Generalization of the Nonlinear Bernoulli Conformable Fractional Differential Equations with Applications

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Abstract: In this work, we study a well-known nonlinear fractional differential equation—the nonlinear Bernoulli conformable fractional differential equation. We classify this equation into different categories and establish a fundamental lemma essential for proving our generalization. This generalization incorporates two methods: the Conformable Leibniz Method and the Conformable Bernoulli Method, both of which provide exact solutions for any nonlinear Bernoulli equation. Finally, we demonstrate the effectiveness of our approach by applying it to selected nonlinear Bernoulli conformable fractional differential equations, including a detailed numerical example.

Key-Words: Conformable derivative, conformable integral, conformable exponential function, Bernoulli equation, nonlinear equation, conformable Leibniz method, conformable Bernoulli method.

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1 Introduction

Fractional calculus extends classical calculus by allowing derivatives and integrals of non-integer orders, making it useful for modeling systems with memory and non-local effects in physics, engineering, and control theory; see, [1], [2], [3].

The differentiation operator $\frac{d}{dx}$ is well known to students who have studied ordinary calculus. For a positive integer n, the n^{th} derivative of a suitable function P, denoted as $D^n P(x) = \frac{d^n P(x)}{dx^n}$, is explicitly defined. The theory of derivatives of non-integer order originated from Leibniz's observation in his letter to L'Hôpital on September 30, 1695, [4], [5]. In this letter, he posed the question of what it would mean to define a derivative of fractional order, [4], [5], [6]. Since then, mathematicians have made significant efforts to define fractional derivatives.

Several fractional derivatives have been introduced, including the Caputo–Fabrizio fractional derivative, [7], Caputo fractional derivative, [8], Riemann–Liouville fractional derivative, [8], and Atangana–Baleanu fractional derivative, [9]. While most of these definitions address fractional calculus, they often present limitations. A common feature among them is linearity; however, not all satisfy fundamental properties such as the chain rule and product rule. To address these challenges, the authors in [10], introduced the conformable fractional derivative—a novel, straightforward, and coherent local derivative that retains most properties of the classical integer-order derivative, relying on a fundamental limit-based definition. Further refinements and significant developments have been made in [11], [12], [13].

The authors in [14], provided a geometric interpretation of the conformable derivative using fractional chords. Since then, this derivative has gained increasing attention, and numerous problems have been solved using its framework. More details and applications related to the conformable fractional derivative can be found in [15], [16].

Differential equations play a crucial role in describing various natural phenomena across disciplines such as physics, biology, engineering, and medicine. More details on these applications can be found in [17], [18], [19], [20], [21], [22], [23].

Bernoulli equations are particularly important as they represent nonlinear differential equations with explicitly known solutions. Unlike linear equations, they do not have singular solutions. Their solutions can be obtained using two different methods: one introduced by Bernoulli himself and the other attributed to Leibniz.

In this work, we begin by presenting the definitions of the conformable derivative and conformable integral, along with some of their fundamental properties. Before studying the nonlinear Bernoulli conformable differential equation, we introduce a fundamental lemma that provides the classical derivative of the conformable fractional integral. We then classify the Bernoulli equation based on the value of n. Furthermore, we propose a generalization of the conformable Leibniz and conformable Bernoulli methods for solving Bernoulli equations, leading to exact solutions. Finally, as an application, we demonstrate the effectiveness of our approach by solving selected Bernoulli conformable fractional differential equations, including a detailed numerical example.

2 Preliminary Concepts

In this section, we introduce fundamental concepts necessary for our study. We begin with the definition of the conformable fractional derivative, which plays a central role in our analysis. Additionally, we establish key properties and theorems that will be essential for the classification and generalization of the nonlinear Bernoulli conformable fractional differential equation. These preliminaries provide the mathematical foundation for the methods and results presented in the subsequent sections.

Definition 1 *Let $P : [0, \infty) \to \mathbb{R}$ be a function. The conformable fractional derivative of P of order α is defined as*

$$D^\alpha P(x) = \lim_{\epsilon \to 0} \frac{P(x + \epsilon x^{1-\alpha}) - P(x)}{\epsilon},$$

for all $\alpha \in (0, 1)$ and $x > 0$. We denote $D^\alpha P(x)$ as the conformable fractional derivative of P with order α. A function P is said to be α-differentiable if its conformable fractional derivative exists. Furthermore, if $\lim_{x\to 0^+} D^\alpha P(x)$ exists and P is α-differentiable in some interval $(0, a)$, where $a > 0$, we define:

$$D^\alpha P(0) = \lim_{x \to 0^+} D^\alpha P(x).$$

As a consequence of the above definition, the following theorems are obtained.

Theorem 2 *Let $P : [0, \infty) \to \mathbb{R}$ be a function. If P is α-differentiable at $x_0 > 0$ for some $\alpha \in (0, 1)$, then P is continuous at x_0.*

Proof 3 *See, [10].*

It is straightforward to verify that the conformable derivative D^α satisfies the subsequent properties provided in the next theorem.

Theorem 4 *Let P and Q be α-differentiable functions at a point $x > 0$, where $\alpha \in (0, 1)$. Then, the conformable derivative satisfies the following properties:*

1. ***Linearity:***
$$D^\alpha(aP + bQ) = aD^\alpha(P) + bD^\alpha(Q),$$
for all $a, b \in \mathbb{R}$.

2. ***Power Rule:***
$$D^\alpha(x^n) = nx^{n-\alpha}, \quad \text{for all } n \in \mathbb{R}.$$

3. ***Derivative of a Constant:***
$$D^\alpha(\lambda) = 0,$$
for any constant function $P(x) = \lambda$.

4. ***Product Rule:***
$$D^\alpha(PQ) = QD^\alpha(P) + PD^\alpha(Q).$$

5. ***Quotient Rule:***
$$D^\alpha\left(\frac{P}{Q}\right) = \frac{QD^\alpha(P) - PD^\alpha(Q)}{Q^2},$$
for $Q(x) \neq 0$.

6. ***Relation to the Classical Derivative:***
$$D^\alpha(P)(x) = x^{1-\alpha}\frac{d}{dx}P(x),$$
for any differentiable function P.

Proof 5 *See, [10].*

It is worth mentioning that most fractional derivatives, except for Caputo-type derivatives, do not satisfy property (3) in Theorem 4. Additionally, all fractional derivatives fail to obey the familiar product rule (property (4) in Theorem 4), quotient rule (property (5) in Theorem 4), and chain rule for two functions. Furthermore, fractional derivatives lack corresponding versions of Rolle's Theorem and the Mean Value Theorem. However, the conformable derivative satisfies all these properties and adheres to the two aforementioned theorems. For more details, we refer the reader to [10].

