{1}{48}\left(12b_2-7b_1^2\right). \quad (15)$$

From (12) and (14) it follows that

$$a_1=-b_1 \quad (16)$$

and

$$32\left[\eta(1-3\kappa)+2(\sigma+\kappa)\right]^2 s_2^2 = a_1^2+b_1^2. \quad (17)$$

If we add (13) to (15), we have

$$\left[\eta^2(3\kappa+1)+\eta(1-11\kappa)+6(2\sigma+\kappa)\right]s_2^2 = \frac{1}{4}(a_2+b_2)-\frac{7}{48}\left(a_1^2+b_1^2\right). \quad (18)$$

Substituting the value of $a_1^2+b_1^2$ from (17) in (18), we have

$$\left[\eta^2(3\kappa+1)+\eta(1-11\kappa)+6(2\sigma+\kappa)+\frac{14}{3}\left[\eta(1-3\kappa)+2(\sigma+\kappa)\right]^2\right]s_2^2=\frac{1}{4}(a_2+b_2). \quad (19)$$

Using the triangle inequality and Lemma 2.5 for the relations (12) and (19), we respectively get:
$|s_2| \le \frac{1}{2|\eta(1-3\kappa)+2(\sigma+\kappa)|}$ and $|s_2| \le \frac{1}{\sqrt{\left|\eta^2(3\kappa+1)+\eta(1-11\kappa)+6(2\sigma+\kappa)+\frac{14}{3}[\eta(1-3\kappa)+2(\sigma+\kappa)]^2\right|}}$.

Moreover, if we subtract (15) from (13), we have

$$\left[2\eta(1-4\kappa)+6(2\sigma+\kappa)\right]\left(s_3-s_2^2\right) = \frac{1}{4}(a_2-b_2)-\frac{7}{48}\left(a_1^2-b_1^2\right). \quad (20)$$

Then, in view of (16), last equation becomes

$$s_3=s_2^2+\frac{a_2-b_2}{8\left[\eta(1-4\kappa)+3(2\sigma+\kappa)\right]}. \quad (21)$$

The above equation with (12) becomes
$s_3 = \frac{a_1^2}{16[\eta(1-3\kappa)+2(\sigma+\kappa)]^2} + \frac{a_2-b_2}{8[\eta(1-4\kappa)+3(2\sigma+\kappa)]}$

And using (19) in (21)
$s_3 = \frac{a_2+b_2}{4\left[\eta^2(3\kappa+1)+\eta(1-11\kappa)+6(2\sigma+\kappa)+\frac{14}{3}[\eta(1-3\kappa)+2(\sigma+\kappa)]^2\right]} + \frac{a_2-b_2}{4[2\eta(1-4\kappa)+6(2\sigma+\kappa)]}$.

Using the triangle inequality and Lemma 2.5 for the last two equation, we get respectively:
$|s_3| \le \frac{1}{4|\eta(1-3\kappa)+2(\sigma+\kappa)|^2} + \frac{1}{2|\eta(1-4\kappa)+3(2\sigma+\kappa)|}$
and
$|s_3| \le \frac{1}{\left|\eta^2(3\kappa+1)+\eta(1-11\kappa)+6(2\sigma+\kappa)+\frac{14}{3}[\eta(1-3\kappa)+2(\sigma+\kappa)]^2\right|} + \frac{1}{2|\eta(1-4\kappa)+3(2\sigma+\kappa)|}$.

Also, from (21) we have
$s_3-\Xi s_2^2 = \frac{a_2-b_2}{8[\eta(1-4\kappa)+3(2\sigma+\kappa)]}+(1-\Xi)s_2^2$
$= \frac{a_2-b_2}{8[\eta(1-4\kappa)+3(2\sigma+\kappa)]} + \frac{(1-\Xi)(a_2+b_2)}{4\left[\eta^2(3\kappa+1)+\eta(1-11\kappa)+6(2\sigma+\kappa)+\frac{14}{3}[\eta(1-3\kappa)+2(\sigma+\kappa)]^2\right]}$
$= \left(\Theta(\Xi)+\frac{1}{8[\eta(1-4\kappa)+3(2\sigma+\kappa)]}\right)a_2 + \left(\Theta(\Xi)-\frac{1}{8[\eta(1-4\kappa)+3(2\sigma+\kappa)]}\right)b_2$,
where

$\Theta(\Xi) = \frac{1-\Xi}{4\left[\eta^2(3\kappa+1)+\eta(1-11\kappa)+6(2\sigma+\kappa)+\frac{14}{3}[\eta(1-3\kappa)+2(\sigma+\kappa)]^2\right]}$.

