Matrix Transforms into the Subsets of Maddox Spaces Defined by Speed

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Abstract: - Let λ and μ be monotonically increasing strictly positive sequences, i.e., the speeds of convergence. Earlier the notions of boundedness, convergence, and zero-convergence with the speed are known. In this paper, the notions of paranormed boundedness, convergence, and zero-convergence with speed have been defined. The matrix transforms from the sets of bounded, convergent, and zero-convergent sequences with the speed λ into the sets of paranormed bounded, paranormed convergent, and paranormed zero-convergent sequences with the speed μ are studied.

Key-Words: - Matrix transforms, paranormed spaces, Maddox spaces, boundedness with speed, convergence with speed, paranormed boundedness with speed, paranormed convergence with speed, paranormed zero-convergence with speed.

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1 Introduction

Let X,Y be two sequence spaces and $A=(a_{nk})$ be an arbitrary matrix with real or complex entries. Throughout this paper we assume that indices and summation indices run from 0 to ∞ unless otherwise specified. If for each $x=(x_k)\in X$ the series

$$A_nx:=\sum_k a_{a_{nk}nk}x_k$$

converge and the sequence $Ax=(A_nx)$ belongs to Y, we say that *A* transforms *X* into *Y*. By (X,Y) we denote the set of all matrices, which transform *X* into *Y*. Let ω be the set of all real or complex-valued sequences. Further, we need the following well-known sub-spaces of ω: c - the space of all convergent sequences, c_0 - the space of all sequences converging to zero, l_∞ - the space of all bounded sequences, and

$$l_\alpha:=\left\{x=(x_k):\sum_n|x_n|^\alpha<\infty\right\},\alpha>0.$$

For estimation and comparison of speeds of convergence of sequences are used different methods, see, for example, [1], [2], [3], [4], [5], [6], [7]. We use the method, introduced in [6] (see also [1]). Let $\lambda:=(\lambda_k)$ be a positive (i.e.; $\lambda_k>0$ for every k) monotonically increasing sequence. Following [6] (see also [1]), a convergent sequence $x=(x_k)$ with

$$\lim_k x_k:=s \text{ and } v_k:=\lambda_k(x_k-s) \qquad (1)$$

is called bounded with the speed λ (shortly, λ-bounded) if $v_k=O(1)$ (or $(v_k)\in l_\infty$), and convergent with the speed λ (shortly, λ-convergent) if the finite limit

$$\lim_k v_k:=b$$

exists (or $(v_k)\in c$). A convergent sequence $x=(x_k)$ with the finite limit s is called α-absolutely convergent with speed λ (or shortly, α-absolutely λ-convergent), if $(v_k)\in l_\alpha$. We denote the set of all λ-bounded sequences by l_∞^λ, the set of all λ-convergent sequences by c^λ, and the set of all α-absolutely λ-convergent sequences by l_α^λ. Moreover, let

$$c_0^\lambda:=\left\{x=(x_k):x\in c^\lambda \text{ and } \lim_k\lambda_k(x_k-s)=0\right\}.$$

It is not difficult to see that

$$l_\alpha^\lambda \subset c_0^\lambda \subset c^\lambda \subset l_\infty^\lambda \subset c.$$

In addition to it, for unbounded sequence λ these inclusions are strict. For $\lambda_k = O(1)$, we get $b=0$, and hence

$$c^\lambda = l_\infty^\lambda = c.$$

Therefore, the most important case is $\lambda_k \neq O(1)$, because in this case relation (1) allows to evaluate the quality of convergence of converging sequences. Indeed, let x^1 and x^2 be two convergent sequences with the finite limit s. If $(v_k) \in c$ (or $v_k = O(1)$) for $x = x^1$, and $(v_k) \notin c$ (or $v_k \neq O(1)$)` for $x = x^2$, then the sequence x^1 converges "better" (more precisely, faster) than sequence x^2. Thus λ, in the case $\lambda_k \neq O(1)$, measures the speed of convergence of the observed sequences.

Matrix transformations, and boundedness and convergence with speed are widely used in approximation theory to transform non-convergent sequences into convergent ones, or to transform convergent sequences into "better" convergent sequences , [5], [8], [9], [10]. Besides, in [1], matrix transformations and boundedness with speed are used for the estimation of the order of approximation of Fourier expansions in Banach spaces by one author of the present paper.