A mathematical comparison with numerical simulations was carefully examined in a study by Feng Gao and Chunmei Chi, titled "Improvement on Conformable Fractional Derivative and Its Applications in Fractional Differential Equations", [24], which further supports our findings.

Example 6 *The conformable fractional derivatives of specific functions are as follows:*

1. $D^\alpha\left(e^{\frac{1}{\alpha}x^\alpha}\right) = e^{\frac{1}{\alpha}x^\alpha}$.
2. $D^\alpha\left(\sin\frac{1}{\alpha}x^\alpha\right) = \cos\frac{1}{\alpha}x^\alpha$.
3. $D^\alpha\left(e^{cx}\right) = cx^{1-\alpha}e^{cx}, \quad c \in \mathbb{R}$.
4. $D^\alpha\left(\sin bx\right) = bx^{1-\alpha}\cos bx, \quad b \in \mathbb{R}$.
5. $D^\alpha\left(\frac{1}{\alpha}x^\alpha\right) = 1$.
6. $D^\alpha\left(\cos bx\right) = -bx^{1-\alpha}\sin bx, \quad b \in \mathbb{R}$.
7. $D^\alpha\left(\cos\frac{1}{\alpha}x^\alpha\right) = -\sin\frac{1}{\alpha}x^\alpha$.
8. $D^\alpha(1) = 0$.

Remark 7 *It is important to note that a function can be α-differentiable at a point while not being differentiable in the classical sense. For example, consider the function $P(x) = 2\sqrt{x}$. Then, the conformable fractional derivative at $x = 0$ is given by:*
$$D^{\frac{1}{2}}(P)(0) = \lim_{x\to 0^+} D^{\frac{1}{2}}(P)(x) = 1.$$
However, the classical derivative does not exist at $x = 0$, i.e.,
$$D^1(P)(0) \text{ does not exist.}$$

The authors in [10], introduced the definition of the α-fractional integral of a function P, starting from a given point $a \geq 0$.

Definition 8 *The conformable integral of fractional order α is defined as:*
$$\begin{aligned} I_\alpha^a(P)(x) &= \int_a^x P(t)\, d_\alpha t \\ &= I_1^a(x^{\alpha-1}P) = \int_a^x \frac{P(t)}{t^{1-\alpha}}dt, \end{aligned}$$
for $\alpha \in (0, 1)$, where the integral is understood as the standard Riemann improper integral.

Example 9 *The conformable integrals of certain functions are as follows:*

1. $I^0_{\frac{1}{2}}\left(\sqrt{x}\cos x\right) = \int_0^x \cos t\, dt = \sin x$.
2. $I^0_{\frac{1}{2}}\left(\cos 2\sqrt{x}\right) = \sin 2\sqrt{x}$.

One of the fundamental results is given below for completeness.

Theorem 10 *Let P be any continuous function defined in the domain of I_α and let $x \geq a$. Then, the conformable derivative of its conformable integral satisfies:*

$$D^\alpha I_\alpha^a(P)(x) = P(x).$$

Proof 11 *See, [10].*

Definition 12 *The conformable exponential function is defined for every $x \geq 0$ as:*

$$E_\alpha(\xi, x) = e^{\xi \frac{x^\alpha}{\alpha}},$$

where $\alpha \in (0,1)$ and $\xi \in \mathbb{R}$.

3 Generalization of the Nonlinear Bernoulli Conformable Equation

In this section, we study, analyze, and generalize one of the most well-known nonlinear equations—the first-order nonlinear Bernoulli conformable fractional differential equation.

Before proceeding with the generalization, we first introduce the following fundamental lemma, which is essential for obtaining the analytical solution of the nonlinear Bernoulli conformable differential equation.

Lemma 13 *Let P be a continuous function defined in the domain of I_α, where $\alpha \in (0,1)$. For $x > a$, we have:*

$$\frac{d}{dx}\left(I_\alpha^a(P)(x)\right) = \frac{P(x)}{x^{1-\alpha}}.$$

Proof 14 *Since P is continuous, the conformable integral $I_\alpha^a(P)(x)$ is differentiable. Thus, we have:*

$$\begin{aligned} D^\alpha\left(I_\alpha^a(P)\right)(x) &= x^{1-\alpha}\frac{d}{dx}I_\alpha^a(P)(x) \\ &= x^{1-\alpha}\frac{d}{dx}\int_a^x \frac{P(t)}{t^{1-\alpha}}dt \\ &= x^{1-\alpha}\cdot\frac{P(x)}{x^{1-\alpha}} = P(x). \end{aligned}$$

Therefore, we obtain:

$$x^{1-\alpha}\frac{d}{dx}I_\alpha^a(P)(x) = P(x).$$

Consequently,

$$\frac{d}{dx}I_\alpha^a(P)(x) = \frac{P(x)}{x^{1-\alpha}}.$$

This completes the proof.

Theorem 15 *Let $0 < \alpha < 1$, $n \in \mathbb{R}$, and $y \in \mathbb{R}^n$. Let D^α denote the conformable derivative of fractional order α, and let y_h and y_p represent the homogeneous and particular solutions, respectively. Additionally, let $P, Q : \mathbb{R} \to \mathbb{R}$ be nonzero, α-differentiable functions that are continuous in the domain of I_α^0. Then, the nonlinear Bernoulli conformable fractional differential equation:*

$$D^\alpha y + P(x)y = Q(x)y^n, \quad P(x), Q(x) \neq 0$$

has the following two special cases:

- ***Case 1:*** *If $n = 1$, the equation reduces to:*

  $$D^\alpha y + R(x)y = 0, \quad R(x) = P(x) - Q(x). \tag{1}$$

  This is a first-order linear homogeneous conformable fractional differential equation, whose solution is given by:

  $$y(x) = y_h(x) = y_0 e^{-I_\alpha^0(R)(x)},$$

  where y_0 is an arbitrary constant determined by initial conditions.

- ***Case 2:*** *If $n = 0$, the equation simplifies to:*

  $$D^\alpha y + P(x)y = Q(x). \tag{2}$$

  This is a first-order linear nonhomogeneous conformable fractional differential equation, whose solution is given by:

  $$\begin{aligned} y(x) &= y_h(x) + y_p(x) \\ &= y_0 e^{-I_\alpha^0(P)(x)} + \lambda(x)e^{-I_\alpha^0(P)(x)}, \end{aligned}$$

  where the function $\lambda : \mathbb{R} \to \mathbb{R}$ is obtained from the condition:

  $$\lambda(x) = I_\alpha^0\left(Q(x)e^{I_\alpha^0(P)(x)}\right).$$

  Here, y_0 is an arbitrary constant determined by the given conditions.