Then, in view Lemma 2.5 for $|a_2|$ and $|b_2|$, and Lemma 2.6, we obtain

$$\left|s_3-\Xi s_2^2\right| \le \begin{cases} \frac{1}{2|\eta(1-4\kappa)+3(2\sigma+\kappa)|} & |\Theta(\Xi)| < \frac{1}{8|\eta(1-4\kappa)+3(2\sigma+\kappa)|}, \\ 4\left|\Theta(\Xi)\right| & |\Theta(\Xi)| \ge \frac{1}{8|\eta(1-4\kappa)+3(2\sigma+\kappa)|}. \end{cases}$$

Which completes the proof.

We utilize the subsequent lemma to establish the Fekete-Szeg"o functional in the following Theorems.

Lemma 2.8 ([29]) *If $h(\varepsilon)=1+s_1\varepsilon+s_2\varepsilon^2+\cdots\in\Gamma$, $\varepsilon\in\Delta$, then there exist some λ,μ with $|\lambda|\le 1$, $|\mu|\le 1$, such that $2s_2=s_1^2+\lambda(4-s_1^2)$ and*

$$4s_3 = s_1^3+2s_1\lambda(4-s_1^2) -(4-s_1^2)s_1\lambda^2+2(4-s_1^2)(1-|\lambda|^2)\mu. \quad (22)$$

Theorem 2.9 *Let $h\in\Gamma$ given by (2) belongs to the subclass $\mathcal{W}_\Gamma(\alpha,\varphi)$. Then*
$|s_2| \le \min\left\{\frac{1}{2|e^{i\varphi}+\alpha+1|}, \frac{1}{\sqrt{\left|2(e^{i\varphi}+1)+\alpha(\alpha+1)+\frac{14}{3}(e^{i\varphi}+\alpha+1)^2\right|}}\right\}$,

$$|s_3| \leq \min\left\{\frac{1}{4|e^{i\varphi}+\alpha+1|^2}+\frac{1}{2|3(e^{i\varphi}+1)+\alpha|}, \frac{1}{\left|2(e^{i\varphi}+1)+\alpha(\alpha+1)+\frac{14}{3}(e^{i\varphi}+\alpha+1)^2\right|}+\frac{1}{2|3(e^{i\varphi}+1)+\alpha|}\right\}$$

and

$$\left|s_3-\varrho s_2^2\right| \leq \begin{cases} \frac{1}{2|3(e^{i\varphi}+1)+\alpha|} & |1-\varrho| < \frac{2|e^{i\varphi}+\alpha+1|^2}{|3(e^{i\varphi}+1)+\alpha|}, \\ \frac{|1-\varrho|}{4|e^{i\varphi}+\alpha+1|^2} & |1-\varrho| \geq \frac{2|e^{i\varphi}+\alpha+1|^2}{|3(e^{i\varphi}+1)+\alpha|}. \end{cases} \quad (23)$$

Proof. Let $h \in \mathcal{W}_\Gamma(\alpha,\varphi)$. So, by using the subordinations (6) and (7), we can write

$$\left(\frac{h(\varepsilon)}{\varepsilon}\right)^\alpha + \frac{1+e^{i\varphi}}{2}\left(\frac{\varepsilon h''(\varepsilon)}{h'(\varepsilon)}\right) = G(r(\varepsilon)) \quad (24)$$

and $\left(\frac{d(\varpi)}{\varpi}\right)^\alpha + \frac{1+e^{i\varphi}}{2}\left(\frac{\varpi d''(\varpi)}{d'(\varpi)}\right) = G(t(\varpi))$.

