Energy Estimates and Existence Results for a Mixed Boundary Value Problem for a Complete Sturm-Liouville Equation Exploiting a Local Minimization Principle

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Abstract: In our work, we are going to look for local minima for the Euler functional corresponding to a mixed boundary value problem for a complete Sturm-Liouville equation where the coefficients can also be negative, to obtain the existence results and energy estimates for solutions for the problem. In particular, we establish the existence of a non-zero solution for a specific localization of the parameter and show that the solution exists for positive values of the parameter, under the condition that the nonlinear component exhibits sublinearity both at the origin and at infinity. The proof relies on a local minimum theorem for differentiable functionals. We also consider the existence of solutions for our problem under algebraic conditions with the classical Ambrosetti-Rabinowitz.

Key-Words: Sturm-Liouville, Mixed boundary value problems, Generalized solution, Variational methods.

Received: September 22, 2024. Revised: January 11, 2025. Accepted: February 13, 2025. Published: April 2, 2025.

1 Introduction

Optimization problems involving of maximizing or minimizing real functions are ubiquitous in the mathematical modeling of real-world systems and encompass a very broad range of applications across various fields, including economics, finance, chemistry, materials science, astronomy, physics, structural and molecular biology, engineering, computer science, and medicine. Boundary value problems (BVPs) play a crucial role in the mathematical analysis of constrained physical systems subjected to external forces. Consequently, BVPs frequently arise in various disciplines such as economics, finance, and engineering, covering diverse problem domains including fluid mechanics, electromagnetics, quantum mechanics, and elasticity. Mixed boundary value problems in Sturm-Liouville theory have applications in the vibrations of a beam and strings, heat conduction in rods and plates, electromagnetic wave propagation, quantum mechanics and Schrödinger equation, fluid dynamics and diffusion problems, and electrical circuits and signal processing.

The goal of this paper is to demonstrate the existence results for the following problem

$$\begin{cases} -z'' + \alpha(\varsigma)z' + \delta(\varsigma)z = \gamma h(\varsigma, z(\varsigma)), & \varsigma \in (a,b), \\ z(a) = z(b) = 0 \end{cases} \tag{P^h}$$

where $\gamma > 0$, h is an L^1-Carathéodory function and $\alpha, \delta \in L^\infty([a,b])$ are such that

$$\operatorname{ess}\inf_{\varsigma\in[a,b]} \delta(\varsigma) > -\left(\frac{\pi}{2(b-a)}\right)^2. \tag{1}$$

Recent mathematical research has extensively studied Sturm-Liouville problems with mixed boundary conditions, yielding important findings Readers are encouraged to consult [1], [2], [3], [4], [5], [6], and the related references. For instance, [3] utilized variational methods to establish the existence of nontrivial solutions for a mixed boundary value problem involving the Sturm-Liouville equation. Additionally, [1], applied multiple critical points theorems to demonstrate the existence of three solutions for a Sturm-Liouville mixed boundary value problem.

Notably, in the above references, the coefficients of the differential equations are nonnegative. Here, in the mixed boundary value problem for a complete Sturm-Liouville equation (P^h) the coefficients can also be negative. The key observation about nonnegative coefficients in these equations is important because they typically ensure the system remains physically meaningful, such as maintaining a positive temperature or concentration of a substance. These types of models are common in various industrial processes, especially when dealing with heat transfer, phase changes, or material behavior under specific conditions. The example of mixed boundary value problems (where boundary conditions are specified in different ways) related to the solidification and melting of materials is particularly interesting because it combines both thermal and mechanical properties. In these processes, the material can undergo phase transitions, such as from liquid to solid, which involve non-linear behaviors and complex boundary conditions (see, for instance, [7], and the references).

It should be mentioned that differential equations are widespread in every field of science and engineering, varying from physics to economics. Thus, significant research has been done on growing numerical methods for solving differential equations. With the unprecedented availability of computational power, neural networks hold promise in redefining how computational problems are solved or upgrading existing numerical methods. An interesting question in scientific computing whether machine learning can be also applied to solve eigenvalue problems. That is, to train an algorithm using spectral data and examine its ability to predict an unknown function, which is the coefficient of a differential operator and is associated with the physical properties of the problem under consideration. We refer to [8], [9], [10], [11], [12], [13], [14], [15], for applications of neural networks to solve eigenvalue problems.

In this paper, we build upon the results obtained in [16], where, unlike in other available studies, the coefficients α and δ are allowed to change sign. The authors in [16] used critical point theory to investigate the existence of infinitely many distinct positive solutions for the equation (P^h). Furthermore, in [17], the existence of multiple solutions for the problem (P^h) was investigated using some algebraic conditions on the nonlinear term, in particular, requiring that the growth of the antiderivative of the nonlinear term exceeds quadratic growth in a suitable interval and is less than quadratic growth in a subsequent suitable interval. Additionally, some other results were presented guaranteeing the existence of four distinct non-trivial solutions to the problem (P^h) under suitable conditions on the nonlinear term at both zero and infinity.

The primary innovation of this paper lies in our assumption that the coefficients α and δ can vary in sign, which distinguishes it from existing literature. For instance, we can examine the well-known Laguerre differential equation (see, [16]):

$$z''(\varsigma) + \frac{2-\varsigma}{\varsigma} z'(\varsigma) + \frac{1}{4\varsigma} z(\varsigma) = h(\varsigma, z(\varsigma)), \; \varsigma \in (-3, -2). \tag{2}$$

Indeed, equation (2) represents a complete Sturm-Liouville differential equation with its coefficients defined as follows: $\alpha(\varsigma) = \dfrac{2-\varsigma}{\varsigma}$ and $\delta(\varsigma) = \dfrac{2-\varsigma}{\varsigma}$ which are negative and meet our established hypotheses.

As an illustration, we present the following specific case of our findings; refer to Remark 10 for additional details.

Theorem 1. *Let $h : \mathbb{R} \to \mathbb{R}$ be a continuous function such that*

$$\lim_{\zeta \to 0^+} \frac{h(\zeta)}{\zeta} = +\infty \text{ and } \lim_{|\zeta| \to \infty} \frac{h(\zeta)}{|\zeta|} = 0.$$

Then, there exists $\gamma^* > 0$ such that, for each $\gamma \in (0, \gamma^*)$, the following problem

$$\begin{cases} -z'' + \alpha(\varsigma) z' + \delta(\varsigma) z = \gamma h(\varsigma, z(\varsigma)), & \varsigma \in (a, b), \\ z(a) = z(b) = 0 \end{cases} \tag{3}$$

admits at least 1 nontrivial generalized solution $z \in \{z \in W^{1,2}([a,b]) : z(a) = 0\}$ such that

$$\lim_{\gamma \to 0^+} \|z\|_{\mathrm{E}} = 0$$

where $\|\cdot\|_{\mathrm{E}}$ is defined in the next section, and the real function

$$\gamma \to \frac{1}{2}\|z\|_{\mathrm{E}}^2 - \gamma \int_a^b e^{-\Phi(\varsigma)} H(z(\varsigma)) \mathrm{d}\varsigma$$

is negative and strictly decreasing in $(0, \gamma^*)$.