In general, the problems of improvement of the quality of convergence of sequences by matrix transformations have been studied by several authors for example, [1], [11], [12], [13], [14], [15], [16], [17] and [18]. Moreover, in [17] and [18], the λ-convergence and the λ-boundedness in abstract spaces, considering instead of a matrix with real or complex entries a matrix, whose elements are bounded linear operators from a Banach space X into a Banach space Y, have been studied. We note that the results connected with convergence, absolute convergence, α-absolute convergence, and boundedness with speed can be used in several applications. For example, in theoretical physics, such results can be used for accelerating the slowly convergent processes, a good overview of such investigations can be found, for example, from the sources, [19] and [20].

Let $p = (p_k)$ be a sequence of strictly positive numbers, and let

$$c_0(p) := \left\{ x = (x_k) : \lim_k |x_k|^{p_k} = 0 \right\},$$

$$l_\infty(p) := \left\{ x = (x_k) : |x_k|^{p_k} = O(1) \right\},$$

$$c(p) := \left\{ x = (x_k) : \lim_k |x_k - l|^{p_k} = 0 \text{ for some } l \in C \right\}.$$

The sets $c_0(p)$, $l_\infty(p)$ and $c(p)$ are known as Maddox spaces, [21], [22], [23], [24]. These sets are also the paranormed spaces (see, for example, [24]). Now we introduce the notions of the paranormed boundedness with speed, the paranormed convergence with speed, and the paranormed zero-convergence with speed.

Definition 1. We say that a convergent sequence $x = (x_k)$ with the finite limit s is called paranormally bounded with speed λ with respect to p (shortly, paranormally λ-bounded with respect to p), if $(v_k) \in l_\infty(p)$.

Definition 2. We say that a convergent sequence $x = (x_k)$ with the finite limit s is called paranormally convergent with speed with respect to p λ (shortly, paranormally λ-convergent with respect to p), if $(v_k) \in c(p)$.

Definition 3. We say that a convergent sequence $x = (x_k)$ with the finite limit s is called paranormally zero-convergent with speed λ with respect to p (shortly, paranormally λ-zero-convergent with respect to p), if $(v_k) \in c_0(p)$.

We denote the set of all paranormally λ-bounded sequences by $(l_\infty(p))^\lambda$, the set of all paranormally λ-convergent sequences by $(c(p))^\lambda$, and the set of all paranormally λ-zero-convergent sequences by $(c_0(p))^\lambda$. It is easy to see that for $p_k \equiv 1$ we have

$$(l_\infty(p))^\lambda = l_\infty^\lambda, \; (c(p))^\lambda = c^\lambda, \; (c_0(p))^\lambda = c_0^\lambda.$$

Next, we explain Definitions 1-3 by the following example.

Example 1. Let $x = (x_k)$, $\lambda = (\lambda_k)$ and $p = (p_k)$ be defined as follows:

$$x_k := \frac{1}{2(k+1)}, \; \lambda_k := k+1, \text{ and } p_k := k+1.$$

Then

$$\lim_k x_k = 0, \; v_k \equiv \frac{1}{2},$$

and

$$\lim_k |v_k - l|^{p_k} = \lim_k \frac{1}{2^{k+1}} = 0,$$

if $l = 0$ or $l = 1$. This implies $(v_k) \in c_0(p) \subset c(p)$, and hence $x \in (c_0(p))^\lambda \subset (c(p))^\lambda$. Also it is easy to see that $(v_k) \in l_\infty(p)$, and hence $x \in (l_\infty(p))^\lambda$. Therefore, x

is simultaneously paranormally λ-zero-convergent, paranormally λ-convergent and paranormally λ-bounded with respect to *p*.

Let $\mu:=(\mu_k)$ be another speed of convergence, i.e., a monotonically increasing positive sequence. Matrix transforms between the subsets of *c* defined by the speeds λ and μ have been studied by the authors of the present work in several papers. For example, in [25] the sets $(l_\infty^\lambda, l_\alpha^\mu)$, $(c^\lambda, l_\alpha^\mu)$, $(c_0^\lambda, l_\alpha^\mu)$ and $(l_1^\lambda, l_\alpha^\mu)$ for $\alpha>1$ have been characterized. A short overview of the convergence with speed has been presented in [1].

