A New $\mathbb{D}$-Normed Banach Bicomplex $\mathbb{BC}$-Module Derived Using Matrix of Hyperbolic Fibonacci Numbers

NİLAY DEĞİRMEN, BİRSEN SAĞIR
Department of Mathematics
Ondokuz Mayıs University
Samsun
TURKEY

Abstract: In this paper, we present a new matrix $\widetilde{F} = \left(\widetilde{F}_{nm}\right)_{n,m=1}^{\infty}$ consisting hyperbolic Fibonacci numbers. Also, we construct new $\mathbb{D}$-normed Banach bicomplex $\mathbb{BC}$-modules $l_p^k\left(\mathbb{BC},\widetilde{F}\right)$ and $l_\infty^k\left(\mathbb{BC},\widetilde{F}\right)$ using this matrix. Moreover, we demonstrate that these new $\mathbb{BC}$-modules are isometrically isomorphic to $l_p^k(\mathbb{BC})$ and $l_\infty^k(\mathbb{BC})$, respectively. We further study some inclusion theorems for this new type.

Key-Words: Bicomplex number, sequence space, hyperbolic Fibonacci number, inclusion relation, $\mathbb{BC}$-linearity, bicomplex $\mathbb{BC}$-module

Received: September 16, 2024. Revised: January 5, 2025. Accepted: February 9, 2025. Published: April 2, 2025.

1 Introduction and Preliminaries

The bicomplex numbers are first introduced in 1892, [1]. These numbers generalize and extend the complex numbers more firmly and specifically to quaternions. An excellent investigation of the bicomplex space and related context is given in [2]. The authors in [3], have developed the bicomplex version of functional analysis with complex scalars and it was the next significant push in subsequent studies on theory of functions with bicomplex variables.

Now, we introduce a basic review of bicomplex numbers which are necessary for the understanding of this work. Further, we refer to the books, [2], [3], [4], for more detailed knowledge. If we give examples of recent studies on this subject, these are [5], [6].

The set of bicomplex numbers, denoted by $\mathbb{BC}$, consists of the numbers that can be written in the form $\zeta = z_1 + jz_2$ with aid of two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, where i and j are commuting imaginary units, i.e. $ij = ji$, $i^2 = -1$ and $j^2 = -1$. $\mathbb{BC}$ is a commutative algebraic ring and a $\mathbb{BC}-$module with respect to the operations defined on it, as not all bicomplex numbers have a multiplicative inverse. But a so-called idempotent representation

$$\zeta = (z_1 - jz_2)\, e_1 + (z_1 + jz_2)\, e_2 = \zeta_1 e_1 + \zeta_2 e_2$$

importantly gives a better way to think about the bicomplex numbers, and associated mathematical structures. Here $e_1 = \frac{1+ij}{2}$ and $e_2 = \frac{1-ij}{2}$, and $\zeta_1, \zeta_2 \in \mathbb{C}$ are called as idempotent components of $\zeta \in \mathbb{BC}$. Obviously, the equalities $e_1^2 = e_1$, $e_2^2 = e_2$ and $e_1e_2 = e_2e_1 = 0$ are satisfied.

The set of hyperbolic numbers $\mathbb{D}$ is defined by $\mathbb{D} = \{x + ky : x, y \in \mathbb{R}, k = ij\}$. For $\alpha = x + ky \in \mathbb{D}$, we have the equality $\alpha = e_1\alpha_1 + e_2\alpha_2$, where $\alpha_1 = x+y, \alpha_2 = x-y \in \mathbb{R}$. If $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$, then α is called a positive hyperbolic number. Therefore, the set of positive hyperbolic numbers $\mathbb{D}^+$ is denoted by $\mathbb{D}^+ = \{\alpha = e_1\alpha_1 + e_2\alpha_2 : \alpha_1 \geq 0, \alpha_2 \geq 0\}$. For two hyperbolic numbers α and β; if their difference $\beta - \alpha \in \mathbb{D}^+$, then we write $\alpha \precsim \beta$. For $\alpha = e_1\alpha_1 + e_2\alpha_2$, $\beta = \beta_1 e_1 + \beta_2 e_2 \in \mathbb{D}$ with real numbers $\alpha_1, \alpha_2, \beta_1$ and β_2, we have that $\alpha \precsim \beta$ if and only if $\alpha_1 \leq \beta_1$ and $\alpha_2 \leq \beta_2$. In addition, this relation $\precsim$ defines a partial order on $\mathbb{D}$.

Let X be a $\mathbb{BC}-$module. A map $\|.\|_{\mathbb{D}} : X \to \mathbb{D}^+$ is said to be a hyperbolic valued norm on X if it satisfies the following properties:

(i) $\|x\|_{\mathbb{D}} = 0$ if and only if $x = 0$.

