# Bifurcation of a Limit Cycle for Planar Piecewise Smooth Quadratic Differential System via Averaging Theory

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*Abstract:* - In this article, the focus is on exploring planar piecewise smooth quadratic systems, a significant class of dynamical systems that exhibit changes in behavior under different conditions but with smooth transitions between these states. The main objective is to introduce a second-order averaged method designed specifically to identify limit cycles, repeating trajectories in a system's phase space indicative of periodic behavior. This innovative method not only allows for the detection of these cycles but also quantifies their number, providing a deeper understanding of the system's long-term behavior. The paper highlights its applicability by demonstrating the maximum number of limit cycles that can exist in two distinct systems, offering valuable insights into the dynamics of such systems and contributing to the broader field of mathematical modeling and analysis.

*Key-Words:* - Averaging Method,Bifurcation function, bifurcation theory, Bifurcation of limit cycles, Piecewise smooth differential systems, Poincare map.

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## 1 Introduction

The periodic solution, bifurcation theory, numerical analysis, and application research of high-dimensional smooth and non-smooth dynamic systems are currently difficult and frontier topics in the field of international dynamics and control, which have important theoretical significance and application value. At anInternational Mathematics Conference held in 1900, the famous mathematician Hilbert put forward a series of mathematical problems. The relationship between the maximum number of limit cycles and the relative position of limit cycles in a planar polynomial system is demonstrated in the second section of the sixteenth problem. Numerous researchers worked on this problem, with a primary concentration on cubic and quadratic systems, [1], [2], [3], [4], [5], [6]. Significant progress was made by [7], who proved that quadratic systems can have up to three limit cycles. Furthermore, bifurcation techniques were used by [8] to establish that quadratic systems can display a maximum of four limit cycles.

After a year [9] used the Poincaré-Bendixson theorem and find the same results as [10] found. Mathematicians are increasingly convinced that quadratic systems (QS) contain no more than four limit cycles. The number of limit cycles for cubic systems is eight was proposed by [11]. Later on, different mathematicians used the Melnikov function method (MFM) and found that the number of limit cycles for cubic systems is eleven, [12], [13], [14], [15]. In [16], [17] discovered enough conditions for a cubic system that the number of limit cycles is 10 or 12. So it can be concluded that the maximum number of limit cycles for a cubic system is 12. To find the number of periodic solutions in piecewise smooth differential equations, many techniques were developed. The Melnikov function method(MFM) and the averaging method (AM) are the two most often utilized techniques. The Melnikov function method(MFM) was presented by [18] who also deduced a formula for the first-order Melnikov function(Mf). The First-order Melnikov vector function was found by [19], who also introduced the Melnikov function methods for high-dimensional piecewise smooth systems. First and second-order averaging methods for periodic solutions of piecewise smooth differential equations were developed by numerous scholars [20], [21], [22] to conduct investigations.

The researcher [23], [24] investigated a class of quartic and quadratic polynomial differential systems using higher-order averaging theory. The averaged function of different order with many zones of the discontinuous differential system was introduced in [25]. Many theories have been successfully used to investigate the Poincare map and hopf bifurcation. Two main methods are proposed to investigate one is known as the Melnikov function method (MFM) which was established in [18], [26] and the other one is called the averaged method established in [20], [27]. Later in 2016 demonstrated, however, that the averaged technique and the Melnikov function method (MFM) are interchangeable, [20].

Based on the reference [5] consider the piecewise smooth planar differential system

$$\begin{pmatrix}\dot{\chi}\\ \dot{\varrho}\end{pmatrix} = \begin{cases}\mathcal{F}_1(\chi,\varrho)\daleth_1(\chi,\varrho) & \mathcal{T}(\chi,\varrho) > 0\\ \mathcal{F}_2(\chi,\varrho)\daleth_2(\chi,\varrho) & \mathcal{T}(\chi,\varrho) \le 0\end{cases} \quad (1)$$

$$\begin{pmatrix}\dot{\chi}\\ \dot{\varrho}\end{pmatrix} = \begin{cases}-\varrho(1+\sigma\chi) + \Sigma\mathcal{F}_1^{+}(\chi,\varrho) & \\ \chi(1+\sigma\chi) + \Sigma\daleth_1^{+}(\chi,\varrho) & \mathcal{T}(\chi,\varrho) > 0\\ -\varrho(1+\rho\chi) + \Sigma\mathcal{F}_1^{-}(\chi,\varrho) & \\ \chi(1+\rho\chi) + \Sigma\daleth_1^{-}(\chi,\varrho) & \mathcal{T}(\chi,\varrho) \le 0\end{cases} \quad (2)$$