Thus we have

$$\left(\frac{h(\varepsilon)}{\varepsilon}\right)^\alpha + \frac{1+e^{i\varphi}}{2}\left(\frac{\varepsilon h''(\varepsilon)}{h'(\varepsilon)}\right) = 1 + \frac{a_1}{4}\varepsilon + \frac{1}{48}\left(12a_2-7a_1^2\right)\varepsilon^2 + \frac{1}{192}\left(17a_1^3-56a_1a_2+48a_3\right)\varepsilon^3 + \cdots$$

and

$$\left(\frac{d(\varpi)}{\varpi}\right)^\alpha + \frac{1+e^{i\varphi}}{2}\left(\frac{\varpi d''(\varpi)}{d'(\varpi)}\right) = 1 + \frac{b_1}{4}\varpi + \frac{1}{48}\left(12b_2-7b_1^2\right)\varpi^2 + \frac{1}{192}\left(17b_1^3-56b_1b_2+48b_3\right)\varpi^3 + \cdots.$$

Comparing the coefficients in equations (2) and (2), we have

$$\left[e^{i\varphi}+\alpha+1\right]s_2 = \frac{a_1}{4}, \quad (25)$$

$$\left[3\left(e^{i\varphi}+1\right)+\alpha\right]s_3 + \left[\frac{\alpha(\alpha-1)}{2}-2(e^{i\varphi}+1)\right]s_2^2 = \frac{1}{48}\left(12a_2-7a_1^2\right), \quad (26)$$

$$-\left[e^{i\varphi}+\alpha+1\right]s_2 = \frac{b_1}{4}, \quad (27)$$

and

$$\left[4(e^{i\varphi}+1)+\frac{\alpha(\alpha+3)}{2}\right]s_2^2 - \left[3\left(e^{i\varphi}+1\right)+\alpha\right]s_3 = \frac{1}{48}\left(12b_2-7b_1^2\right). \quad (28)$$

From (25) and (27) it follows that

$$a_1 = -b_1 \quad (29)$$

and

$$32\left(e^{i\varphi}+\alpha+1\right)^2 s_2^2 = a_1^2+b_1^2. \quad (30)$$

Substituting the value of $a_1^2+b_1^2$ from (30) after we add (26) to (28), we get

$$4\left[2(e^{i\varphi}+1)+\alpha(\alpha+1)+\frac{14}{3}\left(e^{i\varphi}+\alpha+1\right)^2\right] s_2^2 = a_2+b_2. \quad (31)$$

Using the triangle inequality and Lemma 2.5 for the relations (30) and (31), we respectively get:
$|s_2| \leq \frac{1}{2|e^{i\varphi}+\alpha+1|}$ and
$|s_2| \leq \frac{1}{\sqrt{\left|2(e^{i\varphi}+1)+\alpha(\alpha+1)+\frac{14}{3}(e^{i\varphi}+\alpha+1)^2\right|}}$.

Moreover, if we subtract (28) from (26), we have

$$2\left[3(e^{i\varphi}+1)+\alpha\right]\left(s_3-s_2^2\right) = \frac{1}{4}\left(a_2-b_2\right) - \frac{7}{48}\left(a_1^2-b_1^2\right).$$

Then, in view of (29), last equation becomes

$$s_3 = s_2^2 + \frac{a_2-b_2}{8\left[3(e^{i\varphi}+1)+\alpha\right]}. \quad (32)$$

The above equation with (30) becomes $s_3 = \frac{a_1^2}{16(e^{i\varphi}+\alpha+1)^2} + \frac{a_2-b_2}{8[3(e^{i\varphi}+1)+\alpha]}$.

Using the triangle inequality and Lemma 2.5 for the last relation, we get

$$|s_3| \leq \frac{1}{4|e^{i\varphi}+\alpha+1|^2} + \frac{1}{2|3(e^{i\varphi}+1)+\alpha|}.$$

Similarly, using of (31) in relation (32) we get

$$|s_3| \leq \frac{1}{\left|2(e^{i\varphi}+1)+\alpha(\alpha+1)+\frac{14}{3}(e^{i\varphi}+\alpha+1)^2\right|} + \frac{1}{2|3(e^{i\varphi}+1)+\alpha|}.$$

Also, using (29) and (30), we get $s_2^2 = \frac{a_1^2}{16(e^{i\varphi}+\alpha+1)^2}$. Thus, from (32), we have

$$s_3 - \varrho s_2^2 = \frac{a_2-b_2}{8(3(e^{i\varphi}+1)+\alpha)} + (1-\varrho)s_2^2 = \frac{a_2-b_2}{8(3(e^{i\varphi}+1)+\alpha)} + (1-\varrho)\frac{a_1^2}{16(e^{i\varphi}+\alpha+1)^2}.$$