The present paper is the continuation of the paper, [25]. We find necessary and sufficient conditions for the matrix to transforms from c_0^λ, c^λ and l_∞^λ into $\left(c_0(p)\right)^\mu$, $\left(c(p)\right)^\mu$ or $\left(l_\infty(p)\right)^\mu$.

2 Auxiliary Results

For the proof of the main results, we need some auxiliary results.

Lemma 1 ([26], p. 44, see also [27], Proposition 12). *A matrix* $A=(a_{nk})\in(c_0,c)$ *if and only if*

there exists finite limits $\lim_m a_{nk}:=a_k$, (2)

$$\sum_k |a_{nk}|=O(1). \tag{3}$$

Moreover,

$$\lim_n A_n x=\sum_k{}' a_k x_k \tag{4}$$

for every $x=(x_k)\in c_0$.

Lemma 2 ([26], p. 46-47, see also [27], Proposition 11). *A matrix* $A=(a_{nk})\in(c,c)$ *if and only if conditions* (2), (3) *are satisfied and*

there exists τ with $\lim_n \sum_k{}' a_{nk}:=\tau$. (5)

Moreover, if $\lim_k x_k=s$ for $x=(x_k)\in c$, then

$$\lim_n A_n x=s\tau+\sum_k{}'(x_k-s)a_k.$$

Lemma 3 ([26], p. 51, see also [27], Proposition 10). *The following statements are equivalent:*

(a) $A=(a_{nk})\in(l_\infty,c)$.

(b) *The conditions* (2), (3) *are satisfied and*

$$\lim_n \sum_k |a_{nk}-a_k|=0. \tag{6}$$

(c) *The condition* (2) *holds and*

$\sum_k |a_{nk}|$ *converges uniformly in n.* (7)

Moreover, if one of the statements (a)-(c) *is satisfied, then the equation* (4) *holds for every* $x=(x_k)\in l_\infty$.

Lemma 4 ([28], Theorem 5(iii)). *A matrix* $B=(b_{nk})\in(c_0,c_0(p))$ *if and only if*

$|b_{nk}|^{p_n}=o(1)$ *for every k,* (8)

$$\lim_K \limsup_n \left(K^{-1}\sum_k |b_{nk}|\right)^{p_n}=0. \tag{9}$$

Lemma 5 ([28], Theorem 5(i)). *The following statements are equivalent:*

(a) $B=(b_{nk})\in(l_\infty,c_0(p))$.

(b) *The condition* (9) *holds and*

$\left|\sum_{k\in S} b_{nk}x_k\right|^{p_n}=o(1)$ for every $S\in N$. (10)

(c) $\left(\sum_k |b_{nk}|\right)^{p_n}=o(1)$. (11)

Lemma 6 ([28], Theorem 7(ii)). *A matrix* $A=(a_{nk})\in(c_0,c(p))$ *if and only if*

$$\sum_k |b_{nk}|=O(1), \tag{12}$$

and there exists a sequence $(d_1,d_2...)$ *of complex numbers such that*

$|b_{nk}-d_k|^{p_n}=o(1)$ *for every k,* (13)

$$\lim_K \limsup_n \left(K^{-1}\sum_k |b_{nk}-d_k|\right)^{p_n}=0. \tag{14}$$

3 Main Results

To formulate the main results of the paper, we use the matrix $B=(b_{nk})$, defined by

$$b_{nk}:=\frac{\mu_n(a_{nk}-a_k)}{\lambda_k},$$

if condition (2) holds. Also, we need the sequences

$$\lambda^{-1}:=\left(1/\lambda_k\right),\ e:=(1,1,...),\ e^k:=(1,0,...,1,0,...),$$

where 1 is in the *k*-th position. We note that.

$$e,e^k,\lambda^{-1}\in c^\lambda,\quad e,e^k\in c_0^\lambda.$$

Theorem 1. *A matrix* $A=(a_{nk})\in(c^\lambda,(c(p))^\mu)$ *if and only if*

$$Ae=(\tau_n)\in(c(p))^\mu,\ \tau_n:=A_n e=\sum_k a_{nk}, \tag{15}$$

$$Ae^k\in(c(p))^\mu, \tag{16}$$

$$A\lambda^{-1}\in(c(p))^\mu, \tag{17}$$

$$\sum_k \frac{|a_{nk}|}{\lambda_k} = O(1), \tag{18}$$

condition (12) holds, *and there exists a sequence* $(d_1, d_2 ...)$ *of complex numbers such that conditions* (13) *and* (14) *are satisfied.*