Building on the previously mentioned works, we show that there exists at least 1 non-zero generalized solution for the problem (P^h), which is characterized by a single parameter assuming a growth condition and an algebraic condition on the nonlinear term (see Theorem 3). A specific case is highlighted in Corollary 1. Additionally, we present a related result where the only condition required on the data is sublinearity at the origin, as detailed in Theorem 4. We also note that when the nonlinear term exhibits sublinearity at infinity, the corresponding energy functional is coercive, thereby guaranteeing the existence of at least 1 solution (which may be zero) according to the direct methods theorem, as mentioned in Remark 8. It is important to highlight that in our scenarios, the potential may not be coercive, as illustrated by Example 3.1. Moreover, even in the presence of coercivity, our results confirm the existence of at least one nonzero generalized solution. A key tool utilized in the proofs is a recent critical point theorem established in [18, Theorem 5.1], for functionals of the form $\Gamma_\gamma = \Theta - \gamma\Upsilon$, where $\gamma > 0$ is a parameter (see Theorem 2). Furthermore, we would like to note that in Corollary 1, by adding the classical Ambrosetti and Rabinowitz (AR) condition to the hypotheses of Theorem 4, we obtain a second generalized solution. Finally, we provide Example 3 to illustrate Theorem 5.

2 Preliminaries

Let E be a real Banach space. We say that a continuously Gâteaux differentiable functional $\Gamma : X \to \mathbb{R}$ satisfies the *Palais-Smale condition* (in short (PS)-condition) if every sequence $\{z_n\}$ has a convergent subsequence whenever $\{\Gamma(z_n)\}$ is bounded and $\lim_{n\to\infty} \|\Gamma'(z_n)\|_{X^*} = 0$. For further details on the fundamental functional concepts employed in this paper, we direct the reader to [19], v[20].

Let $\Theta, \Upsilon : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functions. Set

$$\Gamma = \Theta - \Upsilon,$$

and fix $s_1, s_2 \in [-\infty, +\infty]$ with $s_1 < s_2$. We say that Γ verifies the *Palais-Smale condition cut off lower at* s_1 *and upper at* s_2 (in short ${}^{[s_1]}(\mathrm{PS})^{[s_2]}$-condition) if any sequence $\{z_n\}$ has a convergent subsequence if $\{\Gamma(z_n)\}$ is bounded, $\lim_{n\to\infty} \|\Gamma'(z_n)\|_{X^*} = 0$ and $s_1 < \Theta(z_n) < s_2$, $\forall\, n \in \mathbb{N}$.

Clearly, if $s_1 = -\infty$ and $s_2 = +\infty$ it coincides with the classical (PS)-condition. Additionally, if $s_1 = -\infty$ and $s_2 \in \mathbb{R}$, we denote this condition as $(\mathrm{PS})^{[s_2]}$. Conversely, if $s_1 \in \mathbb{R}$ and $s_2 = +\infty$, it is referred to as ${}^{[s_1]}(\mathrm{PS})$. Furthermore, if Γ satisfies ${}^{[s_1]}(\mathrm{PS})^{[s_2]}$-condition, then it also satisfies the ${}^{[\phi_1]}(\mathrm{PS})^{[\phi_2]}$-condition for every $\phi_1, \phi_2 \in [-\infty, +\infty]$ such that $s_1 \le \phi_1 < \phi_2 \le s_2$.

In particular, we deduce that if Γ satisfies the classical (PS)-condition, then it satisfies ${}^{[\phi_1]}(\mathrm{PS})^{[\phi_2]}$-condition for all $\phi_1, \phi_2 \in [-\infty, +\infty]$ with $\phi_1 < \phi_2$. For every $s_1, s_2 \in \mathbb{R}$ with $s_1 < s_2$, set

$$\beta(s_1, s_2) = \inf_{z\in\Theta^{-1}(s_1,s_2)} \frac{\sup_{v\in\Theta^{-1}(s_1,s_2)} \Upsilon(v) - \Upsilon(z)}{s_2 - \Theta(z)} \tag{4}$$

and

$$\phi_2(s_1, s_2) = \sup_{z\in\Theta^{-1}(s_1,s_2)} \frac{\Upsilon(z) - \sup_{v\in\Theta^{-1}(-\infty,s_1]} \Upsilon(v)}{\Theta(z) - s_1}. \tag{5}$$

The proof of the main results in this paper relies on the following theorem, initially established by [18], requires Palais-Smale condition, which is derived from the Ricceri's variational principle, [21, Theorem 2.5], as an variant. This theorem provides a more accurate localization of the minimum and does not require any assumptions about weak lower semicontinuity.

Theorem 2. *Let* E *be a real Banach space and let* $\Theta, \Upsilon : X \to \mathbb{R}$ *be two continuously Gâteaux differentiable functions. Assume that there are* s_1, $s_2 \in \mathbb{R}$ *such that* $s_1 < s_2$ *and* $\beta(s_1, s_2) < \phi_2(s_1, s_2)$, *where* β *and* ϕ_2 *are given by* (4) *and* (5) *and for each*

$$\gamma \in \left(\frac{1}{\phi_2(s_1, s_2)}, \frac{1}{\beta(s_1, s_2)} \right),$$

the function $\Gamma_\gamma = \Theta - \gamma\Upsilon$ *satisfies* ${}^{[s_1]}(\mathrm{PS})^{[s_2]}$*-condition.*

Then for any

$$\gamma \in \left(\frac{1}{\phi_2(s_1, s_2)}, \frac{1}{\beta(s_1, s_2)} \right),$$

there exists $z_{0\gamma} \in \Theta^{-1}(s_1, s_2)$ *such that* $\Gamma_\gamma(z_{0\gamma}) \le \Gamma_\gamma(z)$ *for all* $z \in \Theta^{-1}(s_1, s_2)$ *and* $\Gamma'_\gamma(z_{0\gamma}) = 0$.

We direct interested readers to the papers, [22], [23], where Theorem 2 has been effectively utilized to establish the existence of at least 1 generalized solution for various boundary value problems.

In this section, we will introduce several essential definitions, notations, lemmas, and propositions that will be referenced throughout the paper. Take the Sobolev space

$$\mathrm{E} = \{z \in W^{1,2}([a, b]) : z(a) = 0\}$$

endowed with the following norm:

$$\|z\| = \left(\int_a^b |z(s)|^2 \mathrm{d}s \right)^{\frac{1}{2}} + \left(\int_a^b |z'(s)|^2 \mathrm{d}s \right)^{\frac{1}{2}}.$$

Moreover, for all $z \in \mathrm{E}$, put

$$\|z\|_0 = \|z'\|_{L^2} = \left(\int_a^b |z'(s)|^2 \mathrm{d}s \right)^{\frac{1}{2}}.$$

The following proposition holds true.

