On Certain Classes of Bi-Bazilevic Functions Defined by q-Ruscheweyh Differential Operator

WALEED AL-RAWASHDEH
Department of Mathematics
Zarqa University
Zarqa 13132
JORDAN

Abstract: In this paper, we make use of the concept of fractional q-calculus to introduce two new classes of bi-Bazilevic functions involving q-Ruscheweyh differential operator that are subordinate to Gegenbauer polynomials and q-analogue of hyperbolic tangent functions. This study explores the characteristics and behaviors of these functions, offering estimates for the modulus of the initial Taylor series coefficients a_2 and a_3 within this specific class and their various subclasses. Additionally, this study delves into the classical Fekete-Szegö functional problem concerning functions f that are part of our newly defined class and several of their subclasses.

Key-Words: Bi-Univalent Functions; Bi-Bazilevic; q-Ruscheweyh Differential Operator; Jackson q-Derivative Operator; q-Gamma Function; Fractional q-Calculus Operator; Gegenbauer Polynomials; q-analogue of Hyperbolic Tangent Functions; Coefficient Estimates; Fekete-Szegö Functional Problem; Convolution; Hadamard Product

Received: August 25, 2024. Revised: December 18, 2024. Accepted: January 11, 2025. Published: March 20, 2025.

1 Introduction

The q-analysis has attracted significant interest among mathematicians, especially within the field of operator theory, as highlighted by the extensive research documented in [1]. The progress made in operator theory within this domain has inspired numerous researchers, leading to the release of a diverse array of academic publications. The q-calculus provides crucial instruments that are extensively employed to analyze various categories of analytic functions. A range of geometric characteristics, including coefficient estimates, convexity, near-convexity, distortion bounds, and radii of starlikeness, has been explored in relation to these proposed classes of functions.

Recently, [2], has been published a thorough survey, providing a valuable resource for scholars engaged in the study of geometric function theory. This survey conducts an in-depth examination of the mathematical structures and applications associated with fractional q-derivative operators and fractional q-calculus, with a specific focus on their relevance to geometric function theory. It explores the intricacies of employing these fractional operators and calculus principles to define mathematical functions and their geometric properties. Furthermore, this paper highlights the real-world applications of fractional q-derivative operators in the broad field of geometric function theory, providing a comprehensive examination of the theoretical foundations as well as the practical uses of these mathematical tools within the pertinent area of research.

A variety of researchers have utilized the framework of q-calculus to create new subclasses of analytic and univalent functions. This study aims to deepen the understanding of the characteristics and features of these functions, especially concerning the recently defined q-derivative, thus clarifying the conditions that govern inclusion in the identified subclasses, see, for example, the articles, [3], [4], [5], [6], [7], [8], [9], and the related references included therein.

The collaborative endeavors of these scholars have expanded the comprehension and utilization of geometric function theory, thus promoting additional exploration and creativity within the domain of complex analysis. Recognizing q-calculus as a vital tool for establishing classifications and clarifying geometric properties highlights its importance in the ongoing advancement of geometric function theory.

In this study, the primary emphasis is placed on employing the notion of the q-derivative to formulate a particular differential operator. This operator is presented with the objective of extending the category of q-analogues of the Ruscheweyh operator among univalent functions. Through the application of this newly established operator, we introduce a novel class of bi-Bazilvic functions that are associated with the Legendre polynomials.

Let us consider the collection $\mathcal{H}$, which contains all functions $f(z)$ that are analytic in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. These functions are subject to the normalization condition $f(0) = 1 - f'(0) = 0$. Furthermore, any function f that is a member of the set $\mathcal{H}$ can be expressed in a specific form.

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad \text{where } \; z \in \mathbb{D}. \tag{1}$$

The convolution (or Hadamard product) of two analytic functions $f(z)$ given by Equation (1) and $F(z) = z + \sum_{n=2}^{\infty} b_n z^n$ is defined as:

$$(f * F)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

The process of convolution enables a more profound investigation into mathematical concepts and improves comprehension of the geometric and symmetric characteristics of functions belonging to the space $\mathcal{H}$. The importance of convolution in the fields of operator theory and geometric function theory is extensively recorded in academic literature. For more information about convolution in the geometric function theory, we invite the interested reader to see the monograph, [1], the articles, [8], [10], and the related references provided therein.

Let f and g be analytic functions in the open unit disk $\mathbb{D}$. We say that f is subordinated to g, denoted as $f(z) \prec g(z)$ for every z in $\mathbb{D}$, if there exists a Schwarz function h such that $h(0) = 0$ and $|h(z)| < 1$ for all $z \in \mathbb{D}$. Moreover for $z \in \mathbb{D}$, the relationship $f(z) = g(h(z))$ satisfies. Furthermore, in the case where g is a univalent function in $\mathbb{D}$, the subordination condition $f(z) \prec g(z)$ is equivalent to the conditions $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$. For more information about the Subordination Principle, it is recommended to consult the monographs, [11], [12], [13], and [14].

In this study, the notation $\mathcal{S}$ represents the collection of univalent functions within the open unit disk $\mathbb{D}$ that belong to the set $\mathcal{H}$. Univalent functions are invertible, but their inverses may not be defined throughout the entire unit disk. According to the Koebe one-quarter Theorem, the image of $\mathbb{D}$ under any function $f \in \mathcal{S}$ includes the disk $D(0, 1/4)$. Therefore, for each function $f \in \mathcal{S}$, there exists an inverse $f^{-1} = g$ defined

$$g(f(z)) = z, \quad z \in \mathbb{D}$$

$$f(g(w)) = w, \quad |w| < r(f); \;\; r(f) \geq 1/4.$$

Moreover, the inverse function is given by

$$g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots. \tag{2}$$

Now, we introduce the class Σ in the following manner. A function $f \in \mathcal{H}$ is said to be bi-univalent if both f and f^{-1} are univalent in the unit disk $\mathbb{D}$. Thus, we define Σ as the collection of all bi-univalent functions in $\mathcal{H}$ that are represented by Equation (1). For example, $f_1(z) = z(1-z)^{-1}$, $f_2(z) = -\log(1-z)$ and $f_3(z) = \frac{1}{2}(\log(1+z) - \log(1-z))$ are functions belong to the class Σ. On the other hand, Koebe function, $f_4(z) = \frac{2z - z^2}{2}$ and $f_5(z) = z(1-z^2)^{-1}$ are some of the functions that are not in the class Σ.For additional details regarding univalent and bi-univalent functions, we refer the readers to the articles, [15], [16], [17], the monograph, [11], [18], and the references included therein.

The research in geometric function theory uncovers complex relationships between function coefficients and their geometric properties. By analyzing the constraints on the modulus of these coefficients, we enhance our understanding of function behavior within the mathematical framework. This approach not only deepens our grasp of geometric function theory but also encourages further exploration. For instance, in the class $\mathcal{S}$, the modulus of the coefficient a_n is bounded by n. These constraints provide valuable insights into the geometric characteristics of functions in this class, particularly regarding the second coefficients, which are essential for understanding growth and distortion properties.

The study of coefficient-related characteristics of functions in the class Σ began in the 1970s. A pivotal moment occurred in 1967 when [15], studied the bi-univalent functions and found a bound for the coefficient $|a_2|$. In 1969, [16], furthered this research by establishing that the maximum value of $|a_2|$ for functions in Σ is $\frac{4}{3}$. Later, in 1979, [19], proved that for functions belonging to this class, the inequality $|a_2| \leq \sqrt{2}$ is valid. However, despite the extensive research in coefficient bounds, there is still a considerable lack of understanding regarding the general coefficients $|a_2|$ when $n \geq 4$. The difficulty in estimating these coefficients, especially the general coefficient $|a_n|$, remains an open question in the field. This situation suggests that additional investigation is crucial for a comprehensive understanding of how these coefficients behave in higher-dimensional contexts.