Proof 16
- ***Case 1:*** *To prove the solution of equation (1), we verify that the given function*

  $$y(x) = y_0 e^{-I_\alpha^0(R)(x)}$$

  satisfies the conformable differential equation (1). Substituting this function into equation (1) *and using Lemma 13, we get:*

$$\begin{aligned}
&D^\alpha y + R(x)y \\
&= y_0 x^{1-\alpha}\frac{d}{dx}\left(e^{-I^0_\alpha(R)}\right) + R(x)y_0 e^{-I^0_\alpha(R)(x)} \\
&= -y_0 x^{1-\alpha}\frac{d}{dx}\left(I^0_\alpha(R)(x)\right)e^{-I^0_\alpha(R)(x)} \\
&\quad + R(x)y_0 e^{-I^0_\alpha(R)(x)} \\
&= -y_0 x^{1-\alpha}\frac{R(x)}{x^{1-\alpha}}e^{-I^0_\alpha(R)(x)} + R(x)y_0 e^{-I^0_\alpha(R)} \\
&= -R(x)y_0 e^{-I^0_\alpha(R)(x)} + R(x)y_0 e^{-I^0_\alpha(R)(x)} \\
&= 0.
\end{aligned}$$

Thus, we conclude that the homogeneous solution of equation (1) is:

$$y(x) = y_0 e^{-I^0_\alpha(R)(x)}.$$

This completes the proof of Case 1.

- ***Case 2:*** *To prove the solution of equation (2), we first verify that the homogeneous solution is given by:*

$$y_h = y_0 e^{-I^0_\alpha(P)(x)},$$

as shown in Case 1. To obtain the general solution, we compute the particular solution:

$$\begin{aligned}
y_p &= \lambda(x)e^{-I^0_\alpha(P)(x)} \\
&= I^0_\alpha\left(Q(x)e^{I^0_\alpha(P)(x)}\right)e^{-I^0_\alpha(P)(x)}.
\end{aligned}$$

Now, to complete the proof, we verify that equation (2) is satisfied by:

$$I^0_\alpha\left(Q(x)e^{I^0_\alpha(P)(x)}\right)e^{-I^0_\alpha(P)(x)}.$$

Since the functions $P(x)$ and $Q(x)$ are continuous in the domain of I^0_α, they are integrable (under the Riemann integral). Also, the condition $P(x) \neq 0$ ensures the existence of the conformable integrating factor $e^{I^0_\alpha(P)(x)}$. Now, substituting the candidate solution into equation (2) and using Theorem 10 and Lemma 13, we get:

$$\begin{aligned}
&D^\alpha y_p + P(x)y_p \\
&= D^\alpha\left(\lambda(x)e^{-I^0_\alpha(P)(x)}\right) + P(x)\lambda(x)e^{-I^0_\alpha(P)} \\
&= D^\alpha\left(I^0_\alpha\left(Q(x)e^{I^0_\alpha(P)(x)}\right)e^{-I^0_\alpha(P)(x)}\right) \\
&\quad + P(x)I^0_\alpha\left(Q(x)e^{I^0_\alpha(P)(x)}\right)e^{-I^0_\alpha(P)(x)} \\
&= D^\alpha\left[I^0_\alpha Q(x)e^{I^0_\alpha(P)(x)}\right]e^{-I^0_\alpha(P)(x)} \\
&\quad + D^\alpha\left(e^{-I^0_\alpha(P)(x)}\right)\left[I^0_\alpha Q(x)e^{I^0_\alpha(P)(x)}\right] \\
&\quad + P(x)I^0_\alpha\left(Q(x)e^{I^0_\alpha(P)(x)}\right)e^{-I^0_\alpha(P)(x)} \\
&= Q(x)e^0 - I^0_\alpha\left(Q(x)e^{I^0_\alpha(P)(x)}\right)P(x)e^{-I^0_\alpha(P)} \\
&\quad + P(x)I^0_\alpha\left(Q(x)e^{I^0_\alpha(P)(x)}\right)e^{-I^0_\alpha(P)(x)} \\
&= Q(x).
\end{aligned}$$

Thus, we conclude that the general solution of equation (2) is:

$$y(x) = y_h(x) + y_p(x).$$

This completes the proof of Case 2.

Theorem 17 ***(Conformable Leibniz Method)*** *Let $0 < \alpha < 1$, $n \in \mathbb{R}$, and $y \in \mathbb{R}^n$. Let D^α represent the conformable derivative of fractional order α, and let y_h and y_p denote the homogeneous and particular solutions, respectively. Additionally, let $P, Q : \mathbb{R} \to \mathbb{R}$ be α-differentiable functions that are continuous in the domain of I^0_α. Then, according to the Conformable Leibniz Method, the nonlinear Bernoulli conformable fractional differential equation:*

$$D^\alpha y + P(x)y = Q(x)y^n, \quad n \neq 0, \quad n \neq 1, \tag{3}$$

has a solution given by:

$$\begin{aligned}
y(x) = \Big[&z_0 e^{-I^0_\alpha[(1-n)P](x)} \\
&+ I^0_\alpha\left((1-n)Q(x)e^{I^0_\alpha[(1-n)P](x)}\right) \\
&\times e^{-I^0_\alpha[(1-n)P](x)}\Big]^{\frac{1}{1-n}},
\end{aligned}$$

where z_0 is a constant.

Proof 18 *To solve the nonlinear Bernoulli conformable fractional differential equation, we proceed as follows:*

Step 1: Reducing the Nonlinear Equation. *Dividing both sides of equation (3) by y^n, we obtain:*

$$y^{-n}D^\alpha y + P(x)y^{1-n} = Q(x). \tag{3}$$

Step 2: Applying the Substitution. *Let* $z = y^{1-n}$. *Using Theorem 2.3, we compute:*

$$\begin{aligned} D^\alpha z &= x^{1-\alpha}\frac{d}{dx}z = x^{1-\alpha}\frac{d}{dx}y^{1-n} \\ &= x^{1-\alpha}(1-n)y^{-n}\frac{d}{dx}y \\ &= (1-n)y^{-n}\left(x^{1-\alpha}\frac{d}{dx}y\right) \\ &= (1-n)y^{-n}D^\alpha y. \end{aligned} \tag{4}$$