Proposition 1. *[16, Proposition 2.1] The norms $\|\cdot\|_0$ and $\|\cdot\|$ are equivalent on* E.

Here, we point out the following result.

Proposition 2. *[16, Proposition 2.2] (Poincaré inequalities) For all $z \in \mathrm{E}$, one has*

(1) $\max_{\varsigma\in[a,b]} |z(\varsigma)| \leq (b-a)^{\frac{1}{2}} \|z\|_0$,

(2) $\|z\|_{L^2} \leq \dfrac{2(b-a)}{\pi} \|z'\|_{L^2}$.

Remark 1. *We observe that the Poincaré inequalities hold true in the Sobolev space $W^{1,2}([a,b])$, as given in [24], with different constants.*

Now, let us introduce another norm in the space E, given by

$$\|z\|_{\mathrm{E}} = \left(\int_a^b e^{-\Phi(\varsigma)} |z'(\varsigma)|^2 \mathrm{d}\varsigma + \int_a^b e^{-\Phi(\varsigma)} \delta(\varsigma) |z(\varsigma)|^2 \mathrm{d}\varsigma \right)^{\frac{1}{2}}$$

where $\Phi(\varsigma) = \int_a^\varsigma \alpha(\xi)\mathrm{d}\xi, \ \ \forall \varsigma \in [a,b]$.

Proposition 3. *[16, Proposition 2.3] Assume* (1) *holds. Then $\|\cdot\|_{\mathrm{E}}$ is a norm on the space* E *and it is equivalent to $\|\cdot\|_0$. In particular, one has*

$$m\|z\|_0 \leq \|z\|_{\mathrm{E}} \leq M\|z\|_0 \tag{6}$$

for all $z \in \mathrm{E}$, where m, M with $M \geq m > 0$, are given by

$$m = \begin{cases} \left(\min_{\varsigma\in[a,b]} e^{-\Phi(\varsigma)}\right)^{\frac{1}{2}}, \\ \text{if} \quad \operatorname{ess\,inf}_{\varsigma\in[a,b]} \delta(\varsigma) \geq 0, \\ \left(\min_{\varsigma\in[a,b]} e^{-\Phi(\varsigma)} \left(1 + \operatorname{ess\,inf}_{\varsigma\in[a,b]} \delta(\varsigma) \left(\frac{2(b-a)}{\pi}\right)^2\right)\right)^{\frac{1}{2}}, \\ \text{if} \quad \operatorname{ess\,inf}_{\varsigma\in[a,b]} \delta(\varsigma) < 0 \end{cases}$$

and

$$M = \begin{cases} \left(\max_{\varsigma\in[a,b]} e^{-\Phi(\varsigma)} \left(1 + \operatorname{ess\,sup}_{\varsigma\in[a,b]} \delta(\varsigma) \left(\frac{2(b-a)}{\pi}\right)^2\right)\right)^{\frac{1}{2}}, \\ \text{if} \quad \operatorname{ess\,inf}_{\varsigma\in[a,b]} \delta(\varsigma) \geq 0, \\ \left(\max_{\varsigma\in[a,b]} e^{-\Phi(\varsigma)}\right)^{\frac{1}{2}}, \\ \text{if} \quad \operatorname{ess\,inf}_{\varsigma\in[a,b]} \delta(\varsigma) < 0. \end{cases}$$

Remark 2. *We observe that, since $\|z\|_0$ is equivalent to $\|z\|$, as proved in Proposition 1, thanks to the transitivity property, we obtain the equivalence between $\|z\|_{\mathrm{E}}$ and $\|z\|$.*

Remark 3. *The space* E *is a Hilbert space with the dot product*

$$< z - v > = \int_a^b e^{-\Phi(\varsigma)} z'(\varsigma) v'(\varsigma) \mathrm{d}\varsigma + \int_a^b e^{-\Phi(\varsigma)} \delta(\varsigma) z(\varsigma) v(\varsigma) \mathrm{d}\varsigma$$

that clearly induces the norm $\|z\|_{\mathrm{E}}$.

Remark 4. *Taking into account* (1) *and* (6)*, the following inequality holds:*

$$\max_{\varsigma\in[a,b]} |z(\varsigma)| \leq \frac{(b-a)^{\frac{1}{2}}}{m} \|z\|_{\mathrm{E}}, \ \ \forall z \in \mathrm{E}.$$

Now we recall the definition of classical and generalized solution for the problem (P^h).

Definition 1. *We say that $z : [a,b] \to \mathbb{R}$ is a classical solution if $z \in C^2([a,b])$, $z(a) = z'(b) = 0$,*

$$-z'' + \alpha(\varsigma)z' + \delta(\varsigma)z = \gamma h(\varsigma, z(\varsigma)), \ \ \forall \varsigma \in [a,b].$$

Definition 2. *We say that $z : [a,b] \to \mathbb{R}$ is a generalized solution if $z \in C^1([a,b])$, $z' \in C([a,b])$,*

$$z(a) = z'(b) = 0, \ -z'' + \alpha(\varsigma)z' + \delta(\varsigma)z = \gamma h(\varsigma, z(\varsigma))$$

for almost every $\varsigma \in [a,b]$.

Remark 5. *Classical and generalized solutions coincide when α, δ and h are continuous functions.*

Definition 3. *A function $z \in \mathrm{E}$ is called a generalized solution of the problem (P^h), if*

$$\int_a^b e^{-\Phi(\varsigma)} z'(\varsigma) v'(\varsigma) \mathrm{d}\varsigma + \int_a^b e^{-\Phi(\varsigma)} \delta(\varsigma)(z(\varsigma)) v(\varsigma) \mathrm{d}\varsigma - \gamma \int_a^b e^{-\Phi(\varsigma)} h(\varsigma, z(\varsigma)) v(\varsigma) \mathrm{d}\varsigma = 0$$

holds for any $v \in \mathrm{E}$.

Put

$$H(\varsigma, \zeta) = \int_0^\zeta h(\varsigma, x)\mathrm{d}x \ \text{ for any } (\varsigma, \zeta) \in (a,b) \times \mathbb{R}.$$

We define the functionals Π and Υ for each $z \in \mathrm{E}$, as follows

$$\Theta(z) = \frac{1}{2} \|z\|_{\mathrm{E}}^2 \tag{7}$$

and

$$\Upsilon(z) = \int_a^b e^{-\Phi(\varsigma)} H(\varsigma, z(\varsigma)) \mathrm{d}\varsigma \tag{8}$$

and we put

$$\Gamma_\gamma(z) = \Theta(z) - \gamma \Upsilon(z)$$

for every $z \in \mathrm{E}$.