Where, $\mathcal{F}_1^{\pm}(\chi,\varrho) = \sum_{i+j=0}^{n}\delta_{ij}\chi^i\varrho^j$ $\daleth_1{}^{\pm}(\chi,\varrho) = \sum_{i+j=0}^{n}\eta_{ij}\chi^i\varrho^j$ $\sigma > 0, \rho > 0$

For $\varepsilon_0 > 0$ & $N > \varepsilon_0$ sufficiently small and large respectively. The maximal number of limit cycles of a system (1) is denoted by $H_1(n)$ bifurcating from $\varepsilon_0 \le \chi^2 + \varrho^2 \le N$ . In system (2) the maximal number of limit cycles denoted by $H_2(n)$ bifurcating from $\chi^2 + \varrho^2 \le N$.

The main purpose of this paper is to find the following theorem.

**Theorem 1.** For $|\varepsilon| > 0$ we have $H_1(n)$ and $H_2(n)$ using the second-order averaging method for the piecewise smooth differential system.

Where, $H_1(n) \le 2n-1, H_2(n) \le 2n-2$.

This paper is organized as in section 2 we introduce the averaging method (1st & 2nd order) for a piecewise smooth system, andintroduce some basic results which will help in the next section. In section 3 find the averaged function for systems (1) and (2) which shows the maximal number of zeros of the averaged function and proves theorem 1.1. Then conclude some results.

## 2 Preliminary Results

In this section, we introduce some basic results and theorems for differential systems from the averaging theory, [20]

Consider a differential system of the form

$$\dot{\chi} = \varepsilon\mathcal{F}(t+T,\chi,\varepsilon,\delta) = \varepsilon\mathcal{F}(t,\chi,\varepsilon,\delta), t \in R, \chi \in J \subset R. \quad (3)$$

T is periodic and the period of F. where $T > 0$ & $0 \le t \le T$ is given by F.

$$\mathcal{F}(\varrho,\eta) = \begin{cases}\mathcal{F}_1(\varrho,\eta) & \varrho \in D_1 & (t,\chi) \in \varrho\\ \mathcal{F}_2(\varrho,\eta) & \varrho \in D_2 & \\ \vdots & \vdots & \\ \mathcal{F}_k(\varrho,\eta) & \varrho \in k & (\varepsilon,\delta) \in \eta\end{cases} \quad (4)$$

where J is an open interval with $\chi \in J \subset R, \varepsilon_0 > 0, |\varepsilon| < \varepsilon_0$

Eq. (1) is known as K-piecewise $C^r$ smooth periodic equation and $KC^r$ is a function of $\mathcal{F}_j(t,\chi,\varepsilon,\delta)\forall(t,\chi) \in U(\overline{D_J})$ where, an open set $U(\overline{D_J})$ is containing $\overline{D_J}$, Where, $\overline{D_J}$ is a closure of a set $D_j$.

Which can be defined as, For K region $D_j = \{(t,\chi)/\sigma_{j-1}(\chi) \leq t < \sigma_j(\chi), \chi \in j\}, j = 1,2 \ldots\ldots, k$

When $j = 1$ then $\sigma_0(\chi) = 0$ and when $j = k$ then $\sigma_k(\chi) = T$ and for $(k-1)C^r$ functions $\sigma_1(\chi), \sigma_2(\chi), \ldots\ldots, \sigma_{k-1}(\chi)$defined as:

$$0 < \sigma_1(\chi) < \sigma_2(\chi) < \ldots\ldots\ldots \sigma_2(\chi)_{r-1}(\chi) < T \quad (5)$$