Thus, we can rewrite:

$$y^{-n}D^\alpha y = \frac{D^\alpha z}{(1-n)}. \tag{5}$$

Substituting this into equation (3), we obtain:

$$\frac{D^\alpha z}{(1-n)} + P(x)z = Q(x). \tag{6}$$

Step 3: Transforming into a Linear Equation. *Multiplying both sides by* $(1-n)$, *we get:*

$$D^\alpha z + [(1-n)P(x)]z = (1-n)Q(x). \tag{7}$$

This is a first-order linear nonhomogeneous conformable fractional differential equation. Since the Leibniz transformation is valid for all cases where $n \neq 0$ *and* $n \neq 1$, *we apply Case 2 of Theorem 15. The solution is:*

$$\begin{aligned} z(x) &= z_h(x) + z_p(x) \\ &= z_0 e^{-I_\alpha^0[(1-n)P](x)} \\ &\quad + I_\alpha^0\left((1-n)Q(x)e^{I_\alpha^0[(1-n)P](x)}\right) \\ &\quad \times e^{-I_\alpha^0[(1-n)P](x)}. \end{aligned} \tag{8}$$

Step 4: Expressing the Final Solution. *Since* $z = y^{1-n}$, *we conclude:*

$$y = z^{\frac{1}{1-n}}. \tag{9}$$

Thus, the solution of equation (3) is:

$$\begin{aligned} y(x) = \Bigg[& z_0 e^{-I_\alpha^0[(1-n)P](x)} \\ & + I_\alpha^0\left((1-n)Q(x)e^{I_\alpha^0[(1-n)P](x)}\right) \\ & \times e^{-I_\alpha^0[(1-n)P](x)}\Bigg]^{\frac{1}{1-n}}. \end{aligned} \tag{10}$$

This completes the proof of Theorem 17.

Theorem 19 ***(Conformable Bernoulli Method)*** *Let* $0 < \alpha < 1$, $n \in \mathbb{R}$, *and* $y \in \mathbb{R}^n$. *Let* D^α *denote the conformable derivative of fractional order* α. *Additionally, let* $P, Q : \mathbb{R} \to \mathbb{R}$ *be* α*-differentiable functions that are continuous in the domain of* I_α^0. *Then, according to the Conformable Bernoulli Method, the nonlinear Bernoulli conformable fractional differential equation:*

$$D^\alpha y + P(x)y = Q(x)y^n, \quad n \neq 0, \quad n \neq 1, \tag{4}$$

has a solution given by:

$$\begin{aligned} & y(x) = u(x)v(x) = e^{-I_\alpha^0(P)(x)} \\ & \times \left[(1-n)I_\alpha^0\left(Q(x)\left(e^{-I_\alpha^0(P)(x)}\right)^{n-1}\right) + c\right]^{\frac{1}{1-n}}, \end{aligned} \tag{11}$$

where c *is a constant determined by initial conditions. Here,* $u(x)$ *represents the solution of the linear part, while* $v(x)$ *is the general solution of the "truncated" Bernoulli equation without the linear part.*

Proof 20 *The nonlinear Bernoulli conformable fractional differential equation (4) can be decomposed into two separate equations. We assume the solution is of the form:*

$$y(x) = u(x)v(x), \tag{12}$$

where $u(x)$ *is a solution, and we require it to satisfy only the linear component:*

$$D^\alpha u + P(x)u = 0. \tag{13}$$

Using Theorem 15, the solution of this linear equation is:

$$u(x) = e^{-I_\alpha^0(P)(x)}. \tag{14}$$

The second function, $v(x)$, *represents the general solution of the "truncated" Bernoulli equation without a linear component:*

$$uD^\alpha v = Q(x)u^n v^n. \tag{15}$$

Substituting $u(x) = e^{-I_\alpha^0(P)(x)}$, *we obtain:*

$$D^\alpha v = Q(x)u^{n-1}v^n$$

$\Rightarrow$

$$v^{-n}D^\alpha v = Q(x)\left(e^{-I_\alpha^0(P)(x)}\right)^{n-1}. \tag{16}$$

Since this is a separable equation, it can be integrated as:

$$v^{1-n} = (1-n)I_\alpha^0\left(Q(x)\left(e^{-I_\alpha^0(P)(x)}\right)^{n-1}\right) + c. \tag{17}$$

Taking the $(1-n)^{th}$-root, we obtain:

$$v(x) = \left[(1-n)I_\alpha^0\left(Q(x)\left(e^{-I_\alpha^0(P)(x)}\right)^{n-1}\right)+c\right]^{\frac{1}{1-n}} \tag{18}$$

Thus, multiplying by $u(x)$ gives the final solution:

$$y(x) = u(x)v(x) = e^{-I_\alpha^0(P)(x)} \times \left[(1-n)I_\alpha^0\left(Q(x)\left(e^{-I_\alpha^0(P)(x)}\right)^{n-1}\right)+c\right]^{\frac{1}{1-n}}. \tag{19}$$

To verify the solution, we use Theorem 4, Theorem 10, and Lemma 13, we compute:

$$\begin{aligned}
D^\alpha y(x) &= D^\alpha(u(x)v(x)) \\
&= v(x)D^\alpha u(x) + u(x)D^\alpha v(x) \\
&= -v(x)P(x)e^{-I_\alpha^0(P)(x)} + u(x)D^\alpha v(x) \\
&= -P(x)u(x)v(x) + u(x) \\
&\quad \times D^\alpha\left[\left[(1-n)I_\alpha^0\left(Q(x)u^{n-1}\right)+c\right]^{\frac{1}{1-n}}\right] \\
&= -P(x)y + u(x) \\
&\quad \times \left[\frac{1}{1-n}\left((1-n)I_\alpha^0\left(Q(x)u^{n-1}\right)+c\right)^{\frac{1}{1-n}-1}\right. \\
&\quad \left.\times D^\alpha\left((1-n)I_\alpha^0\left(Q(x)u^{n-1}(x)\right)+c\right)\right] \\
&= -P(x)y + u(x)\left[\left((1-n)\right.\right. \\
&\quad \left.\left.\times I_\alpha^0\left(Q(x)u^{n-1}(x)\right)+c\right)^{\frac{n}{1-n}} Q(x)u^{n-1}(x)\right] \\
&= -P(x)y + u(x)v^n(x)Q(x)u^{n-1}(x) \\
&= -P(x)y + Q(x)\left(u(x)v(x)\right)^n \\
&= -P(x)y + Q(x)y^n. \qquad (20)
\end{aligned}$$

Substituting this into equation (4), we get:

$$\begin{aligned}
D^\alpha y + P(x)y &= Q(x)y^n \\
-P(x)y + Q(x)y^n + P(x)y & \qquad (21)\\
&= Q(x)y^n \\
Q(x)y^n &= Q(x)y^n. \qquad (22)
\end{aligned}$$

Since the equation is satisfied, this completes the proof of Theorem 19.