In 1933, [20], found an upper bound of the expression $|a_3 - \beta a_2^2|$ for univalent functions f, where the $0 \leq \beta \leq 1$. This pivotal finding gave rise to the Fekete-Szegö problem, which focuses on maximizing the modulus of the functional $\Psi_\beta(f) = a_3 - \beta a_2^2$ for functions f belonging to the class $\mathcal{H}$, with β being any complex number. A significant body of research has since been dedicated to explore the Fekete-Szegö functional and related coefficient estimation issues. Noteworthy contributions to this area can be found in various publications, including, [17], [21], [22], [23], [24], [25], [26], [27], [28], [29], and the references provided therein. These investigations have significantly enhanced the comprehension of the Fekete-Szegö problem and its relevance within the domain of geometric function theory.

2 Preliminaries and Lemmas

The information presented in this section are essential for understanding the principal outcomes of this research. In 1975, [30], introduced the operator $\mathcal{R}$, which is defined using the convolution of two power series. In particular, for a function $f \in \mathcal{H}$, a variable $z \in \mathbb{D}$, and a real number $\alpha > -1$, the Ruscheweyh operator is articulated as follows:

$$\mathcal{R}^\alpha f(z) = f(z) * \frac{z}{(1-z)^{\alpha+1}}.$$

For $\alpha = n \in \mathbb{N} \cup \{0\}$, we get the Ruscheweyh derivative $\mathcal{R}^\alpha$ as follows:

$$\mathcal{R}^\alpha f(z) = z\frac{\left(z^{\alpha-1} f(z)\right)^{(\alpha)}}{\Gamma(\alpha+1)}.$$

Moreover, the power series of $\mathcal{R}^\alpha f$ is given by

$$\mathcal{R}^\alpha f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\alpha+n)}{\Gamma(n)\Gamma(\alpha+1)} a_n z^n.$$

In this context, we revisit the notion of q-difference operators, which are essential in various domains such as hypergeometric series, quantum mechanics, and the study of geometric functions. The origins of q-calculus can be attributed to [31]. Subsequently, [32], employed fractional q-calculus operators to investigate specific classes of analytic functions related to conic regions.

The q-integer number, for $0 < q < 1$ and non-negative integer n, is defined as follows

$$[n]_q = \frac{1-q^n}{1-q} = \sum_{k=0}^{n-1} q^k, \quad \text{with} \quad [0]_q = 0.$$

In general, for any non-negative real number x, we have $[x]_q = \dfrac{1-q^x}{1-q}$. Moreover, the q-shifted factorial is defined by

$$[n]_q! = [n]_q [n-1]_q [n-2]_q \cdots [2]_q [1]_q, \quad \text{with} \quad [0]_q! = 1.$$

It is obvious that $\lim_{q \to 1^-} [n]_q = n$ and $\lim_{q \to 1^-} [n]_q! = n!$.

Let $f \in \mathcal{H}$ given by Equation (1). The q-Jackson derivative operator (or q-difference operator) is defined by

$$\mathcal{D}_q f(z) = \begin{cases} \frac{f(qz)-f(z)}{(q-1)z}, & \text{if} \quad z \neq 0 \\ f'(0), & \text{if} \quad z = 0 \\ f'(z), & \text{as } q \to 1^-. \end{cases}$$

Therefore, for a function $f \in \mathcal{H}$ that is given by Equation (1), it is easy to see that

$$\mathcal{D}_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}.$$

For example, if $n \in \mathbb{N} = \{1, 2, \cdots\}$ and $z \in \mathbb{D}$, then

$$\mathcal{D}_q(z^n) = \frac{(q^n-1)z^{n-1}}{(q-1)} = [n]_q z^{n-1}.$$

Also, $\lim_{q \to 1^-} \mathcal{D}_q(z^n) = \lim_{q \to 1^-} [n]_q z^{n-1} = nz^{n-1}$, which is the ordinary derivative of z^n.

Moreover, for $m \in \mathbb{N}$, we have

$$\mathcal{D}_q^0 f(z) = f(z), \quad \text{and} \quad \mathcal{D}_q^m f(z) = \mathcal{D}_q\left(\mathcal{D}_q^{m-1} f(z)\right).$$

It is known that, for $f, g \in \mathcal{H}$, we have the following rules for the q-difference operator

(i) $\mathcal{D}_q(mf(z) \pm ng(z)) = m\mathcal{D}_q f(z) \pm n\mathcal{D}_q g(z)$, for $m, n \in \mathbb{C}$.

(ii) $\mathcal{D}_q(fg)(z) = f(z)\mathcal{D}_q g(z) + g(z)\mathcal{D}_q f(z)$.

(iii) $\mathcal{D}_q\left(\frac{f(z)}{g(z)}\right) = \frac{g(z)\mathcal{D}_q f(z) - f(z)\mathcal{D}_q g(z)}{g(z)g(qz)}$, where $g(z)g(qz) \neq 0$.

For any $x \in \mathbb{R}$ and $n \in \mathbb{N}$, the q-generalized Pochhammer symbol is defined as follows

$$[x; n]_q = [x]_q [x+1]_q [x+2]_q \cdots [x+n-1]_q.$$

Moreover, for $x > 0$, the q-Gamman function is defined as follows

$$\Gamma_q(x+1) = [x]_q \Gamma_q(x), \quad \text{with} \quad \Gamma_q(1) = 1.$$

Now, we present a q-analogue of the Ruscheweyh differential operator by employing the convolution alongside the q-difference operator $\mathcal{R}_q^\alpha : \mathcal{H} \to \mathcal{H}$. Thus, for any $f \in \mathcal{H}$ and $\alpha > -1$, this linear operator is defined as $\mathcal{R}_q^\alpha f(z) = \mathcal{F}_{q,\alpha+1}(z) * f(z)$, where

$$\mathcal{F}_{q,\alpha+1}(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma_q(n+\alpha)}{[n-1]_q!\Gamma_q(\alpha+1)} z^n.$$

More precisely, the q-Rucheweyh differential operator can be written as follows

$$\mathcal{R}_q^\alpha f(z) = z + \sum_{n=2}^{\infty} \psi_n(q,\alpha) a_n z^n,$$

where

$$\psi_n = \psi_n(q,\alpha) = \frac{\Gamma_q(\alpha+n)}{[n-1]_q!\Gamma_q(\alpha+1)}.$$

It is clear that,

$$\mathcal{R}_q^0 f(z) = f(z), \quad \mathcal{R}_q^1 f(z) = z\mathcal{D}_q f(z), \text{ and}$$

$$\mathcal{R}_q^n f(z) = \frac{z\mathcal{D}_q^n\left(z^{n-1}f(z)\right)}{[n]_q!}, \quad n \in \mathbb{N}$$

It is worth mention that,

$$\lim_{q\to 1^-} \mathcal{F}_{q,\alpha+1}(z) = \frac{z}{(1-z)^{\alpha+1}},$$

and

$$\lim_{q\to 1^-} \mathcal{R}_q^\alpha f(z) = f(z) * \frac{z}{(1-z)^{\alpha+1}} = \mathcal{R}^\alpha f(z).$$

For more information about q-Rucheweyh differential operator and q-derivative operator, we refer the interested readers to consult the articles, [2], [3], [9], [31], [32], [33], [34], [35], [36], [37], [38], [39], and the references provided therein.

For any real numbers δ and x, where $\delta \geq 0$ and $-1 \leq x \leq 1$, and any $z \in \mathbb{D}$ the foolowing generating function of Gegenbauer polynomials

$$G_\delta(x,z) = (z^2 - 2xz + 1)^{-\delta}.$$

In addition, for a fixed x the function $G_\delta(x,z)$ is analytic on the unit disk $\mathbb{D}$ and its Taylor-Maclaurin series is given by

$$G_\delta(x,z) = \sum_{n=0}^{\infty} g_n^\delta(x) z^n.$$

Moreover, the recurrence relation of Gegenbauer polynomials is given by

$$g_n^\delta(x) = \frac{2t(n+\delta-1)g_{n-1}^\delta(x) - (n+2\delta-2)g_{n-1}^\delta(x)}{n}, \tag{3}$$

with initial values,

$$g_0^\delta(x) = 1, \quad g_1^\delta(x) = 2\delta x, \quad \text{and} \quad g_2^\delta(x) = 2\delta(\delta+1)x^2 - \delta. \tag{4}$$

It is widely recognized that Gegenbauer polynomials, along with their special cases, are orthogonal polynomials. A notable example is the Chebyshev polynomials of the second kind, denoted as $T_n(x,z)$ when $\delta = 1$, which can be expressed more specifically as $T_n(x,z) = G_1(x,z)$. For a more comprehensive understanding of the Gegenbauer polynomials and their various special cases, readers are encouraged to consult the readers to the articles, [17], [26], [29], [40], [41], [42], the monograph, [11], [18], [43], and the references therein.

