Quadric Surfaces in Terms of Coordinate Finite *II*-type

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Abstract: - Quadric surfaces of finite type are a class of three-dimensional surfaces in geometry that are defined by second-degree equations in three variables, which are an essential part of the study of conic sections, and they exhibit a wide range of interesting geometric properties and real-world applications. This paper explores the intriguing domain of quadric surfaces, particularly emphasizing those of finite type. This will start by defining the ideas of the second Laplace-Beltrami operators, involving a surface's second fundamental form (II) in the Euclidean space E^3. Then, we characterize the coordinate finite type quadrics involving the second fundamental form.

Key-Words: - Beltrami-Laplace operator, Quadric Surfaces, Surfaces in the Euclidean 3-space, Surfaces of finite Chen-type, Surfaces of coordinate finite type, Ruled surfaces.

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1 Introduction

The study of surfaces of finite Chen type is one of the fields in differential geometry with a high research active, named by Bang-Yen Chen, for his contributions to submanifold theory. These kinds of surfaces are defined within the context mostly of Euclidean spaces, and their defining property includes the behavior of their position vector associated with their mean curvature vector. They have drawn considerable attention due to their applications in pure and applied mathematics and potentially intriguing geometric properties.

This type of research started in the late 1970s when the author mentioned above introduced the concept, which marked a significant approach in the study of submanifolds in differential geometry. Chen's groundbreaking idea was to characterize the interaction between the Laplacian Δ acting on the vector-valued function of a surface, and the intrinsic geometry encoded by the surface’s first fundamental form. This novel approach provided a framework for characterizing surfaces in Euclidean and other ambient spaces using spectral and geometric properties.

The study of quadric surfaces of finite type is crucial in various fields, including geometry, engineering, physics, and computer graphics. Quadric surfaces in 3-dimensional space are a special surface class defined by second-degree equations in three variables. These surfaces are an essential part of the study of conic sections, and they introduce a wide range of interesting geometric properties and real-world applications.

The introduction of surfaces of finite Chen type in [1] sparked significant interest among differential geometers, transforming it into a dynamic and well-studied area of research. This concept has since inspired numerous investigations, focusing on the characterization of finite Chen-type surfaces across various special classes of surfaces, each offering unique geometric properties and applications.

As a result, active research has been done on surfaces of finite Chen type. These studies have revealed how the finite Chen-type condition manifests across different families of surfaces, such as quadric surfaces [2], [3], [4], tubes [5], [6] translation surfaces [7], ruled surfaces [3], [8], [9], surfaces of revolution [10], [11], [12], spiral surfaces [13], cycles of Dupin [14], [15] and helicoidal surfaces [16], [17].

Consider a surface S in the Euclidean 3-space E^3 parametrized by local coordinates v^1, and v^2. The surface is equipped with three fundamental forms, the first fundamental form I, with coefficients (q_{ij}), which describe the metric, the second fundamental form II, with coefficients (h_{ij}), which describe the extrinsic curvature of the surface and the third fundamental form III with coefficients (c_{ij}). For any two functions γ and δ be defined on S. The first Laplace operator regarding the $J = I, II, III$ of S is defined by:

$$\nabla^J(\gamma,\delta) := a^{ij}\, \gamma_{/i}\, \delta_{/j},$$

where $\gamma_{/i}$ is the partial derivative with respect to the parameter v^i and (a^{ij}) is defined to be the inverse tensor of (q_{ij}), (h_{ij}), and (c_{ij}) for $J = I, II$, and III respectively. The second Laplace operator according to the fundamental form J of S is defined by

$$\Delta^J\gamma := -\frac{1}{\sqrt{a}}(\sqrt{a}a^{ij}\gamma_{/i})_{/j}$$

where $a = \det(a_{ij})$.

Considering the position vector $\mathbf{z} = \mathbf{z}(v^1,v^2)$, of S in E^3, authors in [18], showed the relation:

$$\Delta^{III}\mathbf{z} = -\nabla^{II}\left(\frac{2H}{K}, \mathbf{z}\right) - \frac{2H}{K}\mathbf{G}.$$

where K and H are the Gauss and the mean curvature of S respectively, and $\mathbf{G}$ is its Gauss map. Moreover, they proved that a surface satisfying the condition:

$$\Delta^{III}\mathbf{z} = \lambda\mathbf{z}, \quad \lambda\in\mathbb{R},$$

i.e., a surface M: $\mathbf{z} = \mathbf{z}(v^1,v^2)$ for which all coordinate functions are eigenfunctions of Δ^{III} with the same eigenvalue λ, is a part of a sphere ($\lambda = 2$) or a minimal surface ($\lambda = 0$).