Proposition 4. *[16, Proposition 2.4] Function z is a generalized solution of (P^h) if only if z is a critical point of Γ_γ.*

We need the following Proposition for existence our main results.

Proposition 5. *[17, Proposition 2.14] Let $S : \mathrm{E} \longrightarrow \mathrm{E}^*$ be the operator defined by*

$$S(z)(v) = \int_a^b e^{-\Phi(\varsigma)} z'(\varsigma)v'(\varsigma)\mathrm{d}\varsigma + \int_a^b e^{-\Phi(\varsigma)}\delta(\varsigma)(z(\varsigma)v(\varsigma)\mathrm{d}\varsigma$$

for every $z, v \in \mathrm{E}$. Then, S admits a continuous inverse on E^*.

Let $h : (a, b) \times \mathbb{R} \to \mathbb{R}$ are continuous functions. We say that h are of type $(\mathcal{G}_{h,2})$ if it meets the following growth condition.

$(\mathcal{G}_{h,2})$ There exist two positive constants a_1 and a_2 such that

$$|h(\varsigma, \epsilon)| \leq a_1 + a_2|\epsilon|$$

for a.e. $(\varsigma, \epsilon) \in (a, b) \times \mathbb{R}$.

3 Main Results

We state our main result as follows.

For given nonnegative constants θ and σ, with

$$m^2\theta_1^2 \neq 2M^2\sigma^2,$$

we set

$$b_\theta(\sigma) = \frac{\mathcal{A}_\theta - \min_{\varsigma\in[a,b]} e^{-\Phi(\varsigma)} \int_{\frac{a+b}{2}}^b H(\varsigma, \sigma)\mathrm{d}\varsigma}{m^2\theta_1^2 - 2M^2\sigma^2} \tag{9}$$

where

$$\mathcal{A}_\theta = \max_{\varsigma\in[a,b]} e^{-\Phi(\varsigma)} \left(a_1(b-a)\theta + \frac{a_2(b-a)}{2}\theta^2\right).$$

We are now ready to present and demonstrate the primary results of this paper.

Theorem 3. *Assume that h fulfills $(\mathcal{G}_{h,2})$ and assume that there exist three real constants θ_1, θ_2 and σ such that*

$$0 \leq \theta_1 < \sqrt{2}\sigma$$

and

$$\sqrt{2}\frac{M}{m}\sigma < \theta_2$$

such that

(A_1) $h(\varsigma, \vartheta) \geq 0$ *for each* $(\varsigma, \vartheta) \in [a, \frac{a+b}{2}] \times [0, \infty)$,

and

$$b_{\theta_2}(\sigma) < b_{\theta_1}(\sigma). \tag{10}$$

Then for each parameter $\gamma \in \left(\frac{1}{2(b-a)b_{\theta_1}(\sigma)}, \frac{1}{2(b-a)b_{\theta_2}(\sigma)}\right)$, *the problem (P^h) admits at least 1 non-zero generalized solution $z_{0\gamma} \in \mathrm{E}$, such that*

$$\frac{m\theta_1}{\sqrt{b-a}} < \|z_{0\gamma}\|_{\mathrm{E}} < \frac{m\theta_2}{\sqrt{b-a}}.$$

Proof. Our goal is to utilize Theorem 2 to address the problem described in (P^h). We will consider the functionals Θ and Υ as given in (7) and (8), respectively. We will demonstrate that the functionals Θ and Υ meet the necessary conditions outlined in Theorem 2. It is well known (see, for instance, [24]) that is well defined, and are Gâteaux differentiable, and one has

$$\Upsilon'(z)(v) = \int_a^b e^{-\Phi(\varsigma)} h(\varsigma, z(\varsigma))v(\varsigma)\mathrm{d}\varsigma$$

and

$$\Theta'(z)(v) = \int_a^b e^{-\Phi(\varsigma)} z'(\varsigma)v'(\varsigma)\mathrm{d}\varsigma + \int_a^b e^{-\Phi(\varsigma)}\delta(\varsigma)(z(\varsigma))v(\varsigma)\mathrm{d}\varsigma$$

for every $z, v \in \mathrm{E}$. Furthermore, Θ and Υ are C^1-functions. By utilizing the definition of Θ, it follows that

$$\lim_{\|z\|_{\mathrm{E}}\to+\infty} \Theta(z) = +\infty,$$

which indicates that Θ is coercive, while Proposition 5 gives that Θ admits a continuous inverse on E^*. Therefore, we conclude that the regularity assumptions on Θ and Υ, as specified in Theorem 2, are satisfied. Fix the eigenvalue $\gamma \in \left(\frac{1}{2(b-a)b_{\theta_1}(\sigma)}, \frac{1}{2(b-a)b_{\theta_2}(\sigma)}\right)$. As we have seen in [16, Proposition 2.4], the critical points in E of the functional Γ_γ are exactly the generalized solutions of the considered problem (P^h). We now consider the existence of a critical point of the functional Γ_λ in E. In doing so, we confirm that the regularity assumptions on Θ and Υ, as stipulated in Theorem 2, are satisfied. It is important to note that the operator Γ_λ is a C^1 $(\mathrm{E}, \mathbb{R})$ functional, and the critical points of Γ_λ correspond to generalized solutions of the problem outlined in (P^h). Furthermore, under condition

$(\mathcal{G}_{h,2})$, the application of Hölder's inequality leads to

$$H(\varsigma,\zeta) \leq a_1|\zeta| + a_2\frac{|\zeta|^2}{2} \tag{11}$$

for a.e. $\varsigma \in (a,b)$ and for all $\zeta \in \mathbb{R}$, considering relation (11), it can be concluded that

$$\Upsilon(z) = \int_a^b e^{-\Phi(\varsigma)} H(\varsigma, z(\varsigma))\mathrm{d}\varsigma \leq$$

$$\max_{\varsigma\in[a,b]} e^{-\Phi(\varsigma)} \left(\frac{a_1(b-a)^{\frac{3}{2}}}{m}\|z\|_{\mathrm{E}} + \frac{a_2(b-a)^2}{2m^2}\|z\|_{\mathrm{E}}^2 \right).$$

Then, for each $z \in \mathrm{E}$ such that $\Theta(z) \leq s$, we obtain

$$\sup_{z\in\Theta^{-1}(-\infty,s]} \Upsilon(z) \leq \max_{\varsigma\in[a,b]} e^{-\Phi(\varsigma)} \tag{12}$$

$$\left(\frac{a_1(b-a)^{\frac{3}{2}}}{m} (2s)^{\frac{1}{2}} + \frac{a_2(b-a)^2}{2m^2} (2s) \right). \tag{13}$$