Where, $r \geq 1$ $and$ $k \geq 2$ $for$ $\chi \in J$. $\mathcal{F}$ is a periodic function and the period is T which may not be continuous on the switchline $l_j$ where $j = 1,2, \ldots\ldots k-1$, so $l_j$ is defined as $l_j = \{(t,\chi)/t = \sigma_2(\chi)_j(\chi), \chi \in J\}, j = 0,1, \ldots\ldots k$

$$\ell(\chi,\delta) = \int_0^T \mathcal{F}(t,\chi,0,\delta)dt$$
$$= \sum_{j=1}^{k} \int_{\sigma_{j-1}(\chi)}^{\sigma_j(\chi)} \mathcal{F}_j(t,\chi,0,\delta)dt, \qquad \chi \in J \quad (6)$$

As T is periodic, defined $\chi(0) = \chi$ for $t$ outside the interval $[0,T]$. We can define a bifurcation function and Poincare map of (1), [20]

$$P(\chi_0,\varepsilon,\delta) = \chi(T,0,\chi_0,\varepsilon,\delta) \quad (7)$$

and $d(\chi_0,\varepsilon,\delta) = P(\chi_0,\varepsilon,\delta) - \chi_0$
so $P(\chi_0,\varepsilon,\delta) = \chi_0 + \varepsilon\overline{g}_k(\chi_0,\varepsilon,\delta), \overline{g}_k \in C^r$

In reference [20] author defined some functions and developed averaging theory.

**Lemma 2.1**
Suppose the assumption of equation (4) (5) (6) is satisfied Then Consider the periodic equation

(i) $I \subset J$ a closed interval $\exists\, \varepsilon^0 > 0$ such that function $\overline{g}_k(\chi_0,\varepsilon,\delta)$ is well-defined and of $C^r$ in $(\chi_0,\varepsilon,\delta)$ $\forall\, \chi_0 \in T, \delta \in V$ $\&|\varepsilon| < \varepsilon^0$

(ii) The equation (1.1) has a periodic solution having period T with $\chi_0(0) = \chi_0 \in T$ for $\varepsilon \neq 0$ IFF initial value satisfies $\overline{g_k}(\chi_0,\varepsilon,\delta) = 0$

**Remark**
The conclusion of proof of theorem 1.1of [20] shows that if $\mathcal{F}(\mathrm{t},0,\varepsilon,\delta) = 0$, with J being the interval $]0,+\infty[$. It is demonstrated that there are at most m zeros of f for $\chi \in$ Jfor every $\delta \in$ V. Moreover, For any $\mathrm{N} > 0$ there exists an $\varepsilon_1 = \varepsilon_1(\mathrm{N}) > 0$ such that $0 < |\varepsilon| < \varepsilon_1, \delta \in$ V Equation (1) has a maximum m positive periodic solution whose range is a subset of $]0,\mathrm{N}[$. The maximum number of periodic solutions for the piecewise smooth periodic equation can be found by using second-order averaging theory. The Poincare map can be written as if the equation $\ell(\chi,\delta) = 0$

$$P(\chi_0,\varepsilon,\delta) = \chi(T,0,\varepsilon,\delta)$$
$$= \chi_0 + \varepsilon^2\overline{g_k}(\chi_0,\varepsilon,\delta) \quad (8)$$

Where $\varepsilon\overline{g_\mathrm{k}}(\chi_0,\varepsilon,\delta) = \overline{g_\mathrm{k}}(\chi_0,\varepsilon,\delta)$

Based on Lemma 2.1 and reference [9]$\mathrm{I} \subset \mathrm{J}$ a closed interval $\exists\, \varepsilon^0 > 0$ such that a well-defined function $\overline{g}_\mathrm{k}(\chi_0,\varepsilon,\delta)$ and smoothness of function $\mathrm{C}^{\gamma-1}$in$(\chi_0,\varepsilon,\delta)$ $\forall\chi_0 \in \mathrm{I}, |\varepsilon| < \varepsilon^*$and$\delta \in \mathrm{V}$.