From (22) in Lemma 2.8, we have $2a_2 = a_1^2 + \lambda(4 - a_1^2)$ and $2b_2 = b_1^2 + \mu(4 - b_1^2)$, $|\lambda| \le 1$, $|\mu| \le 1$, and using (29), we obtain

$a_2 - b_2 = \frac{4-a_1^2}{2}(\lambda - \mu)$, and thus $s_3 - \varrho s_2^2 = \frac{(4-a_1^2)(\lambda+\mu)}{16(3(e^{i\varphi}+1)+\alpha)} + \frac{(1-\varrho)a_1^2}{16(e^{i\varphi}+\alpha+1)^2}$.

Using the triangle inequality, taking $|\lambda| = m$, $|\mu| = v$, $m, v \in [0,1]$, and assuming that $a_1 = p \in [0,2]$; thus, we get

$$\left|s_3 - \varrho s_2^2\right| \le \frac{(4-p^2)(m+v)}{16\left|3(e^{i\varphi}+1)+\alpha\right|} + \frac{|1-\varrho|\, p^2}{16\left|e^{i\varphi}+\alpha+1\right|^2}. \quad (33)$$

Assume that: $A(p) = \frac{|1-\varrho|p^2}{16|e^{i\varphi}+\alpha+1|^2} \ge 0$ and $C(p) = \frac{(4-p^2)}{16|3(e^{i\varphi}+1)+\alpha|} \ge 0$, the relation (33) can be rewritten as $\left|s_3 - \varrho s_2^2\right| \le A(p) + C(p)(m + v) := \mathcal{W}(m,v), m, v \in [0,1]$.

Therefore,

$\max\{\mathcal{W}(m,v) : m, v \in [0,1]\} = \mathcal{W}(1,1) = A(p) + 2C(p) =: T(p), p \in [0,2]$ where

$$\mathrm{T(p)}=1\frac{}{16|e^{i\varphi}+\alpha+1|^2\left(|1-\varrho|-\frac{2|e^{i\varphi}+\alpha+1|^2}{|3(e^{i\varphi}+1)+\alpha|}\right)p^2+\frac{1}{2|3(e^{i\varphi}+1)+\alpha|}}.$$

Since

$\mathrm{T}'(p) = \frac{1}{8|e^{i\varphi}+\alpha+1|^2}\left(|1-\varrho| - \frac{2|e^{i\varphi}+\alpha+1|^2}{|3(e^{i\varphi}+1)+\alpha|}\right)p$, it is clear that $T'(p) \le 0$ iff $|1-\varrho| \le \frac{2|e^{i\varphi}+\alpha+1|^2}{|3(e^{i\varphi}+1)+\alpha|}$. Hence, the function T is a decreasing on $[0,2]$; therefore, $\max\{T(p) : p \in [0,2]\} = T(0) = \frac{1}{2|3(e^{i\varphi}+1)+\alpha|}$.

Also, $T'(p) \ge 0$ iff $|1-\varrho| \ge \frac{2|e^{i\varphi}+\alpha+1|^2}{|3(e^{i\varphi}+1)+\alpha|}$. So, T is an increasing function over $[0,2]$, so

$\max\{T(p) : p \in [0,2]\} = T(2) = \frac{|1-\varrho|}{4|e^{i\varphi}+\alpha+1|^2}$ and the estimation (23) has been validated.

Theorem 2.10 *Let $h \in \Gamma$ given by (2) belongs to the subclass $\mathcal{K}_\Gamma(\alpha,\varphi)$. Then*

$$|s_2| \le \min\left\{\frac{1}{2|e^{i\varphi}+2\alpha+1|}, \frac{1}{\sqrt{\left|6(e^{i\varphi}+1)+2\alpha(2\alpha+1)+\frac{14}{3}(e^{i\varphi}+2\alpha+1)^2\right|}}\right\},$$

$$|s_3| \le \min\left\{\frac{1}{4|e^{i\varphi}+2\alpha+1|^2} + \frac{1}{6|e^{i\varphi}+\alpha+1|}, \frac{1}{\left|6(e^{i\varphi}+1)+2\alpha(2\alpha+1)+\frac{14}{6}(e^{i\varphi}+2\alpha+1)^2\right|} + \frac{1}{6|e^{i\varphi}+\alpha+1|}\right\}$$