In this paper, the symbol $\mathcal{P}$ denotes the Caratheodory class, which is formally defined as

$$\mathcal{P} = \{\Omega \in \mathcal{H} : \Omega(0) = 1, \ \mathcal{R}(\Omega(z)) > 0, \ z \in \mathbb{D}\}.$$

Expanding on these foundational concepts, our objective is to introduce two novel subclasses. The first subclass is comprised of bi-Bazilevic functions characterized by the q-Ruscheweyh differential operator associated with Gegenbauer polynomials. We denote this class as $\mathcal{B}^\beta(R_q^\alpha, G_\delta(x,z))$, and we next provide a formal definition for this class.

Definition 2.1. *A bi-univalent function f that is given by Equtaion (1) is said to be in the class $\mathcal{B}^\beta(R_q^\alpha, G_\delta(x,z))$ if the following subordinations hold:*

$$\frac{z^{1-\beta}\left(R_q^\alpha f(z)\right)'}{\left(R_q^\alpha f(z)\right)^{1-\beta}} \prec G_\delta(x,z),$$

and

$$\frac{w^{1-\beta}\left(R_q^\alpha g(w)\right)'}{\left(R_q^\alpha g(w)\right)^{1-\beta}} \prec G_\delta(x,w),$$

where the function $g(w) = f^{-1}(w)$ is given by the Equation (2), the parameters $\beta \geq 0$, $0 < q < 1$, $\alpha > -1$, $\delta \geq 0$ and $-1 \leq x \leq 1$.

The second class is comprised of bi-Bazilevic functions characterized by the q-Ruscheweyh differential operator associated with the q-analogue hyperbolic functions. We denote this class as $\mathcal{B}^\beta(R_q^\alpha, G_\delta(t,z))$, and we next provide a formal definition for this class.

Definition 2.2. *A bi-univalent function f that is given by Equtaion (1) is said to be in the class $\mathcal{B}^\beta(R_q^\alpha, \tanh)$ if the following subordinations hold :*

$$\frac{z^{1-\beta}\left(R_q^\alpha f(z)\right)'}{\left(R_q^\alpha f(z)\right)^{1-\beta}} \prec 1 + \tanh(qz),$$

and

$$\frac{w^{1-\beta}\left(R_q^\alpha g(w)\right)'}{\left(R_q^\alpha g(w)\right)^{1-\beta}} \prec 1 + \tanh(qw),$$

where the function $g(w) = f^{-1}(w)$ is given by the Equation (2), the parameters $\beta \geq 0$, $0 < q < 1$, and $\alpha > -1$.

The following lemma, extensively elaborated upon in existing literature (refer to, for example, [27]), represents well-established principles that hold significant importance for our presenting research.

Lemma 2.3. *if the function Ω in the Caratheodory class, then it can be written as*

$$\Omega(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots,$$

for $z \in \mathbb{D}$. Moreover, $|c_n| \leq 2$ for each $n \in \mathbb{N}$.

The lemma presented in the following discussion is extensively referenced in existing literature (refer to, for example, [27]) and is regarded as a foundational principle that significantly influences the research we are conducting.

Lemma 2.4. *Let u and v be real numbers. Let p and q be complex numbers. If $|p| < r$ and $|q| < r$,*

$$|(u+v)p + (u-v)q| \leq \begin{cases} 2r|u|, & \text{if } |u| \geq |v| \\ 2r|v|, & \text{if } |u| \leq |v|. \end{cases}$$

By selecting particular values of β in Definition 2.1, it is possible to obtain the following subclasses.

Example 1. *Let f be a bi-univalent function that is given by Equation (1). If f belongs to the subclass $\mathcal{B}^0(R_q^\alpha, G_\delta(x,z))$, then the following subordinations hold*

$$\frac{z\left(R_q^\alpha f(z)\right)'}{\left(R_q^\alpha f(z)\right)} \prec G_\delta(x,z), \tag{5}$$

and

$$\frac{w\left(R_q^\alpha g(w)\right)'}{\left(R_q^\alpha g(w)\right)} \prec G_\delta(x,w), \tag{6}$$

where the function $g(w) = f^{-1}(w)$ is given by Equation (2), the parameters $0 < q < 1$, $\alpha > -1$, $\delta \geq 0$ and $-1 \leq x \leq 1$.

Example 2. *Let f be a bi-univalent function that is given by Equation (1). If f belongs to the subclass $\mathcal{B}^1(R_q^\alpha, G_\delta(x,z))$ then the following subordinations hold*

$$\left(R_q^\alpha f(z)\right)' \prec G_\delta(x,z), \tag{7}$$

and

$$\left(R_q^\alpha g(w)\right)' \prec G_\delta(x,w), \tag{8}$$

where the function $g(w) = f^{-1}(w)$ is given by Equation (2), the parameters $0 < q < 1$, $\alpha > -1$, $\delta \geq 0$ and $-1 \leq x \leq 1$.

Moreover, as $q \to 1^-$ and taking $\alpha = 0$, we get $R_q^0 f(z) = f(z)$. Therefore, we get the following close-to-starlike subclasses. For more information about classes of starlike analytic functions, we refer the readers, for example, to the articles, [22], [44], [45], [46], [47], [48], [49], [50], and the references provided therein.

Example 3. *Let f be a bi-univalent function that is given by Equation (1). If f belongs to the subclass $\mathcal{B}^*(G_\delta(x,z))$ then the following subordinations hold*

$$\frac{zf'(z)}{f(z)} \prec G_\delta(x,z), \tag{9}$$

and

$$\frac{wg'(w)}{g(w)} \prec G_\delta(x,w), \tag{10}$$

where the function $g(w) = f^{-1}(w)$ is given by Equation (2), the parameters $\delta \geq 0$ and $-1 \leq x \leq 1$.

In addition, by taking $\delta = 1$ in Definition 2.1, we easily obtain the following subsequent subclass.

Example 4. *Let f be a bi-univalent function that is given by Equation (1). If f belongs to the subclass $\mathcal{G}^*(\delta, \phi)$ then the following subordinations hold*

$$\frac{z^{1-\beta}\left(R_q^\alpha f(z)\right)'}{\left(R_q^\alpha f(z)\right)^{1-\beta}} \prec T_n(x,z), \tag{11}$$

and

$$\frac{w^{1-\beta}\left(R_q^\alpha g(w)\right)'}{\left(R_q^\alpha g(w)\right)^{1-\beta}} \prec T_n(x,w), \tag{12}$$

where the function $g(w) = f^{-1}(w)$ is given by Equation (2), the parameters $\beta \geq 0$, $0 < q < 1$, $\alpha > -1$ and $-1 \leq x \leq 1$.

Similarly, By selecting particular values of β in Definition 2.2, it is possible to obtain the subsequent subclasses.

Example 5. *Let f be a bi-univalent function that is given by Equation (1). If f belongs to the subclass $\mathcal{B}^0(R_q^\alpha, \tanh)$ then the following subordinations hold*

$$\frac{z\left(R_q^\alpha f(z)\right)'}{\left(R_q^\alpha f(z)\right)} \prec 1 + \tanh(qz), \tag{13}$$

and

$$\frac{w\left(R_q^\alpha g(w)\right)'}{\left(R_q^\alpha g(w)\right)} \prec 1 + \tanh(qw), \tag{14}$$

where the function $g(w) = f^{-1}(w)$ is given by the Equation (2), the parameters $0 < q < 1$, $\alpha > -1$, $\delta \geq 0$ and $-1 \leq x \leq 1$.