Now, we define

$$s_1 = \frac{m^2}{2(b-a)}\theta_1^2,$$

$$s_2 = \frac{m^2}{2(b-a)}\theta_2^2$$

and

$$w_\sigma(\varsigma) = \begin{cases} \dfrac{2\sigma}{b-a}(\varsigma - a), & \text{if } \varsigma \in [a, \frac{a+b}{2}), \\ \sigma, & \text{if } \varsigma \in [\frac{a+b}{2}, b]. \end{cases}$$

Clearly, $w_\sigma \in \mathrm{E}$. Obviously, one has

$$\|w_\sigma\|_0^2 = \int_a^b |w'_\sigma(\varsigma)|^2\mathrm{d}\varsigma = \int_a^b \left(\frac{2\sigma}{b-a}\right)^2 \mathrm{d}\varsigma = \frac{2\sigma^2}{b-a}.$$

Then, we have $\Theta(0) = \Upsilon(0) = 0$ and

$$\frac{m^2\sigma^2}{b-a} = \frac{1}{2}m^2\|w_\sigma\|_0^2 \leq \Theta(w_\sigma) =$$
$$\frac{1}{2}\|w_\sigma\|_{\mathrm{E}}^2 \leq \frac{1}{2}M^2\|w_\sigma\|_0^2 = \frac{M^2\sigma^2}{b-a}.$$

By using condition (A_1), we have

$$\Upsilon(w_\sigma) = \int_a^b e^{-\Phi(\varsigma)} H(\varsigma, w_\sigma)\mathrm{d}\varsigma \geq$$

$$\min_{\varsigma\in[a,b]} e^{-\Phi(\varsigma)} \int_{\frac{a+b}{2}}^b H(\varsigma,\sigma)\mathrm{d}\varsigma.$$

Considering that

$$0 \leq \theta_1 < \sqrt{2}\sigma$$

and

$$\sqrt{2}\frac{M}{m}\sigma < \theta_2,$$

by a direct computation, one has $s_1 < \Theta(w_\sigma) < s_2$. Taking Remark 4 into account, for each $z \in \mathrm{E}$ such that $\Theta(z) = \frac{1}{2}\|z\|_{\mathrm{E}}^2 < s_1$, one has

$$|z(\varsigma)| \leq \frac{(b-a)^{\frac{1}{2}}}{m}\|z\|_{\mathrm{E}} \leq \frac{(b-a)^{\frac{1}{2}}}{m}(2s_1)^{\frac{1}{2}} =$$

$$\left(\frac{2(b-a)}{m^2}s_1\right)^{\frac{1}{2}} = \theta_1, \quad \forall z \in \mathrm{E}.$$

Taking into account that $\max_{\varsigma\in[a,b]} |z(\varsigma)| \leq \theta_1$ for all $z \in \mathrm{E}$ such that $\|z\|_{\mathrm{E}}^2 < 2s_1$, and by same argument as above

$$\Theta^{-1}(-\infty, s_2) \subseteq \{z \in \mathrm{E},\ \|z\|_\infty \leq \theta_2\}.$$

From equation (12), it can be inferred that

$$\sup_{z\in\Theta^{-1}(-\infty,s_1)} \Upsilon(z) \leq \max_{\varsigma\in[a,b]} e^{-\Phi(\varsigma)} \left(\frac{a_1(b-a)^{\frac{3}{2}}}{m}(2s_1)^{\frac{1}{2}} + \frac{a_2(b-a)^2}{2m^2}(2s_1) \right)$$
$$= \max_{\varsigma\in[a,b]} e^{-\Phi(\varsigma)} \left(a_1(b-a)\theta_1 + \frac{a_2(b-a)}{2}\theta_1^2 \right) \tag{14}$$

and

$$\sup_{z\in\Theta^{-1}(-\infty,s_2)} \Upsilon(z) \leq \max_{\varsigma\in[a,b]} e^{-\Phi(\varsigma)} \left(\frac{a_1(b-a)^{\frac{3}{2}}}{m}(2s_2)^{\frac{1}{2}} + \frac{a_2(b-a)^2}{2m^2}(2s_2) \right)$$
$$= \max_{\varsigma\in[a,b]} e^{-\Phi(\varsigma)} \left(a_1(b-a)\theta_2 + \frac{a_2(b-a)}{2})\theta_2^2 \right). \tag{15}$$

Conversely,

$$\beta(s_1,s_2) = \inf_{z\in\Theta^{-1}(s_1,s_2)} \frac{\sup_{v\in\Theta^{-1}(s_1,s_2)} \Upsilon(v) - \Upsilon(z)}{s_2 - \Theta(z)}$$
$$\leq \frac{\sup_{v\in\Theta^{-1}(-\infty,s_2)} \Upsilon(v) - \Upsilon(w_\sigma)}{s_2 - \Theta(w_\sigma)}$$
$$\leq \frac{\mathcal{A}_{\theta_2} - \min_{\varsigma\in[a,b]} e^{-\Phi(\varsigma)} \int_{\frac{a+b}{2}}^b H(\varsigma,\sigma)\mathrm{d}\varsigma}{\dfrac{m^2}{2(b-a)}\theta_2^2 - \dfrac{M^2\sigma^2}{b-a}}$$

and

$$\phi_2(s_1,s_2) = \sup_{z\in\Theta^{-1}(s_1,s_2)} \frac{\Upsilon(z) - \sup_{v\in\Theta^{-1}(-\infty,s_1]} \Upsilon(v)}{\Theta(z) - s_1}$$
$$\geq \frac{\Upsilon(w_\sigma) - \sup_{v\in\Theta^{-1}(-\infty,s_1]} \Upsilon(v)}{\Theta(w_\sigma) - s_1}$$
$$\geq \frac{\min_{\varsigma\in[a,b]} e^{-\Phi(\varsigma)} \int_{\frac{a+b}{2}}^b H(\varsigma,\sigma)\mathrm{d}\varsigma - \mathcal{A}_{\theta_1}}{\dfrac{M^2\sigma^2}{b-a} - \dfrac{m^2}{2(b-a)}\theta_1^2}.$$

Using the notation (9), it follows from (14) and (15) that

$$\beta(s_1, s_2)$$

$$\leq \frac{\mathcal{A}_{\theta_2} - \min_{\varsigma\in[a,b]} e^{-\Phi(\varsigma)} \int_{\frac{a+b}{2}}^{b} H(\varsigma,\sigma)\mathrm{d}\varsigma}{\frac{m^2}{2(b-a)}\theta_2^2 - \frac{M^2\sigma^2}{b-a}} = 2(b-a)b_{\theta_2}(\sigma)$$

and

$$\phi_2(s_1, s_2) \geq \frac{\min_{\varsigma\in[a,b]} e^{-\Phi(\varsigma)} \int_{\frac{a+b}{2}}^{b} H(\varsigma,\sigma)\mathrm{d}\varsigma - \mathcal{A}_{\theta_1}}{\frac{M^2\sigma^2}{b-a} - \frac{m^2}{2(b-a)}\theta_1^2} = 2(b-a)b_{\theta_1}(\sigma).$$