and

$$\left|s_3 - \psi s_2^2\right| \le \begin{cases} \frac{1}{6|e^{i\varphi}+\alpha+1|} & |1-\psi| < \frac{4|e^{i\varphi}+2\alpha+1|^2}{3|e^{i\varphi}+\alpha+1|}, \\ \frac{|1-\varrho|}{8|e^{i\varphi}+2\alpha+1|^2} & |1-\psi| \ge \frac{4|e^{i\varphi}+\alpha+1|^2}{3|e^{i\varphi}+\alpha+1|}. \end{cases}$$

Proof. Let $h \in \mathcal{K}_\Gamma(\alpha,\varphi)$. So from subordinations (8) and (9), we can write

$(h'(\varepsilon))^\alpha + \frac{1+e^{i\varphi}}{2}(\varepsilon h''(\varepsilon)) = G(r(\varepsilon))$ and $(d'(\varpi))^\alpha + \frac{1+e^{i\varphi}}{2}(\varpi d''(\varpi)) = G(t(\varpi))$.

Thus we have
$(h'(z))^\alpha + \frac{1+e^{i\varphi}}{2}(\varepsilon h''(z))$
$= 1 + \frac{a_1}{4}\varepsilon + \frac{1}{48}\left(12a_2 - 7a_1^2\right)\varepsilon^2 + \frac{1}{192}\left(17a_1^3 - 56a_1a_2 + 48a_3\right)\varepsilon^3 + \cdots$.
and
$(d'(\varpi))^\alpha + \frac{1+e^{i\varphi}}{2}(\varpi d''(\varpi))$
$= 1 + \frac{b_1}{4}\varpi + \frac{1}{48}\left(12b_2 - 7b_1^2\right)\varpi^2 + \frac{1}{192}\left(17b_1^3 - 56b_1b_2 + 48b_3\right)\varpi^3 + \cdots$.

Comparing the coefficients in equations (2) and (2), we have

$$\left[e^{i\varphi} + 2\alpha + 1\right] s_2 = \frac{a_1}{4}, \quad (34)$$

$$3\left[e^{i\varphi} + \alpha + 1\right] s_3 + \left[2\alpha(\alpha-1)\right] s_2^2 = \frac{1}{48}\left(12a_2 - 7a_1^2\right), \quad (35)$$

$$-\left[e^{i\varphi} + 2\alpha + 1\right] s_2 = \frac{b_1}{4}, \quad (36)$$

and

$$2\left[3(e^{i\varphi}+1) + \alpha(\alpha+2)\right] s_2^2 - 3\left[e^{i\varphi} + \alpha + 1\right] s_3 = \frac{1}{48}\left(12b_2 - 7b_1^2\right). \quad (37)$$

Using the equations (34), (35), (36) and (37), we will obtain the conclusions that Theorem 2.10 asserts by using the same technique for proving Theorem 2.9.

3 Corollaries

If we set $\kappa = 1$ in Theorems 2.7, we get the next corollary.

Corollary 3.1 *If $h \in \mathcal{Y}_\Gamma(1,\eta,\sigma)$, then*

$$|s_2| \le \min\left\{\frac{1}{4|\sigma-\eta+1|}, \frac{1}{\sqrt{\left|4\eta^2-10\eta+6(2\sigma+1)+\frac{56}{3}(\sigma-\eta+1)^2\right|}}\right\},$$

$$|s_3| \le \min\left\{\frac{1}{16|\sigma-\eta+1|^2}+\frac{1}{6|2\sigma-\eta+1|}, \frac{1}{\left|4\eta^2-10\eta+6(2\sigma+1)+\frac{56}{3}(\sigma-\eta+1)^2\right|}+\frac{1}{6|2\sigma-\eta+1|}\right\}$$

and

$$\left|s_3-\Xi s_2^2\right| \le \begin{cases} \frac{1}{6|2\sigma-\eta+1|} & |\Theta(\Xi)| < \frac{1}{24|2\sigma-\eta+1|}, \\ 4\,|\Theta(\Xi)| & |\Theta(\Xi)| \ge \frac{1}{24|2\sigma-\eta+1|}. \end{cases}$$

where

$$\Theta(\Xi) = \frac{(1-\Xi)}{4\left[4\eta^2-10\eta+6(2\sigma+1)+\frac{56}{3}(\sigma-\eta+1)^2\right]}.$$

If we set $\eta = 0$ in Theorems 2.7, we get the next corollary.