Example 6. *Let f be a bi-univalent function that is given by Equation (1). If f belongs to the subclass $\mathcal{B}^1(R_q^\alpha, \tanh)$ then the following subordinations hold*

$$\left(R_q^\alpha f(z)\right)' \prec 1 + \tanh(qz), \tag{15}$$

and

$$\left(R_q^\alpha g(w)\right)' \prec 1 + \tanh(qw), \tag{16}$$

where the function $g(w) = f^{-1}(w)$ is given by Equation (2), the parameters $0 < q < 1$, $\alpha > -1$, $\delta \geq 0$ and $-1 \leq x \leq 1$.

Moreover, as $q \to 1^-$ and taking $\alpha = 0$, we get $R_q^0 f(z) = f(z)$. Therefore, we get the following close-to-starlike subclasses.

Example 7. *Let f be a bi-univalent function that is given by Equation (1). If f belongs to the subclass $\mathcal{B}^*(\tanh)$ then the following subordinations hold*

$$\frac{zf'(z)}{f(z)} \prec 1 + \tanh(qz), \tag{17}$$

and

$$\frac{wg'(w)}{g(w)} \prec 1 + \tanh(qw), \tag{18}$$

where the function $g(w) = f^{-1}(w)$ is given by Equation (2), the parameters $\delta \geq 0$ and $-1 \leq x \leq 1$.

This study aims to investigate two novel classes of bi-Bazilevic functions, which are characterized by the q-Ruscheweyh operator within the open unit disk $\mathbb{D}$. The primary goal is to derive estimates for the moduli of the initial coefficients $|a_2|$ and $|a_3|$, which are associated with the Taylor-Maclaurin series representation of functions within these categories. Furthermore, the research addresses the Fekete-Szegö functional problem relevant to these functions, thereby contributing to a deeper understanding of their fundamental properties. Additionally, several established corollaries are provided based on the selection of parameters used in defining the specific classes under consideration.

3 Main Results and Corollaries on Coefficient Estimations

This section of the paper focuses on finding the bounds pertaining to the modulus of the initial coefficients of functions that belong to the class $\mathcal{B}^\beta(R_q^\alpha, G_\delta(x,z))$ and to the class $\mathcal{B}^\beta(R_q^\alpha, \tanh)$, along with several of their various subclasses, as delineated in Equation (1).

Theorem 3.1. *Let f be a bi-univalent function that is given by Equation (1). If the function f belongs to the class $\mathcal{B}^\beta(R_q^\alpha, G_\delta(x,z))$, then the following inequalities hold:*

$$|a_2| \leq \frac{4\delta^{3/2}|x|^{3/2}}{\sqrt{\left|\begin{array}{l}4x^2\delta^2\left(2(\beta+2)\psi_3+\right.\\ \left.(\beta-1)(\beta+2)\psi_2^2 - 2(\beta+1)^2\psi_2^2 g_2^\delta(x)\right.\end{array}\right|}}, \tag{19}$$

and

$$|a_3| \leq \frac{2\delta|x|}{(\beta+2)\psi_3} + \frac{4\delta^2 x^2}{(\beta+1)^2\psi_2^2}. \tag{20}$$

Proof. Le f be a bi-univalent function that belongs to the class $\mathcal{B}(\beta, \delta, R_\eta^\alpha, \beta(t))$. Consulting the Definition 2.1 and Subordination Principle, two Schwarz functions $u(z)$ and $v(w)$ can be identified within the open unit disk $\mathbb{D}$ such that

$$\frac{z^{1-\beta}\left(R_q^\alpha f(z)\right)'}{\left(R_q^\alpha f(z)\right)^{1-\beta}} = G_\delta(x, u(z)), \tag{21}$$

and

$$\frac{w^{1-\beta}\left(R_q^\alpha g(w)\right)'}{\left(R_q^\alpha g(w)\right)^{1-\beta}} = G_\delta(x, v(w)). \tag{22}$$

now, comparing the coefficients of both sides of Equation (21) and Equation (22), we obtain the following set of equations

$$(\beta+1)\psi_2 a_2 = g_1^\delta(x)u_1, \tag{23}$$

$$\begin{aligned}&\frac{(\beta-1)(\beta+2)}{2}\psi_2^2 a_2^2 + (\beta+2)\psi_3 a_3\\ &= g_1^\delta(x)u_2 + g_2^\delta(x)u_1^2,\end{aligned} \tag{24}$$

$$-(\beta+1)\psi_2 a_2 = g_1^\delta(x)v_1, \tag{25}$$

and

$$\begin{aligned}&\left(2(\beta+2)\psi_3 + \frac{(\beta-1)(\beta+2)}{2}\psi_2^2\right)a_2^2 - (\beta+2)\psi_3 a_3\\ &= g_1^\delta(x)v_2 + g_2^\delta(x)v_1^2.\end{aligned} \tag{26}$$

Therefore, using Equation (23) and Equation (25), we easily derive the the subsequent equation

$$a_2 = \frac{g_1^\delta(x)u_1}{(\beta+1)\psi_2} = \frac{-g_1^\delta(x)v_1}{(\beta+1)\psi_2} \tag{27}$$

Moreover, we easily get the following equation

$$2(\beta+1)^2\psi_2^2 a_2^2 = \left(g_1^\delta(x)\right)^2(u_1^2+v_1^2). \tag{28}$$

On one hand, adding Equation (24) to Equation (26), we derive the subsequent equation

$$\begin{aligned}&\left((\beta-1)(\beta+2)\psi_2^2 + 2(\beta+2)\psi_3\right)a_2^2\\ &= g_1^\delta(x)(u_2+v_2) + g_2^\delta(x)(u_1^2+v_1^2).\end{aligned} \tag{29}$$

Therefore, using equation (28), the last equation can be written as

$$\begin{aligned}&\left((\beta-1)(\beta+2)\psi_2^2 + 2(\beta+2)\psi_3\right)a_2^2\\ &= g_1^\delta(x)(u_2+v_2) + \frac{2g_2^\delta(x)(\beta+1)^2\psi_2^2}{\left(g_1^\delta(x)\right)^2}a_2^2.\end{aligned} \tag{30}$$

Equivalently, the last equation can be written as

$$a_2^2 = \frac{\left(g_1^\delta(x)\right)^3(u_2+v_2)}{\left\{\begin{array}{l}\left[(\beta-1)(\beta+2)\psi_2^2 + 2(\beta+2)\psi_3\right]\left(g_1^\delta(x)\right)^2\\ -2g_2^\delta(x)(\beta+1)^2\psi_2^2\end{array}\right\}}. \tag{31}$$

Therefore, considering Equation (4) and then using the constraints $|u_2| \leq 1$ and $|v_2| \leq 1$, , we easily get the desired estimation of $|a_2|$ presented in Inequality (19).

Secondly, our objective is to ascertain the coefficient estimate for $|a_3|$. By substituting Equation (26) into Equation (24), we can formulate the following equation.

$$\begin{aligned}&2(\beta+2)\psi_3 a_3 - 2(\beta+2)\psi_3 a_2^2 \\ &= g_1^\delta(x)(u_2 - v_2) + g_2^\delta(u_1^2 - v_1^2).\end{aligned} \tag{32}$$

Now, using Equation (27), we get $u_1 = -v_1$. Consequently, the last equation can be expressed in the following manner

$$a_3 = \frac{g_1^\delta(x)(u_2 - v_2)}{2(\beta+2)\psi_3} + a_2^2. \tag{33}$$

Moreover, consulting Equation (28), Equation (33) can be written as follows

$$a_3 = \frac{g_1^\delta(x)(u_2 - v_2)}{2(\beta+2)\psi_3} + \frac{\left(g_1^\delta(x)\right)^2(u_1^2 + v_1^2)}{2(\beta+1)^2\psi_2^2} \tag{34}$$

Finally, considering the Equation (4) then using the constraints $|u_j| \leq 1$ and $|v_j| \leq 1$ for $j = 1, 2$, the last equation provides the required bound for $|a_3|$, as indicated by Inequality (20). Therefore, the proof of Theorem 3.1 is now complete. □

The subsequent theorem establishes the estimates concerning the modulus of the initial coefficients of functions that are part of the class $\mathcal{B}^\beta(R_q^\alpha, \tanh)$. The methods employed in its proof are akin to those utilized in the proof of Theorem 3.1.