Remark 21 *It is important to note that equations (4) and (3) represent the same nonlinear Bernoulli conformable fractional differential equation; however, we obtain its exact solution using two different methods. Furthermore, both methods yield the same exact solution.*

Verification:
We now demonstrate that the two methods provide identical solutions. Assuming that the constants z_0 and c are equal, we have:

$$\begin{aligned}
y(x) &= \left[z_0 e^{-I_\alpha^0((1-n)P)(x)}\right. \\
&\quad + I_\alpha^0\left((1-n)Q(x)e^{I_\alpha^0((1-n)P)(x)}\right) \\
&\quad \left.\times\ e^{-I_\alpha^0((1-n)P)(x)}\right]^{\frac{1}{1-n}} \\
&= \left(z_0\left(e^{-I_\alpha^0(P)(x)}\right)^{(1-n)}\right. \\
&\quad + I_\alpha^0\left((1-n)Q(x)\left(e^{I_\alpha^0(P)(x)}\right)^{(1-n)}\right) \\
&\quad \left.\times\left(e^{-I_\alpha^0(P)(x)}\right)^{(1-n)}\right)^{\frac{1}{1-n}} \\
&= \left(\left(e^{-I_\alpha^0(P)(x)}\right)^{(1-n)}\left[I_\alpha^0\left((1-n)Q(x)\right.\right.\right. \\
&\quad \left.\left.\left.\times\left(e^{I_\alpha^0(P)(x)})^{(1-n)}\right) + z_0\right]\right)^{\frac{1}{1-n}} \\
&= e^{-I_\alpha^0(P)(x)}\left[I_\alpha^0\left((1-n)Q(x)\right.\right. \\
&\quad \left.\left.\times\left(e^{I_\alpha^0(P)(x)}\right)^{(1-n)}\right) + z_0\right]^{\frac{1}{1-n}}. \\
&= e^{-I_\alpha^0(P)(x)}\left[I_\alpha^0\left((1-n)Q(x)\right.\right. \\
&\quad \left.\left.\times\left(e^{-I_\alpha^0(P)(x)}\right)^{(n-1)}\right) + z_0\right]^{\frac{1}{1-n}} \\
&= u(x)v(x).
\end{aligned}$$

Thus, we conclude that the solutions obtained from both methods are indeed identical.

4 Applications

To demonstrate the effectiveness of the proposed theorems, we apply them to selected conformable fractional differential equations.

• To illustrate Case 1 of Theorem 15, consider the homogeneous conformable fractional differential equation:

$$D^\alpha y + \left(\cos\frac{x^\alpha}{\alpha}\right)y = 0, \quad y(0) = 2. \tag{5}$$

Applying Theorem 15, the analytical solution is given by:

$$\begin{aligned} y(x) &= y_0 e^{-I_\alpha^0\left(\cos\frac{x^\alpha}{\alpha}\right)} \\ &= y_0 e^{-\int_0^x \left(\cos\frac{t^\alpha}{\alpha}\right) d_\alpha t} \\ &= y_0 e^{-\left[\sin\frac{t^\alpha}{\alpha}\right]_0^x} \\ &= y_0 e^{-\left(\sin\frac{x^\alpha}{\alpha}\right)}. \end{aligned}$$

Using the initial condition $y(0) = 2$, we obtain $y_0 = 2$. Hence, the final solution for equation (5) is:

$$y(x) = 2e^{-\left(\sin\frac{x^\alpha}{\alpha}\right)}. \tag{23}$$

• To illustrate Case 2 of Theorem 15, consider the nonhomogeneous conformable fractional differential equation:

$$D^\alpha y + \frac{x^\alpha}{\alpha} y = \left(\sin\frac{x^\alpha}{\alpha}\right) y, \quad y(0) = \frac{3}{e}. \tag{6}$$

Applying Theorem 15, the analytical solution is given by:

$$\begin{aligned} y(x) &= y_0 e^{-I_\alpha^0\left[\frac{x^\alpha}{\alpha} - \left(\sin\frac{x^\alpha}{\alpha}\right)\right]} \\ &= y_0 e^{-\int_0^x \left[\frac{t^\alpha}{\alpha} - \left(\sin\frac{t^\alpha}{\alpha}\right)\right] d_\alpha t} \\ &= y_0 e^{-\left[1 + \left(\cos\frac{t^\alpha}{\alpha}\right)\right]_0^x} \\ &= y_0 e^{\left[-1 - \left(\cos\frac{x^\alpha}{\alpha}\right) + 1\right]} \\ &= y_0 e^{-\left(\cos\frac{x^\alpha}{\alpha}\right)}. \end{aligned}$$

Using the initial condition $y(0) = \frac{3}{e}$, we solve for y_0:

$$\frac{y_0}{e} = \frac{3}{e} \quad \Rightarrow \quad y_0 = 3.$$

Thus, the final solution for equation (6) is:

$$y(x) = 3e^{-\left(\cos\frac{x^\alpha}{\alpha}\right)}. \tag{24}$$

• To illustrate Theorem 17 (Conformable Leibniz Method) and Theorem 19 (Conformable Bernoulli Method), consider the nonlinear Bernoulli conformable fractional differential equation:

$$D^\alpha y - \frac{x^\alpha}{2} y = -\frac{x}{2} e^{-\frac{x^{2\alpha}}{2\alpha}} y^3, \quad y(0) = 1. \tag{7}$$

Solution of Equation (7) Using the Leibniz Method:
To transform equation (7) into a linear conformable fractional differential equation, we proceed as follows:

Step 1: Dividing by y^3
Dividing both sides by y^3, we obtain:

$$y^{-3} D^\alpha y - \frac{x^\alpha}{2} y^{-2} = -\frac{x}{2} e^{-\frac{x^{2\alpha}}{2\alpha}}. \tag{25}$$

Step 2: Substituting $z = y^{-2}$ and Applying Theorem 15
Using $z = y^{-2}$, we differentiate using the conformable derivative:

$$\begin{aligned} D^\alpha z = x^{1-\alpha}\frac{d}{dx} z &= x^{1-\alpha}\frac{d}{dx} y^{-2} \\ &= x^{1-\alpha}(-2) y^{-3}\frac{d}{dx} y \\ &= (-2) y^{-3}\left(x^{1-\alpha}\frac{d}{dx} y\right) \\ &= (-2) y^{-3} D^\alpha y. \end{aligned}$$

Thus, we rewrite:

$$y^{-3} D^\alpha y = \frac{D^\alpha z}{-2}. \tag{26}$$

Substituting into the equation, we obtain:

$$\frac{D^\alpha z}{-2} - \frac{x^\alpha}{2} z = -\frac{x}{2} e^{-\frac{x^{2\alpha}}{2\alpha}}. \tag{27}$$

Step 3: Multiplying by (-2)
Multiplying both sides by (-2), we get:

$$D^\alpha z + x^\alpha z = x e^{-\frac{x^{2\alpha}}{2\alpha}}, \tag{8}$$

which is a first-order linear nonhomogeneous conformable fractional differential equation. Using Case 2 of Theorem 15, the solution of equation (8) is:

$$\begin{aligned} z(x) &= z_h(x) + z_p(x) \\ &= z_0 e^{-I_\alpha^0 x^\alpha} + I_\alpha^0\left(x e^{-\frac{x^{2\alpha}}{2\alpha}} e^{I_\alpha^0 x^\alpha}\right) e^{-I_\alpha^0 x^\alpha}, \end{aligned}$$

where $z_h(x)$ and $z_p(x)$ are the homogeneous and particular solutions, respectively.