(ii) $\|\mu x\|_{\mathbb{D}} = |\mu|_k \cdot \|x\|_{\mathbb{D}}$ for all $x \in X$, $\mu \in \mathbb{BC}$, where $|.|_k$ is a module defined as $|z|_k^2 = \left(|z_1|^2 + |z_2|^2\right) + k\left(-Im\left(z_1.\overline{z_2}\right)\right) \in \mathbb{D}$.

(iii) $\|x + y\|_{\mathbb{D}} \precsim \|x\|_{\mathbb{D}} + \|y\|_{\mathbb{D}}$ for all $x, y \in X$.

The following statements are true for any $\alpha, \beta, \gamma \in \mathbb{D}, z, w \in \mathbb{BC}$:

(i) $\mathbb{BC}$ is complete hyperbolic valued normed space with respect to $|.|_k$.

(ii) If $z = \beta_1 e_1 + \beta_2 e_2$ is idempotent representation of z, then $|z|_k = |\beta_1|\, e_1 + |\beta_2|\, e_2$.

(iii) $\left|\frac{z}{w}\right|_k = \frac{|z|_k}{|w|_k}$ and if $\alpha \in \mathbb{D}^+$, then $|\alpha|_k = \alpha$.

Let $A \subset \mathbb{D}$. If A is a set $\mathbb{D}-$bounded from above, we define the notion of its $\mathbb{D}-$supremum, denoted by $\sup_{\mathbb{D}} A$. The least $\mathbb{D}-$upper bound here means that $\sup_{\mathbb{D}} A \precsim \alpha$ for any $\mathbb{D}-$upper bound α even if not all of the $\mathbb{D}-$upper bounds are comparable. Consider the sets $A_1 = \{a_1 : a_1e_1 + a_2e_2 \in A\}$ and $A_2 = \{a_2 : a_1e_1 + a_2e_2 \in A\}$. If A is $\mathbb{D}-$bounded from above, then $\sup_{\mathbb{D}} A = \sup A_1 e_1 + \sup A_2 e_2$.

Fibonacci numbers are the terms of the integer sequence

$$\{1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, ..., F_n, ...\}$$

defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$ for each $n \in \{3, 4, ...\}$ with $F_1 = 1, F_1 = 1$. Also, $\lim\limits_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2}$.

The theory of sequence spaces has always been of great interest in the study on summability which has applications in many different fields such as functional anaylsis, numerical analsis, approximation theory, the theory of orthogonal series. Special theorems and results in summability theory motivated the authors to study various sequence spaces and their geometric properties. There are also sequence spaces constructed using some special matrices. In [7], the author studied superposition operators on some sequence spaces obtained with a Fibonacci matrix. In [8], the authors introduced the sequence space $l_p(F)$ using a regular matrix of Fibonacci numbers and the discussed its topological and geometric properties. In addition, [9], is a good resource of containing infinite systems of differential equations in some Banach spaces derived by Fibonacci numbers. It will be a pivotal attempt to apply this work in the future to the current article we obtained using hyperbolic Fibonacci numbers.

On the other hand, hyperbolic Fibonacci numbers are defined by the recurrence relation

$$\widetilde{F}_n = F_n + kF_{n+1}$$

with $\widetilde{F}_1 = 1 + k$, $\widetilde{F}_2 = 1 + 2k$ where $k^2 = 1$, [10].

It is worth noting that $\sum\limits_{m=0}^{n} \widetilde{F}_{m+1} = \widetilde{F}_{n+3} - 1 - 2k$.

Inspired by these studies, we extend the sequence space $l_p(F)$, where Fibonacci numbers are used, to the bicomplex case by defining a matrix of hyperbolic Fibonacci numbers and examine some properties of new type space.

2 Main Results

We will give our findings in this section.

In the first one, we define a bicomplex matrix consisting hyperbolic Fibonacci numbers denoted by $\widetilde{F} = \left(\widetilde{F}_{nm}\right)_{n,m=1}^{\infty}$ where

$$\widetilde{F}_{nm} = \begin{cases} \frac{\widetilde{F}_m}{\widetilde{F}_{n+2}-1-2k}, & 1 \le m \le n \\ 0, & n > m \end{cases}.$$

If the idempotent representation $\widetilde{F}_n = (F_n + F_{n+1})e_1 + (F_n - F_{n+1})e_2 = F_{n+2}e_1 - F_{n-1}e_2$ is used, the bicomplex matrix also can be represented as

$$\widetilde{F}_{nm} = \begin{cases} \frac{F_{m+2}}{F_{n+4}-3}e_1 - \frac{F_{m-1}}{F_{n+1}+1}e_2, & 1 \le m \le n \\ 0, & n > m \end{cases}.$$

In [11], the spaces $l_p^k(\mathbb{BC})$ are established and geometric properties of them are investigated. Now, by using these spaces, we introduce a new space for $0 < p \le \infty$ as follows:

$$l_p^k\left(\mathbb{BC},\widetilde{F}\right) = \left\{\phi = (\phi_n) \in w(\mathbb{BC}) : \widetilde{F}\phi \in l_p^k(\mathbb{BC})\right\},$$

where $\widetilde{F}\phi = \left\{\widetilde{F}_n(\phi)\right\}_{n=0}^{\infty}$ is $\widetilde{F}$-transform of ϕ and $\widetilde{F}_n(\phi) = \sum\limits_{m=0}^{n} \frac{\widetilde{F}_{m+1}}{\widetilde{F}_{n+3}-1-2k}\phi_m$ for all $n \in \mathbb{N}$.