Ultimately, the assumption (10) leads to the conclusion that

$$\beta(s_1, s_2) < \phi_2(s_1, s_2).$$

Now, based on [25, Lemma 2.6], the functional Γ_γ fulfills the classical (PS)-condition, which implies that it also meets the $^{[s_1]}(\mathrm{PS})^{[s_2]}$-condition for each s_1 and s_2 with $s_1 < s_2 < +\infty$. Therefore, by using Theorem 2, for each

$$\gamma \in \left(\frac{1}{2(b-a)b_{\theta_1}(\sigma)}, \frac{1}{2(b-a)b_{\theta_2}(\sigma)}\right),$$

the functional Γ_γ admits at least 1 critical point $z_{0\gamma}$ such that

$$s_1 < \Theta(z_{0\gamma}) < s_2,$$

which is equivalent to

$$\frac{m\theta_1}{\sqrt{b-a}} < \|z_{0\gamma}\|_{\mathrm{E}} < \frac{m\theta_2}{\sqrt{b-a}}.$$

□

The following corollaries are derived from Theorem 3.

Corollary 1. *Assume that h fulfills $(\mathcal{G}_{h,2})$ and there exist two positive constants θ and σ with*

$$\sqrt{2}\frac{M}{m}\sigma < \theta$$

such that (A_1) holds and

$$\frac{\min_{\varsigma\in[a,b]} e^{-\Phi(\varsigma)} \int_{\frac{a+b}{2}}^{b} H(\varsigma,\sigma)\mathrm{d}\varsigma}{\frac{M^2\sigma^2}{b-a}} > \frac{\mathcal{A}_\theta}{\frac{m^2}{2(b-a)}\theta^2}. \tag{16}$$

Thus, for each parameter γ that belongs to

$$\left(\frac{\frac{M^2\sigma^2}{b-a}}{\min_{\varsigma\in[a,b]} e^{-\Phi(\varsigma)} \int_{\frac{a+b}{2}}^{b} H(\varsigma,\sigma)\mathrm{d}\varsigma}, \frac{\frac{m^2}{2(b-a)}\theta_1^2}{\mathcal{A}_\theta}\right),$$

the problem (P^h) admits at least 1 non-zero generalized solution $z_{0\gamma} \in \mathrm{E}$ such that

$$\|z_{0\gamma}\|_{\mathrm{E}} < \frac{m\theta}{\sqrt{b-a}}.$$

Proof. We utilize Theorem 3. Take $\theta_1 = 0$ and $\theta_2 = \theta$. Using (9), we obtain

$$b_\theta(\sigma) = \frac{\mathcal{A}_\theta - \min_{\varsigma\in[a,b]} e^{-\Phi(\varsigma)} \int_{\frac{a+b}{2}}^{b} H(\varsigma,\sigma)\mathrm{d}\varsigma}{m^2\theta_1^2 - M^2\sigma^2}$$

and

$$b_0(\sigma) = \frac{\min_{\varsigma\in[a,b]} e^{-\Phi(\varsigma)} \int_{\frac{a+b}{2}}^{b} H(\varsigma,\sigma)\mathrm{d}\varsigma}{M^2\sigma^2}.$$

Now the inequality (16) immediately yields $2(b-a)b_\theta(\sigma) < 2(b-a)b_0(\sigma)$. □

We will now illustrate Corollary 1 by providing the following example.

Example 1. *We consider the following problem*

$$\begin{cases} -z'' + z' - z = \gamma h(z(\varsigma)), & \varsigma \in (0,1), \\ z(0) = z(1) = 0 \end{cases} \tag{17}$$

where

$$h(\zeta) = 1 + 2\zeta$$

for each $\zeta \in [0,\infty)$. Based on the expression for h, we can conclude that

$$H(\zeta) = \zeta + \zeta^2$$

for each $\zeta \in [0,\infty)$. Through straightforward calculations, we derive $m = \sqrt{1-\frac{4}{\pi^2}}$ and $M = \sqrt{e}$. Hence, $(\mathcal{G}_{h,2})$ is holds. Choose $\theta = 10^3$ and $\sigma = 10^{-2}$. Since

$$\frac{101}{2e^2} > \frac{2\pi^2(10^3+10^6)}{(\pi^2-4)10^6},$$

therefore, if condition (16) is satisfied, all the requirements of Corollary 1 are met. Consequently, it follows that for each

$$\gamma \in \left(\frac{2e^2}{101}, \frac{(\pi^2-4)10^6}{2\pi^2(10^3+10^6)}\right),$$

the problem (17) admits at least 1 non-zero generalized solution $z_{0\gamma} \in \mathrm{E}$ such that

$$\|z_{0\gamma}\|_{\mathrm{E}} < \sqrt{1-\frac{4}{\pi^2}}10^3.$$

A direct implication of Corollary 1 is the following result which it gives some properties about the solution, namely the solution is bounded it converges to 0 at 0^+ in E.

Theorem 4. *Assume that h fulfills $(\mathcal{G}_{h,2})$ such that (A_1) holds and*

$$\lim_{\zeta\to 0^+}\frac{\int_a^b H(\varsigma,\zeta)\mathrm{d}\varsigma}{\zeta^2}=+\infty. \tag{18}$$

Let $\theta>0$ and set

$$\gamma_\theta^\star=\frac{\frac{m^2}{2(b-a)}\theta^2}{\mathcal{A}_\theta}.$$

Then, for each $\gamma\in(0,\gamma_\theta^\star)$, the problem (P^h) admits at least 1 non-zero generalized solution $z_{0\gamma}\in\mathrm{E}$ such that $\|z_{0\gamma}\|_{\mathrm{E}}<\frac{m\theta}{\sqrt{b-a}}$ and

$$\lim_{\gamma\to 0^+}\|z_{0\gamma}\|_{\mathrm{E}}=0.$$

Proof. Fix $\gamma\in(0,\gamma_\theta^\star)$. From (18) there exists a positive constant σ with

$$\sqrt{2}\frac{M}{m}\sigma<\theta$$

such that

$$\frac{\frac{M^2\sigma^2}{b-a}}{\min_{\varsigma\in[a,b]}e^{-\Phi(\varsigma)}\int_{\frac{a+b}{2}}^b H(\varsigma,\sigma)\mathrm{d}\varsigma}<\gamma<\frac{\frac{m^2}{2(b-a)}\theta^2}{\mathcal{A}_\theta}.$$