2 Basics

A parametric representation of a surface S in the Euclidean space E^3 is the following:

$$\mathbf{y}(u^1,u^2) = \{y_1(u^1,u^2), y_2(u^1,u^2), y_3(u^1,u^2)\},$$
$$(u^1,u^2) \in \mathrm{U} \subset \mathbb{R}^2$$

The fundamental form I of S is:

$$I = q_{ij}\, du^i du^j, \ \ i, j = 1, 2,$$

where

$$q_{ij} = \langle \mathbf{y}_{/i}\, , \mathbf{y}_{/j} \rangle$$

are called the components of I, $\langle\, ,\, \rangle$ is the Euclidean inner product, and $\mathbf{y}_{/i} = \dfrac{\partial \mathbf{y}}{\partial u^i}$.

For some enough differentiable function $\varphi(u^1,u^2)$ on $\mathrm{U} \subset \mathbb{R}^2$, the Laplacian Δ^I is found to be [19]:

$$\Delta^I\varphi = -\frac{1}{\sqrt{q}}(\sqrt{q}q^{ij}\varphi_{/i})_{/j}, \tag{1}$$

where $q = \det(q_{ij})$, and q^{ij} represents the inverse tensor of q_{ij}.

Denote by $\mathbf{G}$ the Gauss map of S, then we have that:

$$\mathbf{G} = \frac{\mathbf{y}_{/1}\times\mathbf{y}_{/2}}{\sqrt{q}},$$

where × represents the Euclidean cross product.

Then the second metric of S is

$$II = h_{ij}\, du^i du^j,$$

where

$$h_{ij} = \langle \mathbf{G}\, , \mathbf{y}_{/i/j} \rangle$$

are called the components of the metric II. Hence the Laplacian Δ^{II} is found to be:

$$\Delta^{II}\varphi = -\frac{1}{\sqrt{h}}(\sqrt{h}h^{ij}\varphi_{/i})_{/j},$$

where $h = \det(h_{ij})$, and h^{ij} represents the inverse tensor of h_{ij}.

Definition 1. For the vector-valued function $\mathbf{y} = \{y_1, y_2, y_3\}$, then

$$\Delta^J\mathbf{y} = \{\Delta^J y_1, \Delta^J y_2, \Delta^J y_3\}, \ J = I, II.$$

Now we will get into the heart of the topic of this research, firstly we give the following basic definition:

Definition 2. A surface S whose position vector $\mathbf{y}$ satisfies a relation of the form

$$\Delta^{II}\mathbf{y} = A\mathbf{y}. \tag{2}$$

is said to be of coordinate finite Chen type with respect to the second fundamental form, where A is a square matrix of order 3.

In this article, we pay attention to quadrics whose position vector $\mathbf{y}$ satisfies a relation of the form (2).

3 Quadric Surfaces

Let S be a quadric surface in E^3. Then S is either a ruled surface or is one of the following two kinds:

Kind I. S is of the form:

$$z^2 = c + ax^2 + by^2 \quad a, b, c \in \mathbb{R},$$
$$ab \neq 0, \ c > 0, \tag{3}$$

Kind II. S is of the form:

$$z = ax^2 + by^2 \quad a, b \in \mathbb{R}, \ a, b > 0. \tag{4}$$

If S is ruled, then we have [3]:

Theorem 1. There are no ruled surfaces in the Euclidean 3-space that satisfy the relation (2).

For the first kind mentioned above, we prove that a quadric surface of the form (3) satisfies relation (2), exactly when $a = b = -1$, that is, S is a part of a sphere. Next, we prove that for a quadric surface of the form (4) condition (2) cannot be satisfied.

Interesting research also, one can follow the idea in [20] by defining the first and second Laplace operators using the definition of the fractional vector operators. Application within this subject can be found in [21], [22].

3.1 Quadrics of the Form (3)

Putting $u = x$ and $v = y$, then $z = \pm\sqrt{c+au^2+bv^2}$. Thus a parameterization of this form is locally represented by

$$\boldsymbol{y}(u,v) = \{u, v, \sqrt{c+au^2+bv^2}\}. \quad (5)$$

We put

$$\omega = c + axu^2 + byv^2.$$

We have

$$\boldsymbol{y}_u = \left\{1, 0, \frac{au}{\sqrt{\omega}}\right\}, \boldsymbol{y}_v = \left\{0, 1, \frac{bv}{\sqrt{\omega}}\right\}.$$

The coefficients of the metric I are

$$q_{11} = \langle \boldsymbol{y}_u, \boldsymbol{y}_v \rangle = 1 + \frac{(au)^2}{\omega},$$

$$q_{22} = \langle \boldsymbol{y}_v, \boldsymbol{y}_v \rangle = 1 + \frac{(bv)^2}{\omega}.$$

So, we obtain

$$I = \left(1 + \frac{(au)^2}{\omega}\right) du^2 + 2\frac{abuv}{\omega} du\, dv + \left(1 + \frac{(bv)^2}{\omega}\right) dv^2.$$