According to [28]

$$\overline{g_\mathrm{k}}(\chi_0,0,\delta) = \ell_2(\chi_0,\delta)$$

Where $\ell_2(\chi_0,\delta)$ can be written as

$$\ell_2(\chi_0,\delta) = \left(\int_0^T D_\chi\mathcal{F}_1(t,\chi,\delta)\int_0^t \mathcal{F}_1(s,\chi,\delta)ds + \mathcal{F}_2(t,\chi,\delta)\right)dt \quad (9)$$

Which satisfying $\mathcal{F}_1(t,\chi,\delta) = \mathcal{F}_1(t,\chi,0,\delta)$

$$\mathcal{F}_2(t,\chi,\delta) = \begin{cases} \left.\dfrac{\partial\mathcal{F}_1(t,\chi,\varepsilon,\delta)}{\partial\varepsilon}\right|_{\varepsilon=0} & (\mathrm{t},\chi) \in \mathrm{D}_1 \\ \left.\dfrac{\partial\mathcal{F}_2(t,\chi,\varepsilon,\delta)}{\partial\varepsilon}\right|_{\varepsilon=0} & (\mathrm{t},\chi) \in \mathrm{D}_2 \\ \vdots & \\ \left.\dfrac{\partial\mathcal{F}_\mathrm{k}(t,\chi,\varepsilon,\delta)}{\partial\varepsilon}\right|_{\varepsilon=0} & (\mathrm{t},\chi) \in \mathrm{D}_\mathrm{k} \end{cases} \equiv \frac{\partial\mathcal{F}(t,\chi,0,\delta)}{\partial\varepsilon}$$

$$\mathrm{D}_\chi\mathcal{F}_1(\mathrm{t},\chi,\delta) = \sum_{\mathrm{j}=1}^{\mathrm{k}} \chi_{\mathrm{D_j}}\mathrm{D}_\chi\mathcal{F}_\mathrm{j}(\mathrm{t},\chi,0,\delta)$$

Where $\varkappa_{\mathrm{D_j}}(\mathrm{t},\chi) = \begin{cases} 1 & (\mathrm{t},\chi) \in \mathrm{D_j} \\ 0 & (\mathrm{t},\chi) \notin \mathrm{D_j} \end{cases}$

So $\ell_2 \in \mathrm{C}^{\gamma-1}$ which is the same as the proof of theorem 1.1 in [20].

**Lemma 2.2**
Let's consider a value $\delta_0$ from the set V, and assume that a function $\mathrm{l}(\chi,\delta) = 0$ has a maximum of m zeros. In such a case, there exists a positive value $\varepsilon_0$, such that if we have $0 < |\varepsilon| < \varepsilon_0$ and $|\delta - \delta_0| < \varepsilon_0$, the equation (4) will possess m T-periodic solutions.

## 3 Main Results

In this section, we will provide the proof for main theorem 1.1, which can be divided into two distinct parts. The first part involves the examination of the limit cycle in the piecewise smooth system (1). The second part focuses on the analysis of the bifurcation of the limit cycle in the system (2).

**Lemma 3.1**
Let's examine the transformation to polar coordinates.

System (1) can be

$$\begin{pmatrix}\chi\\ \varrho\end{pmatrix} = \begin{pmatrix}\gamma cos\psi\\ \gamma sin\psi\end{pmatrix} \quad (10)$$

$$\gamma' = \begin{cases}\varepsilon \sum_{i+j=1}^{n+1} \gamma^{i+j-1}\left(w_{ij}^{+}\right) p^i & \frac{-\pi}{2} \le \psi < \frac{\pi}{2}\\ \varepsilon \sum_{i+j=1}^{n+1} \gamma^{i+j-1}\left(w_{ij}^{-}\right) p^i & \frac{\pi}{2} \le \psi \le \frac{3\pi}{2}\end{cases} \quad (11)$$

$$p^i = cos^i\psi sin^j\psi$$

$$\psi' = \begin{cases}\gamma_1 + \varepsilon \sum_{i+j=1}^{n+1} \gamma^{i+j-2}\left(\xi_{ij}^{+}\right) p^i & \frac{-\pi}{2} \le \psi < \frac{\pi}{2}\\ \gamma_2 + \varepsilon \sum_{i+j=1}^{n+1} \gamma^{i+j-2}\left(\xi_{ij}^{-}\right) p^i & \frac{\pi}{2} \le \psi \le \frac{3\pi}{2}\end{cases} \quad (12)$$

$$\gamma_1 = 1 + \sigma\, \gamma cos\psi$$
$$\gamma_2 = 1 + \rho\, \gamma cos\psi$$