Theorem 3.2. *Let f be a bi-univalent function that is given by Equation (1). If the function f belongs to the class $\mathcal{B}^\beta(R_q^\alpha, \tanh)$, then the following inequalities hold*

$$|a_2| \leq \frac{\sqrt{2}q}{\sqrt{qK + 2(\beta+1)^2\psi_2^2}}, \tag{35}$$

and

$$|a_3| \leq \frac{q}{(\beta+2)\psi_3} + \frac{2q^2}{qK + 2(\beta+1)^2\psi_2^2}, \tag{36}$$

where

$$K = 2(\beta+2)\psi_3 + (\beta-1)(\beta+2)\psi_2^2.$$

Proof. Let f be a function that is part of the class $\mathcal{B}^\beta(R_q^\alpha, \tanh)$. Based on Definition 2.2 and the Subordination Principle, it is possible to identify two Schwarz functions, $u(z)$ and $v(w)$, that are defined in the open unit disk $\mathbb{D}$ such that

$$\frac{z^{1-\beta}\left(R_q^\alpha f(z)\right)'}{\left(R_q^\alpha f(z)\right)^{1-\beta}} = 1 + \tanh(qu(z)), \tag{37}$$

and

$$\frac{w^{1-\beta}\left(R_q^\alpha g(w)\right)'}{\left(R_q^\alpha g(w)\right)^{1-\beta}} = 1 + \tanh(qv(w)), \tag{38}$$

Now, using these Schwarz functions, we introduce two analytic functions, denoted as $k(z)$ and $h(w)$, defined as follows:

$$h(z) = \frac{1+u(z)}{1-u(z)} \quad \text{and} \quad h(w) = \frac{1+v(w)}{1-v(w)}.$$

It is clear that the functions $k(z)$ and $h(w)$ are analytic in the open unit disk $\mathbb{D}$ and fall within the Caratheodory class. Consequently, we can represent them in the following way:

$$k(z) = \frac{1+u(z)}{1-u(z)} = 1 + k_1 z + k_2 z^2 + \cdots$$

and

$$h(w) = \frac{1+v(w)}{1-v(w)} = 1 + h_1 w + h_2 w^2 + \cdots$$

Moreover, $h(0) = 1$, $k(0) = 1$, $\Re(u) > 0$, $\Re(v) > 0$, $|h_j| \leq 2$ and $|p_j| \leq 2$ for all $j \in \mathbb{N}$.

Hence, we can express $u(z)$ and $v(w)$ in the following ways

$$u(z) = \frac{k(z)-1}{k(z)+1} = \frac{k_1}{2}z + \left(\frac{k_2}{2} - \frac{k_1^2}{4}\right)z^2 + \cdots, \tag{39}$$

and

$$v(w) = \frac{h(w)-1}{h(w)+1} = \frac{h_1}{2}w + \left(\frac{h_2}{2} - \frac{h_1^2}{4}\right)w^2 + \cdots. \tag{40}$$

By referring to Equation (39), we can express the right-hand side of Equation (37) in the following manner:

$$\begin{aligned}1 + \tanh(qu(z)) = 1 &+ \frac{qk_1}{2}z + q\left(\frac{k_2}{2} - \frac{k_1^2}{4}\right)z^2 \\ &+ q\begin{pmatrix}\dfrac{k_2}{2} - \dfrac{k_1 k_2}{2} \\ + \dfrac{(3-2q^2)k_1^3}{24}\end{pmatrix}z^3 + \cdots\end{aligned} \tag{41}$$

In the same way, by referring to Equation (40), we can express the right-hand sides of Equation (38) as follows:

$$\begin{aligned}1 + \tanh(qv(w)) = 1 &+ \frac{qh_1}{2}w + q\left(\frac{h_2}{2} - \frac{h_1^2}{4}\right)w^2 \\ &+ q\begin{pmatrix}\dfrac{h_2}{2} - \dfrac{h_1 h_2}{2} \\ + \dfrac{(3-2q^2)h_1^3}{24}\end{pmatrix}z^3 + \cdots\end{aligned} \tag{42}$$

Thus, we take into account Equation (41) and Equation (42) then analyze the coefficients from both sides of Equations (37) and (38), we obtain the following four equations

$$(\beta+1)\psi_2 a_2 = \frac{q}{2}k_1, \tag{43}$$

$$\frac{(\beta-1)(\beta+2)}{2}\psi_2^2 a_2^2 + (\beta+2)\psi_3 a_3 = q\left(\frac{k_2}{2} - \frac{k_1^2}{4}\right), \tag{44}$$

$$-(\beta+1)\psi_2 a_2 = \frac{q}{2}h_1, \tag{45}$$

and

$$\begin{pmatrix} 2(\beta+2)\psi_3 \\ + \dfrac{(\beta-1)(\beta+2)}{2}\psi_2^2 \end{pmatrix} a_2^2 - (\beta+2)\psi_3 a_3 = q\left(\frac{k_2}{2} - \frac{k_1^2}{4}\right). \tag{46}$$

Now, on one hand, using Equation (43) and Equation (45) we arrive at these two equations

$$k_1 = -h_1, \tag{47}$$

and

$$8(\beta+1)^2\psi_2^2 a_2^2 = q^2\left(k_1^2 + h_1^2\right). \tag{48}$$

On the other hand, adding Equation (44) to Equation (46), then utilizing Equation (48) we derive the following equation

$$a_2^2 = \frac{q^2\left(k_2 + h_2\right)}{2qK + 4(\beta+1)^2\psi_2^2}. \tag{49}$$

Therefore, consulting the last equation and using the constraints $|k_2| \leq 2$ and $|h_2| \leq 2$, simple calculations gives the desired Inequality (35).

Secondly, our objective is to ascertain the coefficient estimate for $|a_3|$. Subtracting Equation (46) from Equation (44), then using Equation (47), we obtain the following equation

$$a_3 = \frac{q(k_2 - h_2)}{4(\beta+2)\psi_3} + a_2^2. \tag{50}$$

Now, by referring to Equation (49), we can express the final equation in this manner.

$$a_3 = \frac{q(k_2 - h_2)}{4(\beta+2)\psi_3} + \frac{q^2\left(k_2 + h_2\right)}{2qK + 4(\beta+1)^2\psi_2^2}. \tag{51}$$

Thus, By applying the constraints $|k_2| \leq 2$ and $|h_2| \leq 2$, we can derive the required estimate of $|a_3|$ that represented in Inequality (36). Therefore, the proof of Theorem 3.2 is now complete. □

The following corollaries are directly obtained from Theorem 3.1, contingent upon the conditions specified in the earlier examples. The techniques employed in deriving these corollaries closely mirror those applied in the proof of aforementioned theorem, which is the rationale behind our decision to exclude the detailed proofs.

Corollary 3.3. *Let f be a bi-univalent function that is given by Equation (1). If the function f belongs to the class $\mathcal{B}^0(R_q^\alpha, G_\delta(x,z))$, then the following inequalities hold:*

$$|a_2| \leq \frac{2\left(\delta|x|\right)^{3/2}}{\sqrt{|8x^2\delta^2\psi_3 + \psi_2^2\delta(2x^2(1-\delta)) - 1|}},$$

and

$$|a_3| \leq \frac{\delta|x|}{\psi_3} + \frac{4\delta^2 x^2}{\psi_2^2}.$$

Corollary 3.4. *Let f be a bi-univalent function that is given by Equation (1). If the function f belongs to the class $\mathcal{B}^1(R_q^\alpha, G_\delta(x,z))$, then the following inequalities hold:*

$$|a_2| \leq \frac{\sqrt{2}\left(\delta|x|\right)^{3/2}}{\sqrt{|3x^2\delta^2\psi_3 - \psi_2^2 g_2^\delta(x)|}},$$

and

$$|a_3| \leq \frac{2\delta|x|}{3\psi_3} + \frac{\delta^2 x^2}{\psi_2^2}.$$

Corollary 3.5. *Let f be a bi-univalent function that is given by Equation (1). If the function f belongs to the class $\mathcal{B}^*(G_\delta(x,z))$, then the following inequalities hold:*

$$|a_2| \leq \frac{2\left(\delta|x|\right)^{3/2}}{\sqrt{|2x^2\delta(3\delta+1) - \delta|}},$$

and

$$|a_3| \leq \delta|x| + 4\delta^2 x^2.$$

Corollary 3.6. *Let f be a bi-univalent function that is given by Equation (1). If the function f belongs to the class $\mathcal{B}^\beta(R_q^\alpha, T_n(x,z))$, then the following inequalities hold:*

$$|a_2| \leq \frac{4|x|^{3/2}}{\sqrt{\left|\begin{matrix} 4x^2\left(2(\beta+2)\psi_3 + (\beta-1)(\beta+2)\psi_2^2\right) \\ -2(\beta+1)^2\psi_2^2 g_2^\delta(x) \end{matrix}\right|}},$$

and

$$|a_3| \leq \frac{2|x|}{(\beta+2)\psi_3} + \frac{4x^2}{(\beta+1)^2\psi_2^2}.$$

The following corollaries are directly obtained from Theorem 3.2, contingent upon the conditions specified in the earlier examples. The techniques employed in deriving these corollaries closely mirror those applied in the proof of aforementioned theorem, which is the rationale behind our decision to exclude the detailed proofs.