The normal vector $\boldsymbol{G}$ of S is:

$$\boldsymbol{G} = \left\{\frac{-au}{\sqrt{W}}, \frac{-bv}{\sqrt{W}}, \frac{\sqrt{\omega}}{\sqrt{W}}\right\} \quad (6)$$

where

$$W = c + a(a+1)u^2 + b(b+1)v^2.$$

We have

$$\boldsymbol{G_u} = \left\{\frac{a^2u^2(a+1) - aW}{W\sqrt{W}}, \frac{ab(a+1)uv}{W\sqrt{W}}, \frac{aWu - a\omega(a+1)u}{W\sqrt{W}\sqrt{\omega}}\right\}$$

$$\boldsymbol{G_v} = \left\{\frac{ab(b+1)uv}{W\sqrt{W}}, \frac{b^2v^2(b+1) - bW}{W\sqrt{W}}, \frac{bWv - b\omega(b+1)v}{W\sqrt{W}\sqrt{\omega}}\right\}$$

then

$$h_{11} = -\langle \boldsymbol{G}_u, \boldsymbol{y}_u \rangle = \frac{a(c+bv^2)}{\omega\sqrt{W}},$$

$$h_{12} = -\frac{1}{2}(\langle \boldsymbol{G}_u, \boldsymbol{y}_v \rangle + \langle \boldsymbol{G}_v, \boldsymbol{y}_u \rangle) = -\frac{abuv}{\omega\sqrt{W}},$$

$$h_{22} = -\langle \boldsymbol{G}_v, \boldsymbol{y}_v \rangle = \frac{b(c+au^2)}{\omega\sqrt{W}}.$$

and

$$\sqrt{|h|} = \frac{\sqrt{abc}}{\sqrt{\omega}\sqrt{W}}.$$

The Second fundamental form of the surface is given by:

$$II = \frac{a(c+bv^2)}{\omega\sqrt{W}} du^2 - \frac{2abuv}{\omega\sqrt{W}} du dv + \frac{b(c+au^2)}{\omega\sqrt{W}} dv^2.$$

Therefore from (1), the Laplace operator Δ^{II} of S is given as follows:

$$\Delta^{II} = -\frac{\sqrt{W}}{c}\left[2u\frac{\partial}{\partial u} + 2v\frac{\partial}{\partial v} + 2uv\frac{\partial^2}{\partial u \partial v} + \frac{(c+au^2)}{a}\frac{\partial^2}{\partial u^2} + \frac{(c+bv^2)}{b}\frac{\partial^2}{\partial v^2}\right] \quad (7)$$

Let (y_1, y_2, y_3) be the components of the vector $\boldsymbol{y}$. Applying (7) for the functions y_1 and y_2, we find:

$$\Delta^{II} y_1 = -\frac{2u\sqrt{W}}{c} \quad (8)$$

$$\Delta^{II} y_2 = -\frac{2b\sqrt{W}}{c} \quad (9)$$

$$\Delta^{II} y_3 = \Delta^{II}(\sqrt{\omega}) = -\frac{2\sqrt{\omega}\sqrt{W}}{c}. \quad (10)$$

Let $A = [a_{ij}]$. Applying relation (2) for the position vector (5) we find:

$$\Delta^{II} y_1 = -\frac{2u\sqrt{W}}{c} = a_{11}y_1 + a_{12}y_2 + a_{13}y_3, \quad (11)$$

$$\Delta^{II} y_2 = -\frac{2b\sqrt{W}}{c} = a_{21}y_1 + a_{22}y_2 + a_{23}y_3, \quad (12)$$

$$\Delta^{II} y_3 = \Delta^{II}(\sqrt{\omega}) = a_{31}y_1 + a_{32}y_2 + a_{33}y_3. \quad (13)$$

From (8) and (11) we have:

$$-\frac{2u\sqrt{T}}{c} = a_{11}u + a_{12}v + a_{13}\sqrt{\omega}. \quad (14)$$

Putting $u = 0$ in the above equation, and taking both sides of the equation to the power 2, then we get:

$$\left(-2u\sqrt{a(a+1)u^2 + c}\right)^2 = \left(ca_{11}u + ca_{13}\sqrt{au^2 + c}\right)^2$$

or

$$c^2u^2a_{11}{}^2 + 2c^2a_{11}a_{13}u\sqrt{au^2+c} + c^2a_{13}{}^2au^2 + c^3a_{13}{}^2 - 4a(a+1)u^4 - 4cu^2 = 0$$

since the above equation hold true for all u, we must have:

$$a(a+1) = 0$$

Since $a \neq 0$, so we must have $a = -1$. From (9) and (12) we have:

$$-\frac{2v\sqrt{T}}{c} = a_{21}u + a_{22}v + a_{23}\sqrt{\omega}\,. \qquad (15)$$

Putting $u = 0$ in (15) and taking both sides of the equation to the power 2, we get:

$$\left(-2v\sqrt{b(b+1)v^2 + c}\right)^2 = \left(ca_{22}v + ca_{23}\sqrt{bv^2 + c}\right)^2$$

or

$$c^2v^2{a_{22}}^2 + 2c^2a_{22}a_{23}v\sqrt{bv^2 + c} + c^2{a_{23}}^2bv^2 + c^3{a_{23}}^2 - 4b(b+1)v^4 - 4cv^2 = 0$$

since the above equation holds true for all v, we must have $b(b+1) = 0$ if and only if $b = -1$.