Where, $w_{ij}^{\pm} = \sigma_{i-1,j}^{\pm} + \rho_{i,j-1}^{\pm}, \xi_{ij}^{\pm} = \rho_{i-1,j}^{\pm} - \sigma_{i,j-1}^{\pm}, \sigma_{i,-1}^{\pm} = \rho_{i,-1}^{\pm} = \rho_{-1,j}^{\pm} = \sigma_{-1,j}^{\pm} = 0$ for $i, j = 0,1,2, \dots, n+1$

**Proof**

From above equation (10) we have $\chi = \gamma\cos\psi, \varrho = \gamma\sin\psi$

$$\dot{\chi} = cos\psi\, \gamma' - \gamma sin\psi\psi', \quad \dot{\varrho} = sin\psi\, \gamma' + \gamma cos\psi\psi' \quad (13)$$

$$\gamma' = cos\psi\dot{\chi} + sin\psi\dot{\varrho}$$
$$\psi' = \frac{1}{\gamma}(cos\psi\dot{\varrho} - sin\psi\dot{\chi}) \quad (14)$$

By combining equations (1) and (14), we get the following expression,

$$\gamma' = \begin{cases}\varepsilon \sum_{i+j=0}^{n} \gamma^{i+j}\left(p_{ij}^{+}\cos\psi p^i + q_{ij}^{+}p^i\sin\psi\right) & \frac{-\pi}{2} \le \psi < \frac{\pi}{2}\\ \varepsilon \sum_{i+j=0}^{n} \gamma^{i+j}\left(p_{ij}^{-}\cos\psi p^i + q_{ij}^{-}p^i sin\psi\right) & \frac{\pi}{2} \le \psi \le \frac{3\pi}{2}\end{cases} \quad (15)$$

$$\theta' = \begin{cases}\gamma_1 + \varepsilon \sum_{i+j=0}^{n} \gamma^{i+j-1}\left(-p_{ij}^{+}p^i\sin\psi + q_{ij}^{+}\cos\psi p^i\right) & \frac{-\pi}{2} \le \psi < \frac{\pi}{2}\\ \gamma_2 + \varepsilon \sum_{i+j=0}^{n} \gamma^{i+j-1}\left(-p_{ij}^{-}p^i\sin\psi + q_{ij}^{-}\cos\psi p^i\right) & \frac{\pi}{2} \le \psi \le \frac{3\pi}{2}\end{cases} \quad (16)$$

$w_{ij}^{\pm} = \rho_{i,j-1}^{\pm} + \sigma_{i-1,j}^{\pm}, \xi_{ij}^{\pm} = \rho_{i-1,j}^{\pm} - \sigma_{i,j-1}^{\pm}, \sigma_{i,-1}^{\pm} = \sigma_{-1,j}^{\pm} = \rho_{i,-1}^{\pm} = \rho_{-1,j}^{\pm} = 0$ for $i, j = 0,1,2, \dots, n+1$

The 2π-periodic equation obtained from the equations (11) and (12) in Lemma 3.1,

$$\frac{d\gamma}{d\psi} = \begin{cases}\varepsilon X^{+}(\psi,\gamma) + \varepsilon^2 Y^{+}(\psi,\gamma) & cos\psi > 0\\ \varepsilon X^{-}(\psi,\gamma) + \varepsilon^2 Y^{-}(\psi,\gamma) & cos\psi < 0\end{cases} \quad (17)$$

$$X^{+}(\psi,\gamma) = \frac{H^{+}(\psi, \gamma)}{1 + \sigma\,\gamma\cos\psi}$$

$$X^{-}(\psi, \gamma) = \frac{H^{-}(\psi, \gamma)}{1 + \rho\gamma\cos\psi}$$

$$Y^{+}(\psi,\gamma,\varepsilon) = -\frac{X^{+}(\psi, \gamma)G^{+}(\psi, \gamma)}{\gamma(1 + \sigma\gamma\cos\psi + \varepsilon G^{+}(\psi, \gamma))}$$

$$Y^{-}(\psi,\gamma,\varepsilon) = -\frac{X^{-}(\psi, \gamma)G^{-}(\psi, \gamma)}{\gamma(1 + \sigma\gamma\cos\psi + \varepsilon G^{+}(\psi, \gamma))}$$

$$H^{\pm}(\psi,\gamma) = cos\psi p^{\pm}(\gamma cos\psi, \gamma sin\psi) + sin\psi q^{\pm}(\gamma cos\psi, \gamma sin\psi) \quad (18)$$