Theorem 1 *The set $l_p^k\left(\mathbb{BC},\widetilde{F}\right)$ is a $\mathbb{BC}-$submodule of $w(\mathbb{BC})$ for $0 < p \le \infty$.*

Proof: It follows from the definition of the set $l_p^k\left(\mathbb{BC},\widetilde{F}\right)$ and the properties of operations in it.

Theorem 2 *For $0 < p < 1$, the set $l_p^k\left(\mathbb{BC},\widetilde{F}\right)$ is a $p_{\mathbb{D}}-$normed Banach bicomplex $\mathbb{BC}-$module with respect to the hyperbolic valued norm defined by*

$$\|\phi\|_{l_p^k\left(\mathbb{BC},\widetilde{F}\right)} = \sum_{n=1}^{\infty} \left|\widetilde{F}_n(\phi)\right|_k^p.$$

Proof: The proof depends on the definitions of the $\mathbb{BC}-$module $l_p^k\left(\mathbb{BC},\widetilde{F}\right)$ and $p_{\mathbb{D}}-$norm given in [11]. We only show the distinguishing feature of the $p_{\mathbb{D}}-$norm for $\|.\|_{l_p^k\left(\mathbb{BC},\widetilde{F}\right)}$ that differs from the classical norm, as follows:

$$\begin{aligned}\|\lambda\phi\|_{l_p^k\left(\mathbb{BC},\widetilde{F}\right)} &= \sum_{n=0}^{\infty}\left|\widetilde{F}_n(\lambda\phi)\right|_k^p \\ &= \sum_{n=0}^{\infty}\left(\left|\sum_{m=0}^{n}\frac{\widetilde{F}_{n+1}}{\widetilde{F}_{n+3}-1-2k}\lambda\phi_m\right|_k\right)^p \\ &= |\lambda|_k^p\sum_{n=0}^{\infty}\left(\left|\sum_{m=0}^{n}\frac{\widetilde{F}_{n+1}}{\widetilde{F}_{n+3}-1-2k}\phi_m\right|_k\right)^p \\ &= |\lambda|_k^p\,\|\phi\|_{l_p^k\left(\mathbb{BC},\widetilde{F}\right)},\end{aligned}$$

for all $\lambda \in \mathbb{BC}$, $\phi \in l_p^k\left(\mathbb{BC},\widetilde{F}\right)$.

Theorem 3 *For $1 \leq p < \infty$, the set $l_p^k\left(\mathbb{BC},\widetilde{F}\right)$ is a $\mathbb{D}-$normed Banach bicomplex $\mathbb{BC}-$module with respect to the hyperbolic valued norm defined by*

$$\|\phi\|_{l_p^k\left(\mathbb{BC},\widetilde{F}\right)} = \left(\sum_{n=0}^{\infty}\left|\widetilde{F}_n(\phi)\right|_k^p\right)^{\frac{1}{p}}.$$

Proof: As a result of some needed calculations, one can easily see that $l_p^k\left(\mathbb{BC},\widetilde{F}\right)$ is a $\mathbb{D}-$normed Banach bicomplex $\mathbb{BC}-$module. We only prove the property of triangle inequality as follows:

$$\begin{aligned}&\|\phi+\psi\|_{l_p^k\left(\mathbb{BC},\widetilde{F}\right)} = \left(\sum_{n=0}^{\infty}\left|\widetilde{F}_n(\phi+\psi)\right|_k^p\right)^{\frac{1}{p}} \\ &= \left(\sum_{n=0}^{\infty}\left|\sum_{m=0}^{n}\frac{\widetilde{F}_{n+1}}{\widetilde{F}_{n+3}-1-2k}(\phi_m+\psi_m)\right|_k^p\right)^{\frac{1}{p}} \\ &= \left(\sum_{n=0}^{\infty}\left|\sum_{m=0}^{n}\frac{\widetilde{F}_{n+1}}{\widetilde{F}_{n+3}-1-2k}\phi_m+\sum_{m=0}^{n}\frac{\widetilde{F}_{n+1}}{\widetilde{F}_{n+3}-1-2k}\psi_m\right|_k^p\right)^{\frac{1}{p}} \\ &\precsim \left(\sum_{n=0}^{\infty}\left|\widetilde{F}_n(\phi)\right|_k^p\right)^{\frac{1}{p}} + \left(\sum_{n=0}^{\infty}\left|\widetilde{F}_n(\psi)\right|_k^p\right)^{\frac{1}{p}} \\ &= \|\phi\|_{l_p^k\left(\mathbb{BC},\widetilde{F}\right)} + \|\psi\|_{l_p^k\left(\mathbb{BC},\widetilde{F}\right)},\end{aligned}$$

for all $\phi,\psi \in l_p^k\left(\mathbb{BC},\widetilde{F}\right)$ by utilizing Lemma 2.5 in [11].