Corollary 3.2 *If $h \in \mathcal{Y}_\Gamma(\kappa,0,\sigma)$, then*

$$|s_2| \le \min\left\{\frac{1}{4|\sigma+\kappa|}, \frac{1}{\sqrt{\left|6(2\sigma+\kappa)+\frac{56}{3}(\sigma+\kappa)^2\right|}}\right\},$$

$$|s_3| \le \min\left\{\frac{1}{16|\sigma+\kappa|^2}+\frac{1}{6|2\sigma+\kappa|}, \frac{1}{\left|6(2\sigma+\kappa)+\frac{56}{3}(\sigma+\kappa)^2\right|}+\frac{1}{6|2\sigma+\kappa|}\right\}$$

and

$$\left|s_3-\Xi s_2^2\right| \le \begin{cases} \frac{1}{6|2\sigma+\kappa|} & |\Theta(\Xi)| < \frac{1}{24|2\sigma+\kappa|}, \\ 4\,|\Theta(\Xi)| & |\Theta(\Xi)| \ge \frac{1}{24|2\sigma+\kappa|}. \end{cases}$$

where $\Theta(\Xi) = \frac{3(1-\Xi)}{4\left(6(2\sigma+\kappa)+\frac{56}{3}(\sigma+\kappa)^2\right)}$.

For $\eta = 0$ in the Corollary 3.1 or $\kappa = 1$ in the Corollary 3.2 simplifies to the following Corollary.

Corollary 3.3 *If $h \in \mathcal{Y}_\Gamma(1,0,\sigma)$, then*

$$|s_2| \le \min\left\{\frac{1}{4|\sigma+1|}, \frac{1}{\sqrt{\left|6(2\sigma+1)+\frac{56}{3}(\sigma+1)^2\right|}}\right\},$$

$$|s_3| \le \min\left\{\frac{1}{16|\sigma+1|^2}+\frac{1}{6|2\sigma+1|}, \frac{3}{\left|6(2\sigma+1)+\frac{56}{3}(\sigma+1)^2\right|}+\frac{1}{6|2\sigma+1|}\right\}$$

and

$$\left|s_3-\Xi s_2^2\right| \le \begin{cases} \frac{1}{6|2\sigma+1|} & |\Theta(\Xi)| < \frac{1}{24|2\sigma+1|}, \\ 4\,|\Theta(\Xi)| & |\Theta(\Xi)| \ge \frac{1}{24|2\sigma+1|}. \end{cases}$$

where $\Theta(\Xi) = \frac{1-\Xi}{4\left(6(2\sigma+1)+\frac{56}{3}(\sigma+1)^2\right)}$.

For $\sigma = 0$ in Corollary 3.3 simplifies to the following Corollary.

Corollary 3.4 *If $h \in \mathcal{Y}_\Gamma(1,0,0)$, then*

$$|s_2| \le \sqrt{\frac{3}{74}} \simeq 0.023\cdots, \quad |s_3| \le \frac{23}{111} \simeq 0.207\cdots$$

and

$$\left|s_3-\Xi s_2^2\right| \le \begin{cases} \frac{1}{6} & \Xi \in \left[\frac{-28}{9}, \frac{46}{9}\right], \\ \frac{3|1-\Xi|}{74} & \Xi \in \mathbb{R} - \left(\frac{-28}{9}, \frac{46}{9}\right). \end{cases}$$

Remark 3.5

- (i) The sufficient conditions for $|s_2|$, $|s_3|$ and $\left|s_3-\Xi s_2^2\right|$ in Corollary 3.4, was obtained by [30].
- (ii) For specific values of parameters α and φ in Theorems 2.9 and Theorems 2.10, we obtain several corollaries for the subclasses $\mathcal{W}_\Gamma(\alpha,\varphi)$ and $\mathcal{K}_\Gamma(\alpha,\varphi)$, respectively.

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