Step 4: Solving for $z_h(x)$ and $z_p(x)$
The homogeneous solution is given by:

$$\begin{aligned} z_h(x) &= z_0 e^{-I_\alpha^0 x^\alpha} \\ &= z_0 e^{-\int_0^x t^{\alpha-1} t^\alpha dt} = z_0 e^{-\int_0^x t^{2\alpha-1} dt} \\ &= z_0 e^{-\left[\frac{t^{2\alpha}}{2\alpha}\right]_0^x} = z_0 e^{-\frac{x^{2\alpha}}{2\alpha}}. \end{aligned}$$

Now, we compute the particular solution:

$$\begin{aligned} z_p(x) &= I_\alpha^0\left(x e^{-\frac{x^{2\alpha}}{2\alpha}} e^{I_\alpha^0 x^\alpha}\right) e^{-I_\alpha^0 x^\alpha} \\ &= I_\alpha^0\left(x e^{-\frac{x^{2\alpha}}{2\alpha}} e^{\frac{x^{2\alpha}}{2\alpha}}\right) e^{-\frac{x^{2\alpha}}{2\alpha}} \\ &= \int_0^x \left(t^{\alpha-1} t e^{-\frac{t^{2\alpha}}{2\alpha}} e^{\frac{t^{2\alpha}}{2\alpha}}\right) dt\, e^{-\frac{x^{2\alpha}}{2\alpha}} \\ &= \left[\frac{t^{\alpha+1}}{\alpha+1}\right]_0^x e^{-\frac{x^{2\alpha}}{2\alpha}} \\ &= \frac{x^{\alpha+1}}{\alpha+1} e^{-\frac{x^{2\alpha}}{2\alpha}}. \end{aligned}$$

Step 5: Constructing the General Solution
Thus, the general solution of equation (8) is:

$$z(x) = z_0 e^{-\frac{x^{2\alpha}}{2\alpha}} + \frac{x^{\alpha+1}}{\alpha+1}\, e^{-\frac{x^{2\alpha}}{2\alpha}}. \qquad (28)$$

Since $z = y^{-2}$, we conclude:

$$y = z^{\left(-\frac{1}{2}\right)}. \qquad (29)$$

Thus, the solution of the nonlinear Bernoulli conformable fractional differential equation (7) is:

$$y(x) = \left(z_0 e^{-\frac{x^{2\alpha}}{2\alpha}} + \frac{x^{\alpha+1}}{\alpha+1}\, e^{-\frac{x^{2\alpha}}{2\alpha}} \right)^{\left(-\frac{1}{2}\right)}. \qquad (30)$$

Step 6: Applying the Initial Condition
Using the initial condition $y(0) = 1$, we obtain:

$$\begin{aligned} 1 &= \left(z_0 e^{-\frac{0^{2\alpha}}{2\alpha}} + \frac{0^{\alpha+1}}{\alpha+1}\, e^{-\frac{0^{2\alpha}}{2\alpha}} \right)^{\left(-\frac{1}{2}\right)} \\ &= (z_0)^{\left(-\frac{1}{2}\right)}. \end{aligned}$$

Thus, $z_0 = 1$, and the final general solution for equation (7) is:

$$y(x) = \left(e^{-\frac{x^{2\alpha}}{2\alpha}} + \frac{x^{\alpha+1}}{\alpha+1}\, e^{-\frac{x^{2\alpha}}{2\alpha}} \right)^{\left(-\frac{1}{2}\right)}. \qquad (31)$$

Hence, the result is obtained.

Solution of Equation (7) Using the Bernoulli Method
Now, we apply the Bernoulli approach to obtain the general solution, given by:

$$y(x) = u(x)v(x), \qquad (32)$$

where $u(x)$ is the solution of the ”truncated” linear equation:

$$D^\alpha u - \frac{x^\alpha}{2} u = 0, \qquad (33)$$

and $v(x)$ is the general solution of the separable equation:

$$u D^\alpha v = -\frac{x}{2} e^{-\frac{x^{2\alpha}}{2\alpha}} (uv)^3. \qquad (34)$$

Step 1: Solving for $u(x)$
Using Theorem 15, the solution for $u(x)$ is:

$$\begin{aligned} u(x) &= e^{-I_\alpha^0\left(-\frac{x^\alpha}{2}\right)} \\ &= e^{\frac{1}{2}\left(\frac{x^{2\alpha}}{2\alpha}\right)}. \end{aligned} \qquad (35)$$

Step 2: Substituting $u(x)$ into the Equation for $v(x)$

Substituting $u(x) = e^{\frac{1}{2}\left(\frac{x^{2\alpha}}{2\alpha}\right)}$, we rewrite the equation for $v(x)$ as:

$$\begin{aligned} u D^\alpha v &= -\frac{x}{2} e^{-\frac{x^{2\alpha}}{2\alpha}} (uv)^3 \\ \Rightarrow \quad v^{-3} D^\alpha v &= -\frac{x}{2} e^{-\frac{x^{2\alpha}}{2\alpha}} u^2. \end{aligned} \qquad (36)$$

Step 3: Solving for $v(x)$
Substituting $u(x)$ into the equation:

$$\begin{aligned} v^{-3} D^\alpha v &= -\frac{x}{2} e^{-\frac{x^{2\alpha}}{2\alpha}} \left(e^{\frac{1}{2}\left(\frac{x^{2\alpha}}{2\alpha}\right)} \right)^2 \\ \Rightarrow \quad v^{-3} D^\alpha v &= -\frac{x}{2}. \end{aligned} \qquad (37)$$

This equation is easily integrable, yielding:

$$v(x) = \left(c + \frac{x^{\alpha+1}}{\alpha+1} \right)^{-\frac{1}{2}}, \qquad (38)$$

where c is a constant.