Theorem 4 *The set $l_\infty^k\left(\mathbb{BC},\widetilde{F}\right)$ is a $\mathbb{D}-$normed Banach bicomplex $\mathbb{BC}-$module with respect to the hyperbolic valued norm defined by*

$$\|\phi\|_{l_\infty^k\left(\mathbb{BC},\widetilde{F}\right)} = \sup_{\mathbb{D}}\left\{\left|\widetilde{F}_n(\phi)\right|_k : n \in \mathbb{N}\right\}.$$

Proof: The proof is a direct application of the definition of $\mathbb{D}-$supremum.

Theorem 5 *The bicomplex $\mathbb{BC}-$module $l_p^k\left(\mathbb{BC},\widetilde{F}\right)$ is isometrically isomorphic to the bicomplex $\mathbb{BC}-$module $l_p^k(\mathbb{BC})$ for $0 < p \leq \infty$.*

Proof: Consider the mapping T defined by $T : l_p^k\left(\mathbb{BC},\widetilde{F}\right) \to l_p^k(\mathbb{BC})$, $\phi \to T\phi = \widetilde{F}(\phi)$. It is trivial that T is $\mathbb{BC}$-linear, injective and $\mathbb{BC}$-isometry, that is,

$T(\phi+\lambda\psi) = T(\phi) + \lambda T(\psi)$ for all $\phi,\psi \in l_p^k\left(\mathbb{BC},\widetilde{F}\right)$, $\lambda \in \mathbb{BC}$,

$\phi \neq \psi$ implies $T(\phi) \neq T(\psi)$,

$\|T\phi\|_{l_p^k(\mathbb{BC})} = \left\|\widetilde{F}\phi\right\|_{l_p^k(\mathbb{BC})} = \|\phi\|_{l_p^k\left(\mathbb{BC},\widetilde{F}\right)}$ for all $\phi \in l_p^k\left(\mathbb{BC},\widetilde{F}\right)$.

So, the bicomplex $\mathbb{BC}-$module $l_p^k\left(\mathbb{BC},\widetilde{F}\right)$ is isometrically isomorphic to the bicomplex $\mathbb{BC}-$module $l_p^k(\mathbb{BC})$ for $0 < p \leq \infty$.

Lemma 1 *For the hyperbolic Fibonacci sequence $\left\{\widetilde{F}_n\right\}$ we have*

$$\sup_{\mathbb{D}}\left\{\widetilde{F}_{m+1}\sum_{n=m}^{\infty}\frac{1}{\widetilde{F}_{n+3}} : m \in \mathbb{N}\right\} \text{ is finite.}$$

Proof: For the proof, we will use Lemma 4.11 in [12], which is stated below:

Let $\lambda = (\lambda_k)_{k=0}^{\infty}$ is a strictly increasing sequence of positive reals tending to ∞, that is $0 < \lambda_0 < \lambda_1 < \ldots$ and $\lambda_k \to \infty$ as $k \to \infty$. If $\frac{1}{\lambda} \in l_1$, then

$$\sup_k\left((\lambda_k-\lambda_{k-1})\sum_{n=k}^{\infty}\frac{1}{\lambda_n}\right) < \infty.$$

We adapt this lemma to the space $l_1^k(\mathbb{BC})$ generated according to the hyperbolic valued norm in [11], and claim the following expression:

Let $w = (w_m)_{m=0}^{\infty}$ is a strictly increasing sequence of positive hyperbolic numbers tending to ∞, that is $0 \prec w_0 \prec w_1 \prec \ldots$ and $w_{m,1} \to \infty$, $w_{m,2} \to \infty$ as $m \to \infty$ where $w_m = w_{m,1}e_1 + w_{m,2}e_2$. If $\frac{1}{w} \in l_1^k(\mathbb{BC})$, then

$$\sup_{\mathbb{D}}\left\{(w_m-w_{m-1})\sum_{n=m}^{\infty}\frac{1}{w_n} : m \in \mathbb{N}\right\} \text{ is finite.}$$

Indeed, if $\frac{1}{w} \in l_1^k(\mathbb{BC})$, then $\sum_{n=1}^{\infty}\left|\frac{1}{w_n}\right|_k$ is convergent. This implies that the series $\sum_{n=1}^{\infty}\left|\frac{1}{w_{n,1}}\right|$ and $\sum_{n=1}^{\infty}\left|\frac{1}{w_{n,2}}\right|$ are convergent. Then, we write $\left(\frac{1}{w_{n,1}}\right), \left(\frac{1}{w_{n,2}}\right) \in l_1$ and so

$$\sup_m\left((w_{m,1}-w_{m-1,1})\sum_{n=m}^{\infty}\frac{1}{w_{n,1}}\right) < \infty \text{ and}$$