Proof. Necessity. Suppose that $A \in (c^\lambda, (c(p))^\mu)$. As $e, e^k, \lambda^{-1} \in c^\lambda$, then conditions (15), (16) and (17) hold. Since, from (1) we get:

$$x_k = \frac{v_k}{\lambda_k} + s; \quad s = \lim_k x_k, \quad (v_k) \in c$$

for every $x = (x_k) \in c^\lambda$, it follows that

$$A_n x = \sum_k \frac{a_{nk}}{\lambda_k} v_k + s\tau_n. \tag{19}$$

As $(\tau_n) \in (c(p))^\mu$ by (15), then the finite limit

$$\tau := \lim_n \tau_n$$

exists. Hence, from (19) we obtain that the matrix

$$A_\lambda := \left(\frac{a_{nk}}{\lambda_k} \right)$$

transforms this sequence $(v_k) \in c$ into c. In addition, for every sequence $(v_k) \in c$, the sequence $(v_k / \lambda_k) \in c_0$. But, for (v_k / λ_k), there exists a convergent sequence $x = (x_k)$ with $s = \lim_k x_k$, such that $v_k / \lambda_k = x_k - s$. So we have proved that, for every sequence $(v_k) \in c$ there exists a sequence $x \in c^\lambda$ such that $v_k = \lambda_k (x_k - s)$. Consequently $A_\lambda \in (c, c)$. This implies, by Lemma 2, that the finite limits a_k and

$$a^\lambda := \lim_n \sum_k \frac{a_{nk}}{\lambda_k} \tag{20}$$

exists, and condition (18) is satisfied, since for A_λ condition (3) and (5) take correspondingly the forms (18) and (20), and the finite limit

$$\phi := \lim_n A_n x = a^\lambda b + \sum_k \frac{a_k}{\lambda_k}(v_k - b) + s\tau \tag{21}$$

exists for every $x \in c^\lambda$, where $b := \lim_k v_k$. Now, using (19) and (21), we obtain

$$\mu_n (A_n x - \phi) = \sum_k b_{nk}(v_k - b) + \mu_n (\tau_n - \tau) s$$

$$+ \mu_n \left(\sum_k \frac{a_{nk}}{\lambda_k} - a^\lambda \right) b. \tag{22}$$

Conditions (15) and (17) imply that $B \in (c_0, c(p))$. Therefore we can conclude by Lemma 6 that condition (12) holds and there exists a sequence $(d_1, d_2 ...)$ of complex numbers such that conditions (13) and (14) are satisfied.

Sufficiency. Assume that all conditions of Theorem 1 are satisfied. First, we notice that relation (19) holds for every $x \in c^\lambda$ and the finite limits τ, a_k and a^λ exist correspondingly by (15), (16), and (17). As condition (18) also holds, then $A_\lambda \in (c, c)$ by Lemma 2, and relations (21) and (22) hold for every $x \in c^\lambda$. Now, due to (12), (13) and (14), $B \in (c_0, c(p))$ by Lemma 6. Hence $A \in (c^\lambda, (c(p))^\mu)$ by (15) and (17).

Theorem 2. *A matrix* $A = (a_{nk}) \in (c^\lambda, (c_0(p))^\mu)$ *if and only if conditions* (8), (9), (16) – (18) hold, and

$$Ae = (\tau_n) \in (c_0(p))^\mu, \tau_n = A_n e = \sum_k a_{nk}. \tag{23}$$

Proof is similar to the proof of Theorem 1. The only difference is that now $B \in (c_0, c_0(p))$. Therefore instead of Lemma 6 we use Lemma 4.

Theorem 3. *A matrix* $A = (a_{nk}) \in (c^\lambda, (l_\infty(p))^\mu)$ *if and only if conditions* (16) – (18) hold,

$$Ae = (\tau_n) \in (l_\infty(p))^\mu, \tau_n = A_n e = \sum_k a_{nk}, \tag{24}$$

and $B \in (c_0, l_\infty(p))$.