Step 4: Constructing the General Solution
Multiplying $u(x)$ and $v(x)$, we obtain:

$$\begin{aligned} y(x) &= u(x)v(x) \\ &= e^{\frac{1}{2}\left(\frac{x^{2\alpha}}{2\alpha}\right)} \left(c + \frac{x^{\alpha+1}}{\alpha+1} \right)^{-\frac{1}{2}} \\ &= \left(e^{-\frac{x^{2\alpha}}{2\alpha}} \right)^{-\frac{1}{2}} \left(c + \frac{x^{\alpha+1}}{\alpha+1} \right)^{-\frac{1}{2}} \\ &= \left(e^{-\frac{x^{2\alpha}}{2\alpha}} \left(c + \frac{x^{\alpha+1}}{\alpha+1} \right) \right)^{-\frac{1}{2}} \\ &= \left(c e^{-\frac{x^{2\alpha}}{2\alpha}} + \frac{x^{\alpha+1}}{\alpha+1} e^{-\frac{x^{2\alpha}}{2\alpha}} \right)^{-\frac{1}{2}}. \end{aligned} \qquad (39)$$

Step 5: Applying the Initial Condition
Using the initial condition $y(0) = 1$, we solve for c:

$$1 = \left(c e^{-\frac{0^{2\alpha}}{2\alpha}} + \frac{0^{\alpha+1}}{\alpha+1} e^{-\frac{0^{2\alpha}}{2\alpha}} \right)^{-\frac{1}{2}}.$$

Since $e^0 = 1$, we simplify:

$$1 = (c)^{-\frac{1}{2}} \quad \Rightarrow \quad c = 1.$$

Step 6: Final General Solution
Thus, the general solution of equation (7) is:

$$y(x) = \left(e^{-\frac{x^{2\alpha}}{2\alpha}} + \frac{x^{\alpha+1}}{\alpha+1} e^{-\frac{x^{2\alpha}}{2\alpha}} \right)^{-\frac{1}{2}}. \qquad (40)$$

Clearly, the obtained solution of equation (7) using both Leibniz and Bernoulli methods supports Remark 21.

5 Numerical Methods

This section presents a numerical example to illustrate the effectiveness of the proposed generalization.

The generalized nonlinear conformable fractional Bernoulli equation is given by:

$$D^{\alpha}y + P(x)y = Q(x)y^n, \quad n \neq 0, \quad n \neq 1. \quad (41)$$

For numerical verification, we set the parameters as follows:

$$\alpha = \frac{1}{2}, \quad n = 3, \quad P(x) = -\frac{\sqrt{x}}{2}, \quad Q(x) = -\frac{x}{2}e^{-x}.$$

Substituting these values, we obtain the following conformable fractional Bernoulli equation:

$$D^{(\frac{1}{2})}y - \frac{\sqrt{x}}{2}y = -\frac{x}{2}e^{-x}y^3, \quad y(0) = 1. \quad (9)$$

Applying Theorems 17 and 19, and referring to the previously derived analytical solution, the solution for equation (9) is:

$$y(x) = \left(ce^{-x} + \frac{2}{3}x^{\frac{3}{2}}e^{-x}\right)^{-\frac{1}{2}}, \quad (42)$$

where c is a constant determined by the initial condition. Using $y(0) = 1$, we solve for c:

$$1 = \left(ce^{-0} + \frac{2}{3}0^{\frac{3}{2}}e^{-0}\right)^{-\frac{1}{2}}.$$

Since $e^0 = 1$ and $0^{\frac{3}{2}} = 0$, we obtain:

$$1 = (c)^{-\frac{1}{2}} \quad \Rightarrow \quad c = 1.$$

Thus, the final solution simplifies to:

$$\begin{aligned} y(x) &= \left(e^{-x} + \frac{2}{3}x^{\frac{3}{2}}e^{-x}\right)^{-\frac{1}{2}} \\ &= \left(e^{-x}\left(1 + \frac{2}{3}x^{\frac{3}{2}}\right)\right)^{-\frac{1}{2}}. \quad (43) \end{aligned}$$

Now we use Mathematica:
The general solution of the nonlinear Bernoulli conformable fractional differential equation (7) with $n = 3$, $\alpha = \frac{1}{2}$ and $c = 1$ which is the solution of equation (7) is as follow :

$$y(x) = \left(c\, e^{-\frac{x^{2\alpha}}{2\alpha}} + \frac{x^{\alpha+1}}{\alpha+1}e^{-\frac{x^{2\alpha}}{2\alpha}}\right)^{\frac{1}{1-n}} \quad (10).$$

The graphical representation of the solution represented in equation (10) can be discussed in Figure 1.

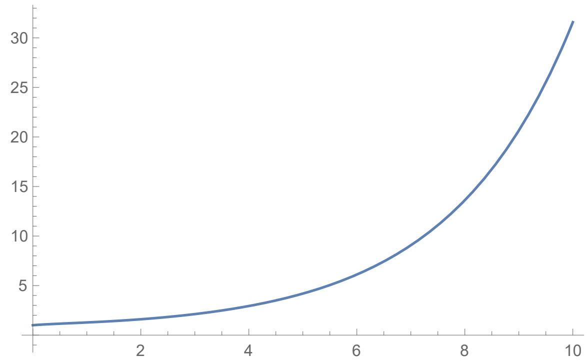

Figure 1: The solution of conformable fractional equation (7) with $n = 3$, $\alpha = \frac{1}{2}$ and $c = 1$

The four streamlines in Figure 2 correspond to different values of the arbitrary constant $c = 1, 2, 3, 4$ from the general solution of the Bernoulli equation (7), given by:

$$y(x) = \left(c\, e^{-\frac{x^{2\alpha}}{2\alpha}} + \frac{x^{\alpha+1}}{\alpha+1}e^{-\frac{x^{2\alpha}}{2\alpha}}\right)^{\frac{1}{1-n}}. \quad (44)$$

These streamlines illustrate the influence of the constant c on the solution behavior, as shown in Figure 2.

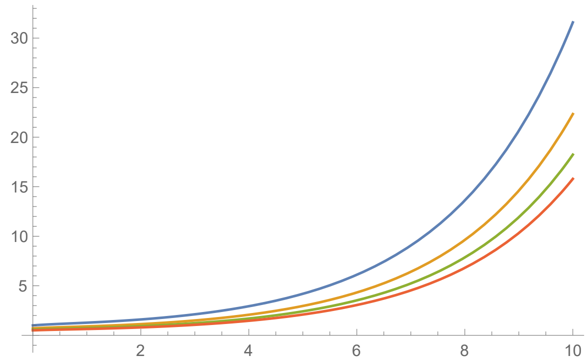

Figure 2: The solution of conformable Bernoulli equation (7) with $n = 3$, $\alpha = \frac{1}{2}$ and $c = (1, 2, 3, 4)$

The four streamlines in Figure 3 correspond to different values of the nonlinearity order $n = 2, 3, 4, 5$ from the general solution of the Bernoulli equation (7), given by:

$$y(x) = \left(c\, e^{-\frac{x^{2\alpha}}{2\alpha}} + \frac{x^{\alpha+1}}{\alpha+1}e^{-\frac{x^{2\alpha}}{2\alpha}}\right)^{\frac{1}{1-n}}. \quad (45)$$

These streamlines illustrate the effect of varying the nonlinearity order n on the solution behavior, as shown in Figure 3.