$\sup_{m}\left(\left(w_{m,2}-w_{m-1,2}\right)\sum_{n=m}^{\infty}\frac{1}{w_{n,2}}\right)<\infty.$

Using idempotent representations of the sequence (w_m) and the series $\sum_{n=m}^{\infty}\frac{1}{w_n}$ and the idempotent decomposition of $\mathbb{D}-$supremum, we get

$$\begin{aligned}&\sup_{\mathbb{D}}\left\{\left(w_m-w_{m-1}\right)\sum_{n=m}^{\infty}\frac{1}{w_n}:m\in\mathbb{N}\right\}\\&=\left[\sup_{m}\left(\left(w_{m,1}-w_{m-1,1}\right)\sum_{n=m}^{\infty}\frac{1}{w_{n,1}}\right)\right]e_1\\&+\left[\sup_{m}\left(\left(w_{m,2}-w_{m-1,2}\right)\sum_{n=m}^{\infty}\frac{1}{w_{n,2}}\right)\right]e_2.\end{aligned}$$

This equality produces the desired result.

Now let's return to our own lemma. For the hyperbolic Fibonacci sequence $\left\{\widetilde{F}_n\right\}$ if we use the idempotent representation $\widetilde{F}_n=F_{n+2}e_1+\left(-F_{n-1}\right)e_2$, it is easy to note that

$$\sum_{n=1}^{\infty}\left|\frac{1}{\widetilde{F}_n}\right|_k=\left(\sum_{n=1}^{\infty}\frac{1}{F_{n+2}}\right)e_1+\left(\sum_{n=1}^{\infty}\frac{1}{F_{n-1}}\right)e_2.$$

Since $\lim_{n\to\infty}\frac{F_{n+1}}{F_n}=\frac{1+\sqrt{5}}{2}$, we obtain that

$$\lim_{n\to\infty}\frac{\frac{1}{F_{n+3}}}{\frac{1}{F_{n+2}}}=\lim_{n\to\infty}\frac{F_{n+2}}{F_{n+3}}=\frac{2}{1+\sqrt{5}}<1,$$

$$\lim_{n\to\infty}\frac{\frac{1}{F_n}}{\frac{1}{F_{n-1}}}=\lim_{n\to\infty}\frac{F_{n-1}}{F_n}=\frac{2}{1+\sqrt{5}}<1,$$

and so the series $\sum_{n=1}^{\infty}\frac{1}{F_{n+2}}$ and $\sum_{n=1}^{\infty}\frac{1}{F_{n-1}}$ converge. This means that $\left(\frac{1}{\widetilde{F}_n}\right)\in l_1^k(\mathbb{BC})$. According to the claim we proved above, $\sup_{\mathbb{D}}\left\{\left(\widetilde{F}_m-\widetilde{F}_{m-1}\right)\sum_{n=m}^{\infty}\frac{1}{\widetilde{F}_n}:m\in\mathbb{N}\right\}$ is finite. Since

$$\begin{aligned}&\left\{\widetilde{F}_{m+1}\sum_{n=m}^{\infty}\frac{1}{\widetilde{F}_{n+3}}:m\in\mathbb{N}\right\}\\&=\left\{\left(\widetilde{F}_{m+3}-\widetilde{F}_{m+2}\right)\sum_{n=m}^{\infty}\frac{1}{\widetilde{F}_{n+3}}:m\in\mathbb{N}\right\}\\&\subset\left\{\left(\widetilde{F}_m-\widetilde{F}_{m-1}\right)\sum_{n=m}^{\infty}\frac{1}{\widetilde{F}_n}:m\in\mathbb{N}\right\},\end{aligned}$$

$\sup_{\mathbb{D}}\left\{\widetilde{F}_{m+1}\sum_{n=m}^{\infty}\frac{1}{\widetilde{F}_{n+3}}:m\in\mathbb{N}\right\}$ is also finite. The proof is completed.

We continue this section with some inclusion relations of $l_p^k(\mathbb{BC})$ and $l_p^k\left(\mathbb{BC},\widetilde{F}\right)$ for $1\leq p\leq\infty$.

Theorem 6 *If $1\leq p\leq\infty$, then the inclusion relation $l_p^k(\mathbb{BC})\subset l_p^k\left(\mathbb{BC},\widetilde{F}\right)$ holds.*