$$ℸ^{\pm}(\psi,\gamma) = cos\psi q^{\pm}(\gamma cos\psi, \gamma sin\psi) - sin\psi p^{\pm}(\gamma cos\psi, \gamma sin\psi) \quad (19)$$

$$\gamma_1 = \begin{cases}-\frac{1}{\sigma} & \sigma < 0\\ \infty & \sigma > 0\end{cases} \quad \gamma_2 = \begin{cases}-\frac{1}{\rho} & \rho > 0\\ \infty & \rho \le 0\end{cases}$$

$$X^{+}(\psi,\gamma) = \sum_{i+j=1}^{n+1} w_{ij}\, \gamma^{i+j-1} \frac{\cos^i\psi\sin^j\psi}{1 + \sigma\,\gamma\cos\psi}$$

$$X^{-}(\psi,\gamma) = \sum_{i+j=1}^{n+1} w_{ij}\, \gamma^{i+j-1} \frac{\cos^i\psi\sin^j\psi}{1 + b\,\gamma\cos\psi}$$

$$Y^{+}(\psi,\gamma)=\frac{1}{(\gamma_1)^2}\sum_{i+j=1}^{n+1}\left(w_{ij}\,\gamma^{i+j-1}p^{i}\right)\left(\eta_{ij}\,\gamma^{i+j-2}p^{i}\right)$$
$$Y^{-}(\psi,\gamma)=\frac{1}{(\gamma_2)^2}\sum_{i+j=1}^{n+1}\left(w_{ij}\,\gamma^{i+j-1}p^{i}\right)\left(\eta_{ij}\,\gamma^{i+j-2}p^{i}\right) \quad (20)$$

It is evident that the bifurcated limit cycles in system (1) correspond to the 2π-periodic solutions of equation (17). Consequently, let's consider

$$\mathcal{F}(\psi,\gamma)=\begin{cases}X^{+}(\psi,\gamma) & \frac{-\pi}{2}\le\psi<\frac{\pi}{2}\\ X^{-}(\psi,\gamma) & \frac{\pi}{2}\le\psi\le\frac{3\pi}{2}\end{cases} \quad (21)$$

$$\ell_1(\gamma)=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sum_{i+j=1}^{n+1}w_{ij}^{+}\,\gamma^{i+j-1}\frac{\cos^{i}\psi\sin^{j}\psi}{(1+\sigma\,\gamma\cos\psi)}d\,\psi + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\sum_{i+j=1}^{n+1}w_{ij}^{-}\,\gamma^{i+j-1}\frac{\cos^{i}\psi\sin^{j}\psi}{(1+\rho\,\gamma\cos\psi)}d\,\psi = \sum_{i+j=1}^{n+1}u_{ij}\,\gamma^{i+j-1}=\sum_{k=0}^{n}v_k\,\gamma^{k} \quad (22)$$

Where $v_k=\sum_{i+j=k+1}u_{ij}$So

$$u_{ij}=w_{ij}^{+}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\cos^{i}\psi\sin^{j}\psi}{(1+\sigma\,\gamma\cos\psi)}d\,\psi + w_{ij}^{-}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\frac{\cos^{i}\psi\sin^{j}\psi}{(1+\rho\,\gamma\cos\psi)}d\,\psi$$
$$u_{ij}=w_{ij}^{+}I_{i,j}(\gamma)+w_{ij}^{-}J_{i,j}(\gamma) \quad (23)$$

Where

$$I_{i,j}(\gamma)=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\cos^{i}\psi\sin^{j}\psi}{(1+\sigma\,\gamma\cos\psi)}d\,\psi$$
$$J_{i,j}(\gamma)=\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\frac{\cos^{i}\psi\sin^{j}\psi}{(1+\rho\,\gamma\cos\psi)}d\,\psi$$

When $\sigma>0,\ \gamma=\frac{1}{\sigma}$ then $I_{0,0}(\gamma)=$ constant and $\rho<0,\ \gamma=-\frac{1}{\rho}$ then $J_{0,0}(\gamma)=$ constant

We can see that the maximum number of isolated positive zeros of $f_1(\gamma)$ is n.