Corollary 3.7. *Let f be a bi-univalent function that is given by Equation (1). If the function f belongs to the class $\mathcal{B}^0(R_q^\alpha, \tanh)$, then the following inequalities hold*

$$|a_2| \leq \frac{q}{\sqrt{2q\psi_3 + (1-q)\psi_2^2}},$$

and

$$|a_3| \leq \frac{q}{2\psi_3} + \frac{q^2}{2q\psi_3 + (1-q)\psi_2^2}.$$

Corollary 3.8. *Let f be a bi-univalent function that is given by Equation (1). If the function f belongs to the class $\mathcal{B}^1(R_q^\alpha, \tanh)$, then the following inequalities hold*

$$|a_2| \leq \frac{q}{\sqrt{3q\psi_3 + 4\psi_2^2}},$$

and

$$|a_3| \leq \frac{q}{3\psi_3} + \frac{q^2}{3q\psi_3 + 4\psi_2^2}.$$

Corollary 3.9. *Let f be a bi-univalent function that is given by Equation (1). If the function f belongs to the class $\mathcal{S}^*(\tanh)$, then the following inequalities hold*

$$|a_2| \leq \frac{1}{\sqrt{2}}, \quad \text{and} \quad |a_3| \leq 1.$$

4 Main Results and Corollaries on Fekete-Szegö problem

In this section, we will explore how to establish the Fekete-Szegö inequalities for functions that fall within the specified class. $\mathcal{B}^\beta(R_q^\alpha, G_\delta(x,z))$ and to the class $\mathcal{B}^\beta(R_q^\alpha, \tanh)$, which are contain bi-Bazilevic functions defined through the q-Ruscheweyh differential operator and associated with Gegenbauer polynomials and q-analogue of hyperbolic tangent function, respectively. Additionally, we aim to establish Fekete-Szegö inequalities for several subclasses within our defined class.

Theorem 4.1. *Let f be a bi-univalent function that is given by Equation (1). If the function f belongs to the class $\mathcal{B}^\beta(R_q^\alpha, G_\delta(x,z))$, then for a real number ζ and $\delta \neq 0$ the following inequality holds*

$$|a_3 - \zeta a_2^2| \leq \begin{cases} \frac{2\delta|x|}{(\beta+2)\psi_3}, & \text{if } |1-\zeta| \leq |\mathcal{Y}| \\ \frac{8\delta^3|x^3||1-\zeta|}{|2\delta^2x^2K - B^2 g_2^\delta(x)|}, & \text{if } |1-\zeta| \geq |\mathcal{Y}|, \end{cases} \tag{52}$$

where

$$K = 2(\beta+2)\psi_3 + (\beta-1)(\beta+2)\psi_2^2, \quad B = (\beta+1)\psi_2, \quad \text{and}$$

$$\mathcal{Y} = \frac{2x^2\delta(K - B^2) - B^2(2x^2+1)}{4x\delta(\beta+2)\psi_3}.$$

Proof. Let ζ be any real number. Consulting Equation (35) and Equation (31), we easily derive the following equation

$$a_3 - \zeta a_2^2 = \frac{g_1^\delta(x)(u_2 - v_2)}{2(\beta+2)\psi_3} + $$
$$+ \frac{(1-\zeta))\left(g_1^\delta(x)\right)^3(u_2+v_2)}{\left\{\begin{array}{l}\left(g_1^\delta(x)\right)^2[2(\beta+2)\psi_3 + (\beta-1)(\beta+2)\psi_2^2] \\ -2(\beta+1)^2\psi_2^2 g_2^\delta(x)\end{array}\right\}}.$$

Therefore, the last equation can be written as follows

$$a_3 - \zeta a_2^2 = \left(g_1^\delta(x)\right)\left\{\begin{array}{l}\left(\Delta + \frac{1}{2(\beta+2)\psi_3}\right)u_2 \\ +\left(\Delta - \frac{1}{2(\beta+2)\psi_3}\right)v_2\end{array}\right\},$$

where

$$\Delta = \frac{(1-\zeta)\left(g_1^\delta(x)\right)^2}{K\left(g_1^\delta(x)\right)^2 - 2B^2 g_2^\delta(x)}.$$

Hence, by consulting Lemma 2.4, we are able to achieve the following inequality

$$|a_3 - \zeta a_2^2| \leq \begin{cases} \frac{\left|g_1^\delta(x)\right|}{(\beta+2)\psi_3}, & \text{if } |\Delta| \leq \frac{1}{2(\beta+2)\psi_3} \\ \frac{2|1-\zeta|\left|g_1^\delta(x)\right|^3}{\left|K\left(g_1^\delta(x)\right)^2 - 2B^2 g_2^\delta(x)\right|}, & \text{if } |\Delta| \geq \frac{1}{2(\beta+2)\psi_3}. \end{cases}$$

Finally, by simplifying the right-hand side of the final inequality, we arrive at the expected result as presented in Inequality (52). This signifies the completion of the proof. ☐

The following corollaries emerge as logical extensions of Theorem 4.1, given the conditions outlined in the preceding examples. The methodology employed to derive this corollary closely resembles that utilized in the earlier theorem; therefore, we have opted to forgo a detailed proof for this corollary.

Corollary 4.2. *Let f be a bi-univalent function that is given by Equation (1). If the function f is obeying the Subordination conditions (5) and (6), then for a real number ζ and $\delta \neq 0$ the following holds*

$$|a_3 - \zeta a_2^2| \leq \begin{cases} \frac{\delta|x|}{2\psi_3}, & \text{if } |1-\zeta| \leq |\mathcal{Y}_1| \\ \frac{8\delta^3|x^3||1-\zeta||g_1^\delta(x)|^3}{|\mathcal{A}|}, & \text{if } |1-\zeta| \geq |\mathcal{Y}_1|, \end{cases}$$

where

$$\mathcal{A} = 2\delta^2x^2(16\psi_3 - 3\psi_2^2) + \delta\psi_2^2(1-2x^2)),$$

$$\mathcal{Y}_1 = \frac{2x^2\delta((4\psi_3 - 3\psi_2^2) - \psi_2^2(2x^2+1)}{8x\delta\psi_3}.$$

Corollary 4.3. *Let f be a bi-univalent function that is given by Equation (1). If the function f is obeying* *the Subordination conditions (7) and (8), then for a real number ζ and $\delta \neq 0$ the following holds*

$$|a_3 - \zeta a_2^2| \leq \begin{cases} \frac{2\delta|x|}{3\psi_3}, & \text{if } |1-\zeta| \leq |\mathcal{Y}_2| \\ \frac{2\delta^3|x^3||1-\zeta|}{|3\delta^2 x^2\psi_3 - \psi_2^2 g_2^\delta(x)|}, & \text{if } |1-\zeta| \geq |\mathcal{Y}_2|, \end{cases}$$

where

$$\mathcal{Y}_2 = \frac{x^2\delta(3\psi_3 - 2\psi_2^2) - \psi_2^2(2x^2+1)}{3x\delta\psi_3}.$$

Corollary 4.4. *Let f be a bi-univalent function that is given by Equation (1). If the function f is obeying the Subordination conditions (9) and (10), then for a real number ζ and $\delta \neq 0$ the following holds*

$$|a_3 - \zeta a_2^2| \leq \begin{cases} \delta|x|, & \text{if } |1-\zeta| \leq |\frac{1-2x^2(1-\delta)}{8x\delta}| \\ \frac{8\delta^2|x^3||1-\zeta|}{|\delta^2x^2 - 2x^2+1|}, & \text{if } |1-\zeta| \geq |\frac{1-2x^2(1-\delta)}{8x\delta}|. \end{cases}$$

In the upcoming theorem, we present the Fekete-Szegö inequality specifically for functions that are part of the designated class $\mathcal{B}^\beta(R_q^\alpha, \tanh)$, which contains the bi-Bazilevic functions defined through the q-Ruscheweyh differential operator and subordinate to q-analogue of hyperbolic tangent function. The proof of this Theorem uses similar techniques as that in the proof of Theorem 3.1.