Proof: Let $p=\infty$. Assume that $(\phi_n)\in l_\infty^k(\mathbb{BC})$. Then, $M=\sup_{\mathbb{D}}\{|\phi_n|_k:n\in\mathbb{N}\}$ is finite. Hence, we deduce that

$$\begin{aligned}&\left|\widetilde{F}_n(\phi)\right|_k=\left|\sum_{m=0}^{n}\frac{\widetilde{F}_{m+1}}{\widetilde{F}_{n+3}-1-2k}\phi_m\right|_k\\&\precsim\sup_{\mathbb{D}}\{|\phi_m|_k:m\in\mathbb{N}\}\sum_{m=0}^{n}\frac{\widetilde{F}_{m+1}}{\widetilde{F}_{n+3}-1-2k}\\&=\sup_{\mathbb{D}}\{|\phi_m|_k:m\in\mathbb{N}\}\frac{\sum_{m=0}^{n}\widetilde{F}_{m+1}}{\widetilde{F}_{n+3}-1-2k}\\&=\sup_{\mathbb{D}}\{|\phi_m|_k:m\in\mathbb{N}\}=M\end{aligned}$$

for all $n\in\mathbb{N}$ and so $\widetilde{F}\phi\in l_\infty^k(\mathbb{BC})$ and $(\phi_n)\in l_\infty^k\left(\mathbb{BC},\widetilde{F}\right)$. This yields the inclusion relation $l_\infty^k(\mathbb{BC})\subset l_\infty^k\left(\mathbb{BC},\widetilde{F}\right)$.

Let $p=1$. Suppose that $(\phi_n)\in l_1^k(\mathbb{BC})$. This means that $\sum_{n=0}^{\infty}|\phi_n|_k$ converges. Thus, we conclude that

$$\begin{aligned}&\sum_{n=0}^{\infty}\left|\widetilde{F}_n(\phi)\right|_k=\sum_{n=0}^{\infty}\left|\sum_{m=0}^{n}\frac{\widetilde{F}_{m+1}}{\widetilde{F}_{n+3}-1-2k}\phi_m\right|_k\\&\precsim\sum_{n=0}^{\infty}\frac{1}{\widetilde{F}_{n+3}-1-2k}\sum_{m=0}^{n}\widetilde{F}_{m+1}|\phi_m|_k\\&=\sum_{m=0}^{\infty}\widetilde{F}_{m+1}|\phi_m|_k\sum_{n=m}^{\infty}\frac{1}{\widetilde{F}_{n+3}-1-2k}\\&\precsim\sup_{\mathbb{D}}\left\{\widetilde{F}_{m+1}\sum_{n=m}^{\infty}\frac{1}{\widetilde{F}_{n+3}-1-2k}:m\in\mathbb{N}\right\}\sum_{m=0}^{\infty}|\phi_m|_k.\end{aligned}$$

Since $\sup_{\mathbb{D}}\left\{\widetilde{F}_{m+1}\sum_{n=m}^{\infty}\frac{1}{\widetilde{F}_{n+3}-1-2k}:m\in\mathbb{N}\right\}$ is finite by Lemma 1 and $(\phi_n)\in l_1^k(\mathbb{BC})$, we obtain that $\sum_{n=0}^{\infty}\left|\widetilde{F}_n(\phi)\right|_k$ is finite. So, $\widetilde{F}\phi\in l_1^k(\mathbb{BC})$ and $(\phi_n)\in l_1^k\left(\mathbb{BC},\widetilde{F}\right)$ as a desired result.

Let $1<p<\infty$ and q be a real number such that $\frac{1}{p}+\frac{1}{q}=1$. Assuming $(\phi_n)\in l_p^k(\mathbb{BC})$ we note that $\sum_{n=0}^{\infty}|\phi_n|_k^p$ converges. By applying Lemma 2.4 in [11], it follows

$$\begin{aligned}&\left|\widetilde{F}_n(\phi)\right|_k^p=\left|\sum_{m=0}^{n}\frac{\widetilde{F}_{m+1}}{\widetilde{F}_{n+3}-1-2k}\phi_m\right|_k^p\\&\precsim\left(\sum_{m=0}^{n}\frac{\widetilde{F}_{m+1}}{\widetilde{F}_{n+3}-1-2k}|\phi_m|_k\right)^p\\&=\left(\sum_{m=0}^{n}\left[\left(\frac{\widetilde{F}_{m+1}}{\widetilde{F}_{n+3}-1-2k}\right)^{\frac{1}{q}}\right]\left[\left(\frac{\widetilde{F}_{m+1}}{\widetilde{F}_{n+3}-1-2k}\right)^{\frac{1}{p}}|\phi_m|_k\right]\right)^p\\&\precsim\left(\sum_{m=0}^{n}\frac{\widetilde{F}_{m+1}}{\widetilde{F}_{n+3}-1-2k}\right)^{\frac{p}{q}}\left(\sum_{m=0}^{n}\frac{\widetilde{F}_{m+1}}{\widetilde{F}_{n+3}-1-2k}|\phi_m|_k^p\right)\\&=\left(\sum_{m=0}^{n}\frac{\widetilde{F}_{m+1}}{\widetilde{F}_{n+3}-1-2k}\right)^{p-1}\frac{1}{\widetilde{F}_{n+3}-1-2k}\sum_{m=0}^{n}\widetilde{F}_{m+1}|\phi_m|_k^p\end{aligned}$$

$= \frac{1}{\widetilde{F}_{n+3}-1-2k} \sum_{m=0}^{n} \widetilde{F}_{m+1} |\phi_m|_k^p$.