Proof is similar to the proof of Theorem 1. The only difference is that now $B \in (c_0, l_\infty(p))$.

Theorem 4. *Let* $\lambda_k \neq O(1)$. *A matrix* $A = (a_{nk}) \in (l_\infty^\lambda, (c_0(p))^\mu)$ *if and only if condition* (2), (9), (10), (18) *and* (23) *are satisfied, and*

$$\lim_n \sum_k \frac{|a_{nk} - a_k|}{\lambda_k} = 0. \tag{25}$$

Proof. Necessity. Assume that $A \in (l_\infty^\lambda, (c_0(p))^\mu)$. As $e^k, e \in l_\infty^\lambda$, then conditions (2) and (23) are satisfied. Considering that equality (19) holds for every $x = (x_k) \in l_\infty^\lambda$ (where $(v_k) \in l_\infty$), we, due to (23), obtain that the matrix A_λ transforms this bounded sequence (v_k) into c. Similar to the proof of necessity of

Theorem 1, it is possible to show that, for every sequence $(v_k)\in l_\infty$, where exists a sequence $(x_k)\in l_\infty^\lambda$ such that $v_k=\lambda_k(x_k-s)$. Hence $A_\lambda\in(l_\infty,c)$. This implies by Lemma 3 ((a) and (b)) that conditions (18) and (25) holds, since for A_λ conditions (3) and (6) take correspondingly the forms (18) and (25), and the finite limit:

$$\phi:=\lim_n A_n x=\sum_k \frac{a_k}{\lambda_k}v_k+s\tau \qquad (26)$$

exists for every $x\in l_\infty^\lambda$. Writing

$$\mu_n(A_n x-\phi)=\sum_k b_{nk}v_k+\mu_n(\tau_n-\tau)s, \qquad (27)$$

We have by (23) that the matrix $B\in(l_\infty,c_0(p))$. Consequently conditions (9) and (10) are satisfied by Lemma 5 ((a) and (b)).

Sufficiency. Let conditions (2), (9), (10), (18), (23) and (25) be satisfied. Then relation (19) also holds for every $x\in l_\infty^\lambda$ and $(\tau_n)\in(c_0(p))^\mu$ by (23). Hence, $A_\lambda\in(l_\infty,c)$, and the finite limits ϕ exists for every $x\in l_\infty^\lambda$ by Lemma 3 ((a) and (b)). This implies that relation (27) holds for every $x\in l_\infty^\lambda$. As conditions (9) and (10) are satisfied, then $B\in(l_\infty,c_0(p))$ by Lemma 5 ((a) and (b)). Thus, due to (23), $A\in(l_\infty^\lambda,(c_0(p))^\mu)$.

Theorem 5. *Let* $\lambda_k\neq O(1)$. *A matrix* $A=(a_{nk})\in(l_\infty^\lambda,(c(p))^\mu)$ *if and only if condition* (2), (15), (18), *and* (25) *are satisfied, and* $B\in(l_\infty,c(p))$.

The proof is similar to the proof of Theorem 4.

Theorem 6. *Let* $\lambda_k\neq O(1)$. *A matrix* $A=(a_{nk})\in(l_\infty^\lambda,(l_\infty(p))^\mu)$ *if and only if conditions* (2), (18), (24), *and* (25) *are satisfied, and* $B\in(l_\infty,l_\infty(p))$.

The proof is similar to the proof of Theorem 4.

Remark 1. Conditions (18) and (25) can be replaced by condition

the series $\sum_k \frac{|a_{nk}|}{\lambda_k}$ converges uniformly in *n*

in Theorems 4-6 by Lemma 3 ((a) and (c)).

Remark 2. Conditions (9) and (10) can be replaced by condition (11) in Theorems 4-6 by Lemma 5 ((a) and (c)).