Theorem 4.5. *Let f be a bi-univalent function that is given by Equation (1). If the function f belongs to the class $\mathcal{B}^\beta(R_q^\alpha, \tanh)$, then for a real number ζ and $\delta \neq 0$ the following inequality holds*

$$|a_3 - \zeta a_2^2| \leq \begin{cases} \frac{q}{2(\beta+2)\psi_3}, & \text{if } |1-\zeta| \leq \mathcal{B} \\ \frac{2q^2|1-\zeta|}{qK + 2(\beta+1)^2\psi_2^2}, & \text{if } |1-\zeta| \geq \mathcal{B}, \end{cases} \tag{53}$$

where

$$\mathcal{B} = \frac{qK + 2(\beta+1)^2\psi_2^2}{2q(\beta+2)\psi_3},$$

$$K = 2(\beta+2)\psi_3 + (\beta-1)(\beta+2)\psi_2^2.$$

Proof. Let ζ be any real number. Consulting Equation (49) and Equation (50), we easily obtain the following equation

$$a_3 - \zeta a_2^2 = \left(\frac{q}{2}\right)\left\{ \begin{array}{c} \left(\Omega + \frac{1}{2(\beta+2)\psi_3}\right)k_2 \\ + \left(\Omega - \frac{1}{2(\beta+2)\psi_3}\right)h_2 \end{array} \right\},$$

where

$$\Omega = \frac{(1-\zeta)q}{qK + 2(1+\beta)^2\psi_2^2)}.$$

Hence, by consulting Lemma 2.4 alongside with the constraints $|k_2| \leq 2$ and $|h_2| \leq 2$, we are able to achieve the following inequality

$$|a_3 - \zeta a_2^2| \leq \begin{cases} \frac{q}{(\beta+2)\psi_3}, & \text{if } |\Omega| \leq \frac{1}{2(\beta+2)\psi_3} \\ \frac{2q^2|1-\zeta|}{qK + 2(1+\beta)^2\psi_2^2)}, & \text{if } |\Omega| \geq \frac{1}{2(\beta+2)\psi_3}. \end{cases}$$

Finally, by streamlining the right-hand side of the concluding inequality, we achieve the anticipated outcome as demonstrated in Inequality (53). This marks the completion of the proof. □

The following corollaries are derived directly from Theorem 4.5, provided that the conditions outlined in the preceding examples are met. The methods utilized in the derivation of these corollaries closely resemble those used in the proof of the aforementioned theorem, which is the reason for our choice to omit the comprehensive proofs.

Corollary 4.6. *Let f be a bi-univalent function that is given by Equation (1). If the function f belongs to the class $\mathcal{B}^0(R_q^\alpha, \tanh)$, then the following inequalities hold*

$$|a_3 - \zeta a_2^2| \leq \begin{cases} \frac{q}{4\psi_3}, & \text{if } \zeta \in [\zeta_1, \zeta_2] \\ \frac{q^2|1-\zeta|}{2q\psi_3 + (1-q)\psi_2^2}, & \text{if } \zeta \notin [\zeta_1, \zeta_2], \end{cases}$$

where

$$\zeta_1 = \frac{(q-1)\psi_2^2}{2q\psi_3} \quad \text{and} \quad \zeta_2 = 2 - \zeta_1.$$

Corollary 4.7. *Let f be a bi-univalent function that is given by Equation (1). If the function f belongs to the class $\mathcal{B}^1(R_q^\alpha, \tanh)$, then the following inequalities hold*

$$|a_3 - \zeta a_2^2| \leq \begin{cases} \frac{q}{6\psi_3}, & \text{if } \zeta \in [\zeta_3, \zeta_4] \\ \frac{q^2|1-\zeta|}{3q\psi_3 + 4\psi_2^2}, & \text{if } \zeta \notin [\zeta_3, \zeta_4], \end{cases}$$

where

$$\zeta_3 = \frac{-4\psi_2^2}{3q\psi_3} \quad \text{and} \quad \zeta_4 = 2 - \zeta_3.$$

Corollary 4.8. *Let f be a bi-univalent function that is given by Equation (1). If the function f belongs to the class $\mathcal{S}^*(\tanh)$, then the following inequalities hold*

$$|a_3 - \zeta a_2^2| \leq \begin{cases} \frac{1}{4}, & \text{if } \zeta \in [0,2] \\ \frac{|1-\zeta|}{2}, & \text{if } \zeta \notin [0,2]. \end{cases}$$

5 Conclusion

This research paper investigates a new category of bi-Bazilevic functions that are defined through the q-Ruscheweyh differential operator and are linked to Legendre polynomials. The author has derived estimates for the initial coefficients and examined the Fekete-Szegö functional problem concerning functions within these specific classes. In conclusion, potential avenues for future research are suggested, particularly the exploration of substituting Legendre polynomials with other types of orthogonal polynomials, such as Gegenbauer polynomials. Furthermore, the findings presented in this study are anticipated to motivate researchers to expand the scope of this investigation to include meromorphic bi-univalent functions.

Acknowledgment:
This research is partially funded by Zarqa University. The author would like to express his sincerest thanks to Zarqa University for the financial support. The author expresses sincere gratitude to the anonymous referees for their insightful comments and recommendations, which significantly enhanced the quality of this paper's presentation.

Declaration of Generative AI and AI-assisted Technologies in the Writing Process:
During the preparation of this work the author used Grammarly for language editing. After using this service, the author reviewed and edited the content as needed and take full responsibility for the content of the publication.

References

[1] A. Aral, V. Gupta, R.P. Agarwal, *Applications of q-Calculus in Operator Theory*; Springer: New York, NY, USA, 2013.

[2] H.M. Srivastava, Operators of basic (or $q-$) calculus and fractional $q-$calculus and their applications in Geometric Function theory of Complex Analysis, *Iran. J. Sci Technol Trans. Sci.* 2020, 44, 327–344.

[3] L. Andrei and V.A. Caus, Subordinations results on a q-Derivative differential operator, *Mathematics* 2024, 12, 208.

[4] R. Ibrahim, J. Suzan and M.D. Obaiys, Studies on generalized differential-difference operator of normalized analytic functions, *Southeast Asian Bull. Math.*, 45(2021), 43–55.

[5] M.F. Khan, I. Al-shbeil, S. Khan, N. Khan, W.U. Haq and J. Gong, Applications of a q-Differential operator to a class of harmonic mappings defined by q-Mittag–Leffler functions, *Symmetry* 2022, 14, 1905.

[6] V. Nezir and N. Mustafa, Analytic functions expressed with q-Poisson distribution series, *Turk. J. Sci.*, 6(2021), 24–30.

[7] C. Ramachandran, D. Kavitha and T. Soupramanien, Certain bound for q-starlike and q-convex functions with respect to symmetric points, *Int. J. Math. Math. Sci.*, 7(2)015, 205682.

[8] H.M. Srivastava, Q.Z. Ahmad, M. Tahir, B. Khan, M. Darus and N. Khan, Certain subclasses of meromorphically-starlike functions associated with the q-derivative operators, *Ukr. Math. J.*, 73(2021), 1260–1273.