So, it becomes clear that

$$\sum_{n=0}^{\infty} \left|\widetilde{F}_n(\phi)\right|_k^p \precsim \sum_{n=0}^{\infty} \left[\frac{1}{\widetilde{F}_{n+3}-1-2k} \sum_{m=0}^{n} \widetilde{F}_{m+1} |\phi_m|_k^p\right]$$
$$= \sum_{m=0}^{\infty} \widetilde{F}_{m+1} |\phi_m|_k^p \sum_{n=m}^{\infty} \frac{1}{\widetilde{F}_{n+3}-1-2k}$$
$$\precsim \sup_{\mathbb{D}} \left\{ \widetilde{F}_{m+1} \sum_{n=m}^{\infty} \frac{1}{\widetilde{F}_{n+3}-1-2k} : m \in \mathbb{N} \right\} \sum_{m=0}^{\infty} |\phi_m|_k^p.$$

This gives rise to the fact $(\phi_n) \in l_p^k\left(\mathbb{BC},\widetilde{F}\right)$, so $l_p^k(\mathbb{BC}) \subset l_p^k\left(\mathbb{BC},\widetilde{F}\right)$.

Theorem 7 *If* $0 < p < \infty$, *then the inclusion relation* $l_p^k\left(\mathbb{BC},\widetilde{F}\right) \subset l_\infty^k\left(\mathbb{BC},\widetilde{F}\right)$ *strictly holds.*

Proof: Assume that $(\phi_n) \in l_p^k\left(\mathbb{BC},\widetilde{F}\right)$. Then, we have $\widetilde{F}\phi \in l_p^k(\mathbb{BC})$ and so $\widetilde{F}\phi \in l_\infty^k(\mathbb{BC})$. This readily follows using the inclusion $l_p^k(\mathbb{BC}) \subset l_\infty^k(\mathbb{BC})$. Hence, by Theorem 6, $(\phi_n) \in l_p^k\left(\mathbb{BC},\widetilde{F}\right)$ and hence the result.

Setting $\phi_n = \frac{1+k}{(1+n)^{\frac{1}{p}}} = \frac{2e_1}{(1+n)^{\frac{1}{p}}}$ for all $n \in \mathbb{N}$, we have $\sup_{\mathbb{D}} \{|\phi_n|_k : n \in \mathbb{N}\} = 2e_1$ is finite and so $(\phi_n) \in l_\infty^k(\mathbb{BC})$. Thus, Theorem 6 yields $(\phi_n) \in l_\infty^k\left(\mathbb{BC},\widetilde{F}\right)$.

However, since

$$\left|\widetilde{F}_n(\phi)\right|_k^p = \left|\sum_{m=0}^{n} \frac{\widetilde{F}_{m+1}}{\widetilde{F}_{n+3}-1-2k} \frac{1+k}{(1+m)^{\frac{1}{p}}}\right|_k^p$$
$$= \left(\sum_{m=0}^{n} \frac{\widetilde{F}_{m+1}}{\widetilde{F}_{n+3}-1-2k} \frac{1+k}{(1+m)^{\frac{1}{p}}}\right)^p$$
$$\succsim \left(\frac{1+k}{\widetilde{F}_{n+3}-1-2k} \frac{1}{(1+n)^{\frac{1}{p}}} \sum_{m=0}^{n} \widetilde{F}_{m+1}\right)^p$$
$$= \frac{(1+k)^p}{1+n} = \frac{2^p}{1+n} e_1$$

and $\sum_{n=0}^{\infty} \frac{2^p}{1+n}$ is divergent, we get $\widetilde{F}\phi \notin l_p^k(\mathbb{BC})$ and $(\phi_n) \notin l_p^k\left(\mathbb{BC},\widetilde{F}\right)$. That is to say that the inclusion $l_p^k\left(\mathbb{BC},\widetilde{F}\right) \subset l_\infty^k\left(\mathbb{BC},\widetilde{F}\right)$ is strict.

Theorem 8 *If* $0 < p < q < \infty$, *then* $l_p^k\left(\mathbb{BC},\widetilde{F}\right) \subset l_q^k\left(\mathbb{BC},\widetilde{F}\right)$ *strictly holds.*

Proof: By assumption $(\phi_n) \in l_p^k\left(\mathbb{BC},\widetilde{F}\right)$, we can write $\widetilde{F}\phi \in l_p^k(\mathbb{BC})$. Since $l_p^k(\mathbb{BC}) \subset l_q^k(\mathbb{BC})$ obviously for $0 < p < q < \infty$, it follows that $\widetilde{F}\phi \in l_q^k(\mathbb{BC})$. Therefore, the proof of the inclusion $l_p^k\left(\mathbb{BC},\widetilde{F}\right) \subset l_q^k\left(\mathbb{BC},\widetilde{F}\right)$ is complete from Theorem 6.