Theorem 7. *A matrix* $A=(a_{nk})\in(c_0^\lambda,(c(p))^\mu)$ *if and only if condition* (2), (12), (15), (18) *are satisfied, and there exists a sequence* $(d_1,d_2...)$ *of complex numbers such that conditions* (13) *and* (14) *are satisfied.*

Proof. Necessity. Suppose that $A\in(c_0^\lambda,(c(p))^\mu)$. As $e^k,e\in c_0^\lambda$, then conditions (2) and (15) are satisfied. Considering that equality (19) holds for every $x=(x_k)\in c_0^\lambda$ (where $(v_k)\in c_0$), we, due to (15), obtain that the matrix A_λ transforms this sequence $(v_k)\in c_0$ into c. Similar to the proof of the necessity of Theorem 1, it is possible to show that, for every sequence $(v_k)\in c_0$, where exists a sequence $(x_k)\in c_0^\lambda$ such that $v_k=\lambda_k(x_k-s)$. Hence $A_\lambda\in(c_0,c)$. This implies by Lemma 1 that condition (18) hold, since for A_λ condition (3) takes the form (18) and relation (26) is valid for every $x\in c_0^\lambda$. Then also (27) holds, and, due to (15), $B\in(c_0,c(p))$. Therefore, by Lemma 6, condition (12) is satisfied and there exists a sequence $(d_1,d_2...)$ of complex numbers such that conditions (13) and (14) hold.

Sufficiency. Let all conditions of Theorem 7 be satisfied. Then relation (19) also holds for every $x\in c_0^\lambda$ and $(\tau_n)\in(c(p))^\mu$ by (15). Hence $A_\lambda\in(c_0,c)$, and relation (26) holds for every $x\in c_0^\lambda$ by Lemma 1. This implies that relation (27) holds for every $x\in c_0^\lambda$. As condition (12) is satisfied, and there exists a sequence $(d_1,d_2...)$ of complex numbers such that conditions (13) and (14) hold, then $B\in(c_0,c(p))$ by Lemma 6. Thus, due to (15), $A\in(c_0^\lambda,(c(p))^\mu)$.

Theorem 8. *A matrix* $A=(a_{nk})\in(c_0^\lambda,(l_\infty(p))^\mu)$ *if and only if conditions* (2), (18), (24) *are satisfied, and* $B\in(c_0,l_\infty(p))$.

The proof is similar to the proof of Theorem 7.

Theorem 9. *A matrix* $A=(a_{nk})\in(c_0^\lambda,(c_0(p))^\mu)$ *if and only if condition* (2), (8), (9), (18), *and* (23) *are satisfied.*

The proof is similar to the proof of Theorem 7. The only difference is that now $B\in(c_0,c_0(p))$. Therefore instead of Lemma 6 we use Lemma 4.

In the following example, we show that there exists a matrix *A* satisfying all the conditions of Theorems 1-9.

Example 2. Let us consider the Zweier matrix $Z_{1/2}=(a_{nk})$, defined by $A=(a_{nk})$, where (see [26], p.14, or [1], p.3) $a_{00}=1/2$ and

$$a_{nk}=\begin{cases}1/2, \text{if } k=n-1 \text{ and } k=n,\\ 0, \text{ if } k<n-1 \text{ and } k>n\end{cases}$$

for $n\geq 1$. We show that $Z_{1/2}\in(X,Y)$, if *X* is one of the sets c_0^λ, c^λ or l_∞^λ, and Y – one of the sets $(c_0(p))^\mu$, $(c(p))^\mu$ or $(l_\infty(p))^\mu$ for $\lambda=(\lambda_k)$, $\mu=(\mu_k)$ and $p=(p_k)$, defined as follows:

$$\lambda_k:=3(k+1),\ \mu_k:=k \text{ and } p_k:=k+1.$$

To prove this, we show that all conditions of Theorems 1-9 are satisfied. As from $B\in(c_0,c_0(p))$ follows $B\in(c_0,l_\infty(p))$, and from $B\in(l_\infty,c_0(p))$ it follows that $B\in(l_\infty,c(p))$ and $B\in(l_\infty,l_\infty(p))$, then by Theorems 1 – 9 and Remark 2 it is sufficient to show that conditions (2), (8), (9), (11) - (18), and (23) – (25) are satisfied.

First, we see that condition (2) holds with $a_k\equiv 0$. Also conditions (15), (16), (23) and (24) are satisfied, since

$$Ae=(1/2,1,1,...)$$

with limit 1 and

$$Ae^k\in(0,...,0,1/2,0,...),$$

where 1 is in the k^{th} position.