[9] M. Ul-Haq, M. Raza, M. Arif, Q. Khan, H. Tang, $q-$analogue of differential subordinations, *Mathematics* 2019, 7, 724

[10] S.D. Purohit, R.K. Raina, Fractional q-calculus and certain subclasses of univalent analytic functions, *Mathematica*, 55(2013), 62–74.

[11] P. Duren, *Univalent functions*, Grundlehren der Mathematischen Wissenschaften 259, Springer-Verlag, New York, 1983.

[12] P. Duren, Subordination in Complex Analysis, *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 599(1977), 22–29.

[13] S. Miller and P. Mocabu, *Differential Subordination: Theory and Applications*, CRC Press, New York, 2000.

[14] Z. Nehari, *Conformal Mappings*, McGraw-Hill, New York, 1952.

[15] M. Lewin, On a coefficient problem for bi-univalent functions, *Proceedings of the American Mathematical Society*, 18(1)(1967), 63-68.

[16] E. Netanyahu, The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$, *Archive for Rational Mechanics and Analysis*, 32(2), 100-112 (1969).

[17] N. Magesh and S. Bulut, Chebyshev polynomial coefficient estimates for a class of analytic bi-univalent functions related to pseudo-starlike functions, *Afrika Matematika*, 29(1-2)(2018), 203-209.

[18] A. W. Goodman, *Univalent Functions*, 2 volumes, Mariner Publishing Co. Inc., 1983.

[19] D.A. Brannan and J.G. Clunie, Aspects of contemporary complex analysis, *Proceedings of the NATO Advanced Study Institute (University of Durham, Durham; July 1–20, 1979)*, Academic Press, New York and London

[20] M. Fekete and G. Szegö, Eine Bemerkung Über ungerade Schlichte Funktionen, *Journal of London Mathematical Society*, s1-8(1933), 85-89.

[21] W. Al-Rawashdeh, On the Study of Bi-Univalent Functions Defined by the Generalized Sălăgean Differential Operator, *European Journal of Pure and Applied Mathematics*, Vol. 17, No. 4, 2024, 3899-3914.

[22] W. Al-Rawashdeh, A new class of generalized starlike bi-univalent functions subordinated to Legendre polynomials, *Int. J. of Analysis and Applications*, 22(2024), 217.

[23] W. Al-Rawashdeh, Applications of Gegenbauer polynomials to a certain Subclass of p-valent functions, WSEAS Transactions on Mathematics, 22(2023), 1025-1030.

[24] M. Cağlar, H. Orhan and M. Kamali, Fekete-Szegö problem for a subclass of analytic functions associated with Chebyshev polynomials, *Boletim da Sociedade Paranaense de Matemática*, 40(2022), 1–6.

[25] J.H. Choi, Y.C. Kim and T. Sugawa, A general approach to the Fekete-Szegö problem, *Journal of the Mathematical Society of Japan*, Vol. 59(2007), 707–727.

[26] M. Kamali, M. Cağlar, E. Deniz and M. Turabaev, Fekete Szegö problem for a new subclass of analytic functions satisfying subordinate condition associated with Chebyshev polynomials, *Turkish J. Math.*, 45 (2021), 1195–1208.

[27] F.R. Keogh and E.P. Merkes, A Coefficient inequality for certain classes of analytic functions, *Proceedings of the American Mathematical Society*, 20(1969), 8-12.

[28] W. Ma and D. Minda, A unified treatment of some special classes of univalent functions, *Proceedings of the Conference on Complex Analysis* (Tianjin, 1992), 157–169, Conf. Proc. Lecture Notes Anal., I, Int. Press, Cambridge, MA.

[29] H. M. Srivastava, M. Kamali and A. Urdaletova, A study of the Fekete-Szegö functional and coefficient estimates for subclasses of analytic functions satisfying a certain subordination conditions and associated with the Gegenbauer polynomials, *AIMS Mathematics*, 7(2)(2021), 2568-2584.

[30] S. Ruscheweyh, New criteria for univalent functions, *Proceedings of the American Mathematical Society*, 49(1975), 109-115.

[31] F.H. Jackson, F.H. On q-functions and a certain difference operator, *Trans. R. Soc. Edinb*, 46(1908), 253-281.

[32] S. Kanas, D. Răducanu, Some subclass of analytic functions related to conic domains, *Math. Slovaca*, 64(2014), 1183–1196.

[33] H. Aldweby and M. Darus, Some subordination results on q-analogue of Ruscheweyh differential operator, *Abst. Appl. Anal.*, 2014, Article ID 985563, 1-6.

[34] Y. Cheng, R. Srivastava and J.L. Liu, Applications of the $q-$Derivative operator to new families of bi-univalent functions related to the Legendre polynomials, *Axioms 2022*, 11, 595, 1-13.

[35] L.I. Cotîrlă and G. Murugusundaramoorthy,Starlike functions based on Ruscheweyh $q-$differential operator defined in Janowski domain, *Fractal Fractional* 2023, 7, 148.

[36] B. Khan, H.M. Srivastava, S. Arjika, A certain q-Ruscheweyh type derivative operator and its applications involving multivalent functions, *Adv. Differ. Equ.* 2021, 2021, 279.

[37] T.M. Seoudy and M.K. Aouf, Coefficent estimates of new classes $q-$starlike and $q-$convex functions of complex order, *Journal of Mathematical Inequalities*, 2016, 10(1), 135–145.

[38] K. Vijaya, Coefficient estimates of bi-univalent bi-Bazilevic Functions defined by $q-$Ruscheweyh differential operator associated with Haradam Polynomials, *Palestine Journal of Mathematics*, 11(2)(2022), 352-361.

[39] H.E.Ö. Uçar, Coefficient inequality for $q-$starlike functions, *Applied Mathematical Comput.* 2016, 276, 122-126.

[40] W. Al-Rawashdeh, Fekete-Szegö functional of a subclass of bi-univalent functions associated with Gegenbauer polynomials, *European Journal of Pure and Applied Mathematics*, Vol. 17, No. 1, 2024, 105-115.

[41] W. Al-Rawashdeh, A class of non-Bazilevic functions subordinate to Gegenbauer Polynomials, *Int. J. of Analysis and Applications*, 22(2024), 29.

[42] K. Kiepiela, I. Naraniecka, and J. Szynal, The Gegenbauer polynomials and typically real functions, *Journal of Computational and Applied Mathematics*, 153(2003), 273–282.

[43] H. M. Srivastava, H. L. Manocha, *A treatise on generating functions*, Halsted Press, John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1984.

[44] N. E. Cho, V. Kumar, S. Kumar and V. Ravichandran, Radius problems for starlike functions associated with the sine function, *Bull. Iranian Math. Society*, Vol.45 (2019), 213–232.

[45] P. Goel and S. Kumar, Certain class of starlike functions associated with modified sigmoid function, *Bull. Malaysian Math. Sci. Society*, Vol.43 (2020), 957–991.

[46] R. Mendiratta, S. Nagpal and V. Ravichandran, On a subclass of strongly starlike functions associated with exponential function, *Bull. Malays. Math. Sci. Society*, Vol. 38 (2015), 365–386.

[47] J. Sokól and J. Stankiewicz, Radius of convexity of some subclasses of strongly starlike functions, *Zeszyty Nauk. Politech. Rzeszowskiej Mathematics*, No. 19 (1996), 101–105.

[48] G. Murugusundaramoorthy and T. Janani, Sigmoid function in the space of univalent λ-pseudo starlike functions, *Int. J. Pure Appl. Math. Sci.*, 101(2015), 33-41.

[49] H. Silverman, Partial sums of starlike and convex functions, *J. Math. Anal. Appl.*, (209)1997, 221–227.

[50] L.A. Wani and A. Swaminathan, Starlike and convex functionsassociated with a Nephroid domain, *Bull. Malays. Math. Sci. Soc.*, (44)2021, 79–104.

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The author has contributed in the present research, at all stages from the formulation of the problem to the final findings and proofs.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

No funding was received for conducting this study.

Conflicts of Interest

The author has no conflicts of interest to declare that are relevant to the content of this article.