The four streamlines in Figure 4 correspond to different values of the fractional order $\alpha = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$

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Hamzeh Alkasasbeh, Lahcen Rabhi

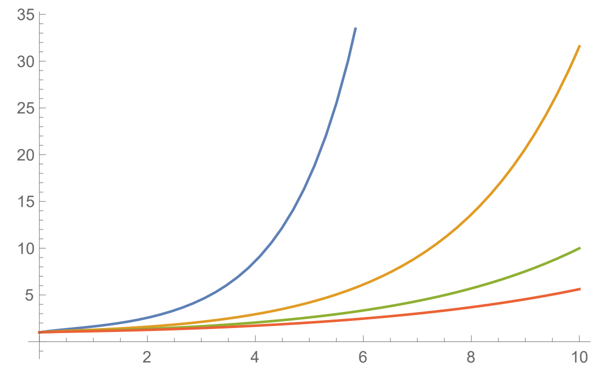

Figure 3: The solution of conformable Bernoulli equation (7) with $c = 1$, $\alpha = \frac{1}{2}$ and $n = (2, 3, 4, 5)$

from the general solution of the Bernoulli equation (7), given by:

$$y(x) = \left(c\, e^{-\frac{x^{2\alpha}}{2\alpha}} + \frac{x^{\alpha+1}}{\alpha+1} e^{-\frac{x^{2\alpha}}{2\alpha}} \right)^{\frac{1}{1-n}}. \tag{46}$$

These streamlines illustrate the impact of varying the fractional order α on the solution behavior, as shown in Figure 4.

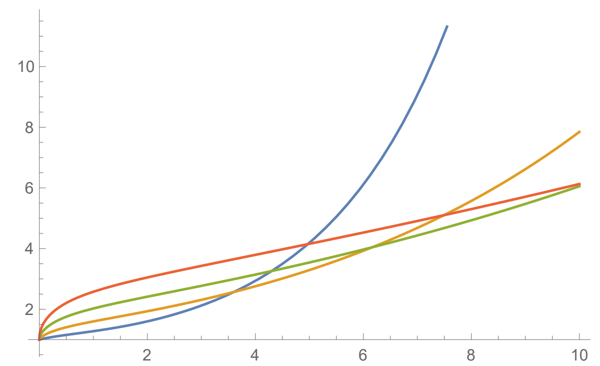

Figure 4: The solution of conformable Bernoulli equation (7) with $c = 1$, $n = 3$ and $\alpha = \left(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\right)$

The four streamlines in Figure 5 correspond to different values of the arbitrary constant $c = 1, 2, 3, 4$, the nonlinearity order $n = 2, 3, 4, 5$, and the fractional order $\alpha = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$ from the general solution of the Bernoulli equation (7), given by:

$$y(x) = \left(c\, e^{-\frac{x^{2\alpha}}{2\alpha}} + \frac{x^{\alpha+1}}{\alpha+1} e^{-\frac{x^{2\alpha}}{2\alpha}} \right)^{\frac{1}{1-n}}. \tag{47}$$

These streamlines illustrate the combined effect of varying the arbitrary constant c, the nonlinearity order n, and the fractional order α on the solution behavior, as shown in Figure 5.

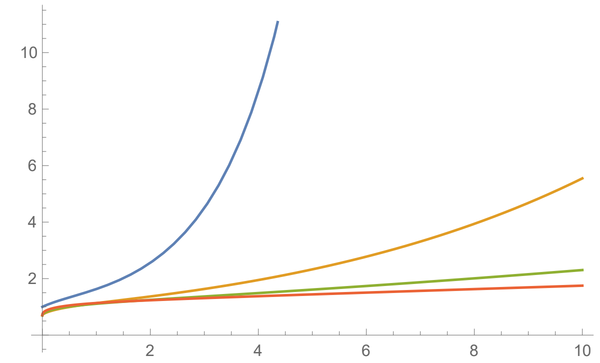

Figure 5: The solution of conformable Bernoulli equation (7) with $(c, n, \alpha) = \left[\left(1, 2, \frac{1}{2}\right), \left(2, 3, \frac{1}{3}\right), \left(3, 4, \frac{1}{4}\right), \left(4, 5, \frac{1}{5}\right)\right]$

6 Discussion

In this section, we analyze the effect of the key parameters—the arbitrary constant c, the fractional order α, and the nonlinearity order n—on the general solution of the nonlinear Bernoulli conformable fractional differential equation. The graphical representations help us draw the following conclusions:

1. **Effect of the constant** c: As c increases, the solution $y(x)$ exhibits (see Figure 2).
2. **Effect of the nonlinearity order** n: As n increases, the solution $y(x)$ evolves more slowly (see Figure 3).
3. **Effect of the fractional order** α: The value of α significantly influences the direction of the solution $y(x)$, making its behavior difficult to predict (see Figure 4).
4. **General observation from Figure 5**: This figure provides an overall view of how the solution $y(x)$ behaves under different parameter values.

Finally, it is important to note that the influence of these parameters is highly dependent on the behavior of the solution $y(x)$.

7 Conclusion

In this study, the Conformable Leibniz Method and the Conformable Bernoulli Method were successfully applied to the nonlinear Bernoulli conformable fractional differential equation, providing an exact solution to the proposed problem. The numerical examples and applications further validate the presented generalization. Moreover, the obtained results demonstrate the potential of extending the Bernoulli equation (the incompressible steady flow energy equation) to non-integer orders, particularly in physics and fluid mechanics, where it explains the conservation of mechanical work-energy. Additionally, the conformable derivative framework generalizes classical calculus while preserving essential properties such as linearity and the Leibniz rule. This approach offers a more flexible and effective tool for handling non-integer order derivatives, making it particularly valuable for modeling real-world phenomena in physics and engineering. Overall, these methods enhance both the analytical and computational efficiency of solving nonlinear fractional differential equations, contributing to the broader application of conformable fractional calculus in mathematical modeling and applied sciences.

Declaration of Generative AI and AI-assisted Technologies in the Writing Process

During the preparation of this work the authors used Grammarly for language editing. After using this service, the authors reviewed and edited the content as needed and take full responsibility for the content of the publication.

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Conflict of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.

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