As

$$A_n\lambda^{-1}=\begin{cases}\dfrac{1}{6}, & \text{if } n=0,\\ \dfrac{1}{6}\left(\dfrac{1}{n}+\dfrac{1}{n+1}\right), & \text{if } n\geq 1,\end{cases}$$

then

$$\lim_n A_n\lambda^{-1}=0.$$

Denoting

$$w_n:=\mu_n\left(A_n\lambda^{-1}-\lim_n A_n\lambda^{-1}\right),$$

We can write $w_0=0$ and

$$w_n=\frac{1}{6}\left(1+\frac{n}{n+1}\right),\ if\ n\geq 1.$$

Taking $l=1/3$, we obtain for $n\geq 1$ that

$$\lim_n|w_n-l|^{p_n}=\lim_n\left|\frac{1}{6}\left(1+\frac{n}{n+1}\right)-\frac{1}{3}\right|^{n+1}$$
$$=\lim_n\frac{1}{[6(n+1)]^{n+1}}=0,$$

hence

$$(w_n)\in(c(p))^\mu.$$

Therefore condition (17) holds.

As

$$\sum_k\frac{|a_{nk}|}{\lambda_k}=A_n\lambda^{-1}$$

and $a_k\equiv 0$, then also conditions (18) and (25) are fulfilled.

For $B=(b_{nk})$ we obtain

$$b_{nk}=\begin{cases}\dfrac{1}{6}, & \text{if } k=n-1,\\ \dfrac{n}{6(n+1)}, & \text{if } k=n,\\ 0, & \text{if } k<n-1 \text{ or } k>n.\end{cases}$$

Hence

$$0<b_{nk}\leq\frac{1}{6} \text{ and } \lim_n b_{nk}=0$$

for every *k*, and

$$\sum_k|b_{nk}|=\frac{1}{6}\left(1+\frac{n}{n+1}\right)<\frac{1}{3}.$$

Therefore conditions (8), (9), (11) and (12) are fulfilled. Taking $d_k\equiv 0$, we can conclude that conditions (13) and (14) also hold.

4 Conclusion

In this paper, we defined the notions of paranormed boundedness with speed, paranormed convergence with speed and paranormed zero-convergence with speed with respect to *p*, where the speed is defined by a monotonically increasing positive sequence μ, and $p=(p_k)$ is a sequence of strictly positive numbers. The sets of all paranormally bounded, paranormally convergent, and paranormally zero-convergent sequences with speed μ with respect to *p* we denote correspondingly by $(l_\infty(p))^\mu$, $(c(p))^\mu$, and $(c_0(p))^\mu$. These sets are the subsets of well-known Maddox spaces $l_\infty(p)$, $c(p)$ or $c_0(p)$.

The notions of ordinary boundedness, convergence, and zero-convergence with speed are known earlier. Let λ be another speed of convergence, and l_∞^λ, c^λ, c_0^λ be respectively the sets of all λ-bounded, all λ-convergent, and all λ-zero-convergent sequences.

Let *A* be a matrix with real or complex entries. We found necessary and sufficient conditions for the transforms $A:c^{\lambda}\to(c(p))^{\mu}$, $A:c^{\lambda}\to(l_{\infty}(p))^{\mu}$, $A:c^{\lambda}\to(c_0(p))^{\mu}$, $A:l_{\infty}^{\lambda}\to(c(p))^{\mu}$, $A:l_{\infty}^{\lambda}\to(l_{\infty}(p))^{\mu}$, $A:l_{\infty}^{\lambda}\to(c_0(p))^{\mu}$, $A:c_0^{\lambda}\to(c(p))^{\mu}$, $A:c_0^{\lambda}\to(l_{\infty}(p))^{\mu}$ and $A:c_0^{\lambda}\to(c_0(p))^{\mu}$. We also present some examples that illustrate the new concepts introduced and the main results of the paper.

In the next paper, we intend to study matrix transformations from *X* to *Y*, where *X* is one of the sets $(l_{\infty}(q))^{\lambda}$, $(c_0(q))^{\lambda}$ or $(c_0(q))^{\lambda}$ (where q is another sequence of strictly positive numbers), and *Y* - one of the sets $(l_{\infty}(p))^{\mu}$, $(c(p))^{\mu}$ or $(c_0(p))^{\mu}$.

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