Using Corollary 1, the problem (P^h) admits at least 1 non-zero generalized solution $z_{0\gamma}$, such that

$$\|z_{0\gamma}\|_{\mathrm{E}}<\frac{m\theta}{\sqrt{b-a}}.$$

Then, for each $\gamma\in(0,\gamma_\theta^\star)$, there exists at least 1 non-zero generalized solution $z_{0\gamma}\in\Theta^{-1}(0,s_2)$ of the problem (P^h) and one has

$$\|z_{0\gamma}\|_{\mathrm{E}}<\frac{m\theta}{\sqrt{b-a}} \tag{19}$$

for each $\gamma\in(0,\gamma_\theta^\star)$. Therefore, from $(\mathcal{G}_{h,2})$, considering equation (19), we can conclude that

$$\left|\int_a^b e^{-\Phi(\varsigma)}h(\varsigma,z_{0\gamma}(\varsigma))z_{0\gamma}(\varsigma)\mathrm{d}\varsigma\right| \le \max_{\varsigma\in[a,b]}e^{-\Phi(\varsigma)}\left(a_1(b-a)\theta+a_2(b-a)^2\theta^2\right) \tag{20}$$

for each $\gamma\in(0,\gamma_\theta^\star)$. Now, $\Gamma'_\gamma(z_{0\gamma})=0$, for each $\gamma\in(0,\gamma_\theta^\star)$ and in particular $\Gamma'_\gamma(z_{0\gamma})(z_{0\gamma})=0$, that is

$$\Theta'(z_{0\gamma})(z_{0\gamma})=\gamma\int_a^b e^{-\Phi(\varsigma)}h(\varsigma,z_{0\gamma}(\varsigma))z_{0\gamma}(\varsigma)\mathrm{d}\varsigma$$

for each $\gamma\in(0,\gamma^\star)$. We have

$$0\le m\|z_{0\gamma}\|_{\mathrm{E}}^2=\Theta'(z_{0\gamma})(z_{0\gamma}),$$

then, from (20), it follows that

$$\lim_{\gamma\to 0^+}\|z_{0\gamma}\|_{\mathrm{E}}^2=\lim_{\gamma\to 0^+}\gamma\Upsilon'(z_{0\gamma}(\varsigma))z_{0\gamma}(\varsigma)=0$$

that implies $\lim_{\gamma\to 0^+}\|z_{0\gamma}\|_{\mathrm{E}}=0$. Thus, the proof is now complete. □

We will now illustrate Theorem 4 with the example below.

Example 2. *We will examine the following problem*

$$\begin{cases}-z''+z'-z=\gamma h(z(\varsigma)), & \varsigma\in(0,1),\\ z(0)=z(1)=0\end{cases} \tag{21}$$

where

$$h(\zeta)=4+2\zeta$$

for each $\zeta\in[0,\infty)$. By performing simple calculations, we find that $m=\sqrt{1-\frac{4}{\pi^2}}$ and $M=\sqrt{e}$. Therefore, we conclude that $(\mathcal{G}_{h,2})$ holds. Choose $\theta=10^3$. Since

$$\lim_{\zeta\to 0^+}\frac{\int_0^1 H(\zeta)\mathrm{d}\varsigma}{\zeta^2}=+\infty,$$

thus, if condition (18) is satisfied, all the prerequisites of Theorem 4 are fulfilled. As a result, it follows that for each

$$\gamma\in\left(0,\frac{(\pi^2-4)10^6}{2\pi^2(4\times 10^3+10^6)}\right),$$

the problem (21) admits at least 1 non-zero generalized solution $z_{0\gamma}\in\mathrm{E}$ such that $\|z_{0\gamma}\|_{\mathrm{E}}<\sqrt{1-\frac{4}{\pi^2}}10^3$ and

$$\lim_{\gamma\to 0^+}\|z_{0\gamma}\|_{\mathrm{E}}=0.$$

Remark 6. *We assert that, given the aforementioned assumptions, the mapping $\gamma\mapsto\Gamma_\gamma(z_{0\gamma})$ is negative and strictly decreasing on the interval $(0,\gamma_\theta^\star)$.*

Proof. We show that the mapping $\gamma\mapsto\Gamma_\gamma(z_{0\gamma})$ is negative and strictly decreasing within the interval $(0,\gamma_\theta^\star)$. Indeed, the restriction of the functional Γ_γ to $\Theta^{-1}(0,s_2)$, where

$$s_2=\frac{m^2}{2(b-a)}\theta_2^2$$

admits a global minimum, which is a critical point (local minimum) of Γ_γ in E. Moreover, since $w_\sigma \in \Theta^{-1}(0, s_2)$ and

$$\frac{\Theta(w_\sigma)}{\Upsilon(w_\sigma)} \leq \frac{\dfrac{M^2\sigma^2}{b-a}}{\min_{\varsigma\in[a,b]} e^{-\Phi(\varsigma)} \int_{\frac{a+b}{2}}^{b} H(\varsigma,\sigma)\mathrm{d}\varsigma} < \gamma,$$

we know

$$\Gamma_\gamma(z_{0\gamma}) \leq \Gamma_\gamma(w_\sigma) = \Theta(w_\sigma) - \gamma\Upsilon(w_\sigma) < 0.$$

Next, we observe that

$$\Gamma_\gamma(\varsigma) = \gamma\left(\frac{\Theta(z)}{\gamma} - \Upsilon(z)\right)$$

for each $z \in$ E and fix $0 < \gamma_1 < \gamma_2 < \gamma_\theta^\star$. Set

$$n_{\gamma_1} = \left(\frac{\Theta(z_{0\gamma_1})}{\gamma_1} - \Upsilon(z_{0\gamma_1})\right) = \inf_{z\in\Theta^{-1}(0,s_2)} \left(\frac{\Theta(z)}{\gamma_1} - \Upsilon(z)\right)$$

and

$$n_{\gamma_2} = \left(\frac{\Theta(z_{0\gamma_2})}{\gamma_2} - \Upsilon(z_{0\gamma_2})\right) = \inf_{z\in\Theta^{-1}(0,s_2)} \left(\frac{\Theta(z)}{\gamma_2} - \Upsilon(z)\right).$$

Clearly, as claimed before, $n_{\gamma_i} < 0$ (for $i = 1, 2$), and $n_{\gamma_2} \leq n_{\gamma_1}$ thanks to $\gamma_1 < \gamma_2$. Then the mapping $\gamma \mapsto \Gamma_\gamma(z_{0\gamma})$ is strictly decreasing in $(0, \gamma_\theta^\star)$ owing to

$$\Gamma_{\gamma_2}(z_{0\gamma_2}) = \gamma_2 n_{\gamma_2} \leq \gamma_2 n_{\gamma_1} < \gamma_1 n_{\gamma_1} = \Gamma_{\gamma_1}(z_{0\gamma_1}).$$