Since the inclusion $l_p^k(\mathbb{BC}) \subset l_q^k(\mathbb{BC})$ is strict, there exists a bicomplex sequence $(w_n) \in l_q^k(\mathbb{BC})$ such that $(w_n) \notin l_p^k(\mathbb{BC})$. We consider a bicomplex sequence defined as follows:

$$\phi_m = \frac{w_m\left(\widetilde{F}_{n+3} - 1 - 2k\right) - w_{m-1}\left(\widetilde{F}_{n+2} - 1 - 2k\right)}{\widetilde{F}_{m+1}}$$

for all $m \in \mathbb{N}$.

We will show at this point that the bicomplex sequence (ϕ_n) is in $l_q^k\left(\mathbb{BC},\widetilde{F}\right)$ but not in $l_p^k\left(\mathbb{BC},\widetilde{F}\right)$. Indeed, we have

$$\widetilde{F}_n(\phi)$$
$$= \sum_{m=0}^{n} \frac{\widetilde{F}_{m+1}}{\widetilde{F}_{n+3}-1-2k} \frac{w_m\left(\widetilde{F}_{m+3}-1-2k\right) - w_{m-1}\left(\widetilde{F}_{m+2}-1-2k\right)}{\widetilde{F}_{m+1}}$$
$$= w_n$$

for all $n \in \mathbb{N}$. So, $\widetilde{F}\phi \notin l_p^k\left(\mathbb{BC},\widetilde{F}\right)$ and $\widetilde{F}\phi \in l_q^k\left(\mathbb{BC},\widetilde{F}\right)$ are valid. Equivalently, $(\phi_n) \in l_q^k\left(\mathbb{BC},\widetilde{F}\right) \setminus l_p^k\left(\mathbb{BC},\widetilde{F}\right)$ and the inclusion $l_p^k\left(\mathbb{BC},\widetilde{F}\right) \subset l_q^k\left(\mathbb{BC},\widetilde{F}\right)$ is strict.

3 Conclusion

In this work, we introduce a new sequence space $l_p^k\left(\mathbb{BC},\widetilde{F}\right)$ using hyperbolic Fibonacci numbers and we discuss some properties of them. It is a relatively new addition to the existing literature and generalizes some known results.

Acknowledgment:
The authors thank the referee for very careful reading of the manuscript and many valuable suggestions that improved the paper by much.

References:

[1] Segre, C., Le rappresentazioni reali delle forme complesse e gli enti iperalgebrici, *Mathematische Annalen*, 40(3), 1892, 413-467.

[2] Price, G. B., *An introduction to multicomplex spaces and functions*, Marcel Dekker Inc, New York, 1991.

[3] Alpay, D., Luna-Elizarrarás, M. E., Shapiro, M., Struppa, D. C., *Basics of functional analysis with bicomplex scalars, and bicomplex Schur analysis*, Springer Science & Business Media, 2014.

[4] Luna-Elizarrarás, M. E., Shapiro, M., Struppa, D. C., Vajiac, A., *Bicomplex holomorphic functions: the algebra, geometry and analysis of bicomplex numbers*, Birkhäuser, 2015.

[5] Eryılmaz, İ., A mean ergodic theorem in bicomplex Lebesgue spaces, *WSEAS Transactions on Mathematics*, 23, 2024, 51-59.

[6] Eryılmaz, İ., Composition operators between $\mathbb{BC}$-rearrangement invariant $\mathbb{BC}$-module spaces, *WSEAS Transactions on Mathematics*, 23, 2024, 438-445.

[7] Oğur, O., Superposition operators on sequence spaces $l_p(F)$ derived by using matrix of Fibonacci numbers, *Linear and Multilinear Algebra*, 68(10), 2020, 2087-2098.

[8] Ercan, S., Bektaş, Ç. A., Some topological and geometric properties of a new $BK-$space derived by using regular matrix of Fibonacci numbers, *Linear and Multilinear Algebra*, 65(5), 2017, 909-921.

[9] İlkhan, M., Kara, E.E., *Infinite systems of differential equations in Banach spaces constructed by Fibonacci numbers*, In: Mohiuddine, S., Acar, T. (eds) Advances in Summability and Approximation Theory. Springer, Singapore, 2018.

[10] Aydın, F. T., Hyperbolic Fibonacci sequence, *Universal Journal of Mathematics and Applications*, 2(2), 2019, 59-64.

[11] Değirmen, N., Sağır, B., On bicomplex $\mathbb{BC}-$modules $l_p^k(\mathbb{BC})$ and some of their geometric properties, *Georgian Mathematical Journal*, 30(1), 2023, 65-79.

[12] Mursaleen, M., Noman, A. K., On some new sequence spaces of non-absolute type related to the spaces l_p and l_∞ I, *Filomat*, 25(2), 2011, 33-51.

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The authors equally contributed in the presentresearch, at all stages from the formulation of theproblem to the final findings and solution.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

No funding was received for conducting this study.

Conflicts of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.