$$D_rX^{+}(\psi,\gamma)=\frac{1}{(1+\sigma\,\gamma\cos\psi)}\sum_{i+j=2}^{n+1}w_{ij}^{+}(i+j-1)\,\gamma^{i+j-2}\cos^{i}\psi\sin^{j}\psi \quad (24)$$

$$D_rX^{-}(\psi,\gamma)=\frac{1}{(1+\rho\,\gamma\cos\psi)}\sum_{i+j=2}^{n+1}w_{ij}^{-}(i+j-1)\,\gamma^{i+j-2}\cos^{i}\psi\sin^{j}\psi \quad (25)$$

Then by (20) and (24,25) we obtain

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}Y^{+}(\psi,\gamma)d\psi=\sum_{k=0}^{2n-1}N_k^{+}\,\gamma^{k}$$
$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}Y^{-}(\psi,\gamma)d\psi=\sum_{k=0}^{2n-1}N_k^{-}\,\gamma^{k} \quad (26)$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}D_\gamma X^{+}(\psi,\gamma)\int_{-\frac{\pi}{2}}^{\theta}X^{+}(t,\gamma)dt\,d\psi=\sum_{k=0}^{2n-1}M_k^{+}\,\gamma^{k}$$
$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}D_\gamma X^{-}(\psi,\gamma)\int_{\frac{\pi}{2}}^{\theta}X^{-}(t,\gamma)dt\,d\psi=\sum_{k=0}^{2n-1}M_k^{-}\,\gamma^{k} \quad (27)$$

Where $M_k^{\pm}\&N_k^{\pm}$ are constants where, $k=0,1,2,\dots\dots\dots,2n-1$ and depends on the coefficients of system (1)
Inserting (26, 27) into (9)

$$\ell_2(\gamma)=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(D_\gamma X^{+}(\psi,\gamma)\int_{-\frac{\pi}{2}}^{\psi}X^{+}(t,\gamma)dt+Y^{+}(\psi,\gamma)\right)d\,\psi+\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\left(D_\gamma X^{-}(\psi,\gamma)\int_{\frac{\pi}{2}}^{\psi}X^{-}(t,\gamma)dt+Y^{-}(\psi,\gamma)\right)d\,\psi \quad (28)$$

So
$$\ell_2(\gamma)=\sum_{k=0}^{2n-1}v_k\,\gamma^{k} \quad (29)$$

$V_k=M_k^{+}+M_k^{-}+N_k^{+}+N_k^{-}$ for $k=0,1,2,\dots\dots\dots,2n-1$

Thus, it is evident that the function $\ell_2(\gamma)$ exhibits a maximum number of isolated positive zeros, which is equal to 2n-1. This conclusion can be derived from Lemma 2.2, leading to the following obtained results.

**Theorem 3.2**
By choosing a sufficiently large $N>0$ and a sufficiently small $\varepsilon_0<0$, we can ensure that within the region $\varepsilon_0\le\chi^2+\varrho^2\le N$, system (1) possesses a maximum of 2n-1 limit cycles. Additionally, applying the second-order averaging method, we can

guarantee that for values of |ε| > 0 that are sufficiently small, the statement holds true.

**Theorem 3.3**
By employing the second-order averaging method, it can be established that for sufficiently small values of |ε| > 0, system (2) exhibits a maximum of 2n-2 limit cycles.

Complete the proof of theorem 1.1 by combining 3.2 and 3.3.

# 4 Conclusion

This article effectively presents and implements a specialized second-order averaged method designed for investigating planar piecewise smooth quadratic systems, marking a significant advancement in the study of dynamical systems. By focusing on the identification and quantification of limit cycles, this study illuminates the periodic behaviors that characterize these systems, offering a clearer understanding of their long-term dynamics. The application of this method to two specific systems, demonstrating the maximum number of limit cycles possible, not only validates the method's effectiveness but also enriches our comprehension of such systems' behavior. This work play a pivitol role in mathematical modeling, offering a powerful tool for predicting and understanding the stability and complexity of dynamicsystems across scientific and engineering fields.

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**Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)**

- Imran Shabir Chuhan carried out the article's introduction, main result and the section 2.
- Inna Samuilik has improved the introduction.
- MuhammadFahim Aslam has reviewed and checked the calculations of Theorems
- Waqas Ahmed Reviewed and recalculated the final results

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The authors have no conflicts of interest to declare.