This completes the proof of our assertion. □

Remark 7. *[16, Proposition 2.6] If h is non-negative, then the generalized solution guaranteed by Theorem 3 is also non-negative.*

Remark 8. *We note that if the nonlinear component h is sublinear at infinity with respect to the second variable, then Theorem 3 guarantees the existence of at least one non-zero generalized solution to the problem (P^h) for each $\gamma > 0$. Moreover, in our approach, the solution obtained is guaranteed to be non-zero, whereas the traditional direct method only guarantees the existence of at least one solution, which may be zero.*

Remark 9. *A detailed analysis of the proof of Theorem 4 confirms that the result holds true even if condition* (18) *is replaced by a broader assumption*

$$\limsup_{\zeta\to 0^+} \frac{\int_a^b H(\varsigma,\zeta)\mathrm{d}\varsigma}{\zeta^2} = +\infty.$$

In the autonomous scenario, this asymptotic condition at zero can be expressed as follows

$$\limsup_{\zeta\to 0^+} \frac{H(\zeta)}{\zeta^2} = +\infty. \tag{22}$$

Thus, based on the analysis provided above, it is reasonable to derive the following result.

Remark 10. *It is important to note that Theorem 1 presented in the Introduction follows directly from Theorem 4 and Remark 6. Indeed, If the following condition holds:*

$$\lim_{\zeta\to 0^+} \frac{h(\zeta)}{\zeta} = +\infty,$$

then the assumption (22) *is automatically fulfilled. Furthermore, the hypothesis*

$$\lim_{|\zeta|\to 0^+} \frac{h(\zeta)}{|\zeta|} = +\infty,$$

guarantees that h exhibits subcritical growth.

In the upcoming section, we illustrate how the preceding results can be applied to move from the existence of at least 1 nontrivial solution to the existence of at least two nontrivial solutions. This goal will be achieved by exploiting the particular nature of the first solution found, which serves as a local minimum. This property will be crucial in demonstrating the existence of a second solution, characterized as a critical point of mountain pass type. To facilitate this, we start with the following theorem that necessitates the well-known Ambrosetti-Rabinowitz condition. As is standard practice, this assumption is essential for establishing that every Palais-Smale sequence is bounded and for confirming that the so-called "mountain pass geometry" is satisfied.

Theorem 5. *Let $h : \mathbb{R} \to \mathbb{R}$ are continuous functions such that*

$$|h(\epsilon)| \leq a_1 + a_2|\epsilon| \quad \textit{for each} \epsilon \in \mathbb{R}. \tag{23}$$

Moreover, we assume that condition (22) *is satisfied in addition to this*

(AR) *there are constants $\mu > 2$ and $r > 0$ such that, for every $|\zeta| \geq r$, the following holds*

$$0 < \mu H(\zeta) \leq \zeta h(\zeta).$$

Then for each $\gamma \in (0, \gamma_\theta^\star)$, the problem described in (3) *has at least two generalized solutions. Furthermore, if $h(0) \neq 0$, it is ensured that these solutions are non-zero.*

Proof. Fix $\gamma \in \Lambda_\Omega$. Utilizing equations (22) and (23), Theorem 4 guarantees that the problem presented in (3) has at least one weak non-zero solution z_1 which serves as a local minimum of the functional Γ_γ as outlined in the proof of Theorem 3. We can further assume that z_1 is a strict local minimum for Γ_γ in E. Consequently, there exists $\phi > 0$ such that

$$\inf_{\|z-z_1\|_{\iota,2}=\phi} \Gamma_\gamma(z) > \Gamma_\gamma(z_1).$$

Furthermore, by applying the (AR)-condition and performing standard calculations, we find that Γ_γ is unbounded from below. Consequently, there exists a z_2 such that $\Gamma_\gamma(z_2) < \Gamma_\gamma(z_1)$, indicating that Γ_γ exhibits mountain pass geometry. At this stage, again utilizing the (AR) condition, we conclude that the functional Γ_γ satisfies the (PS)-condition. As a result, the Ambrosetti-Rabinowitz theorem ensures the existence of a critical point $\tilde{z}$ for Γ_γ such that $\Gamma_\gamma(\tilde{z}) > \Gamma_\gamma(z_1)$. Therefore, z_1 and $\tilde{v}$ are two distinct generalized solutions to the problem (3). □

In conclusion, we will now provide the following example to illustrate Theorem 5.

Example 3. *Consider the problem*

$$\begin{cases} -z'' + z' - (\varsigma)z = \gamma h(z(\varsigma)), & \varsigma \in (0,1), \\ z(0) = z(1) = 0. \end{cases} \tag{24}$$

Then for each $\gamma \in (0, +\infty)$, *the the problem* (24) *admits at least two non-zero generalized solutions.*

Proof. Let $h(\zeta) = 2 + 2\zeta$ for every $\zeta \in \mathbb{R}$. Then h satisfies (23) and, since

$$\lim_{\zeta \to 0^+} \frac{h(\zeta)}{\zeta} = +\infty,$$

additionally, condition (22) is satisfied. Furthermore, considering that

$$\lim_{|\zeta|\to\infty} \frac{\zeta h(\zeta)}{H(\zeta)} = \lim_{|\zeta|\to\infty} \frac{2\zeta + 2\zeta^2}{2\zeta + \zeta^2} = 2,$$

there exist $\mu > 2$ and $r > 0$ such that $0 < \mu H(\zeta) \leq \zeta h(\zeta)$ for each $|\zeta| > r$. Hence, the conclusion is derived from Theorem 5. □

Acknowledgments and Funding

This work was funded by European Union under the NextGeneration EU Programme within the Plan "PNRR - Missione 4 "Istruzione e Ricerca" - Componente C2 Investimento 1.1 "Fondo per il Programma Nazionale di Ricerca e Progetti di Rilevante Interesse Nazionale (PRIN)" by the Italian Ministry of University and Research (MUR), Project title: "Climate risk and uncertainty: environmental sustainability and asset pricing". Project code "P20225MJW8" (CUP: E53D23016470001), MUR D.D. financing decree n. 1409 of 14/09/2022. The authors are grateful to the four referees for providing valuable suggestions.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The authors equally contributed in the present research, at all stages from the formulation of the problem to the final findings and solution.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

This work was funded by European Union under the NextGeneration EU Programme within the Plan "PNRR - Missione 4 "Istruzione e Ricerca" - Componente C2 Investimento 1.1 "Fondo per il Programma Nazionale di Ricerca e Progetti di Rilevante Interesse Nazionale (PRIN)" by the Italian Ministry of University and Research (MUR), Project title: "Climate risk and uncertainty: environmental sustainability and asset pricing". Project code "P20225MJW8" (CUP: E53D23016470001), MUR D.D. financing decree n. 1409 of 14/09/2022.

Conflict of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.