Butting $a = -1, b = -1$ in (14) we finally find:

$$\frac{-2u}{\sqrt{c}} = a_{11}u + a_{12}v + a_{13}\sqrt{c - u^2 - v^2}$$

or

$$\left(\frac{-2}{\sqrt{c}} - a_{11}\right)u - a_{12}v - a_{13}\sqrt{c - u^2 - v^2} = 0,$$

then

$$a_{12} = 0\,, a_{13} = 0\,, a_{11} = \frac{-2}{\sqrt{c}}.$$

And inserting $a = -1\,, b = -1$ in (15) we conclude:

$$\frac{-2v}{\sqrt{c}} = a_{21}u + a_{22}v + a_{23}\sqrt{c - u^2 - v^2}$$

or

$$\left(\frac{-2}{\sqrt{c}} - a_{22}\right)v - a_{21}u - a_{23}\sqrt{c - u^2 - v^2} = 0,$$

then

$$a_{21} = 0\,, a_{23} = 0\,, a_{22} = \frac{-2}{\sqrt{c}}.$$

From (10) and (13) we have:

$$\frac{-2\sqrt{\omega}\sqrt{T}}{c} = a_{31}u + a_{32}v + a_{33}\sqrt{\omega}\,. \quad (16)$$

If we put $a = -1\,, b = -1$ in (16) we get:

$$-2\sqrt{c - u^2 - v^2} = \sqrt{c}\; a_{31}u + \sqrt{c}\; a_{32}v + \sqrt{c}\, a_{33}\sqrt{c - u^2 - v^2}$$

or

$$\left(-2 - \sqrt{c}a_{33}\right)\sqrt{c - u^2 - v^2} - \sqrt{c}\, a_{31}u - \sqrt{c}\, a_{32}v = 0,$$

hence

$$a_{31} = 0\,, a_{32} = 0 \text{ and } a_{33} = \frac{-2}{\sqrt{c}}.$$

So, we proved the following.

Theorem 2. Spheres are the only quadric surfaces of the first kind $z^2 = c + ax^2 + by^2$, $a, b, c \in \mathbb{R}$, $ab \neq 0$, $c > 0$ of coordinate finite type with respect to the second fundamental form.

3. 2 Quadrics of the Form (4)

Putting *u* = *x* and *v* = *y*, then $z = au^2 + bv^2$. And so a parametric representation of this kind is locally given by:

$$\boldsymbol{y}(u, v) = \{u, v, au^2 + bv^2\}. \qquad (17)$$

We have:

$$y_u = \{1, 0, 2au\}, \quad \boldsymbol{y}_v = \{0, 1, 2bv\}.$$

The coefficients of the metric *I* are:

$$q_{11} = 1 + (2au)^2, \quad q_{12} = 4abuv, \quad q_{22} = 1 + (2bv)^2.$$

So, we obtain:

$$\text{I} = [1 + (2au)^2]du^2 + 8abuvdudv + [1 + (2bv)^2]dv^2.$$

The Gauss map ***G*** is:

$$\boldsymbol{G} = \left\{-\frac{2au}{\sqrt{q}}, -\frac{2bv}{\sqrt{q}}, \frac{1}{\sqrt{q}}\right\}, \qquad (18)$$

where $g = 1 + (2au)^2 + (2bv)^2$. We have:

$$\boldsymbol{G}_u = \left\{\frac{-2aq + 4a^2u^2}{q\sqrt{q}}, \frac{4abuv}{q\sqrt{q}}, -\frac{2au}{q\sqrt{q}}\right\},$$

$$\boldsymbol{G}_v = \left\{\frac{4abuv}{q\sqrt{q}}, \frac{-2bq + 4b^2v^2}{q\sqrt{q}}, -\frac{2bv}{q\sqrt{q}}\right\},$$

The components of the second fundamental form are defined as follows:

$$h_{11} = \frac{2a}{\sqrt{q}}, \quad h_{12} = 0, \quad h_{22} = \frac{2b}{\sqrt{q}}$$

The Second fundamental form of the surface is given by:

$$II = \frac{2a}{\sqrt{q}}du^2 + \frac{2b}{\sqrt{q}}dv^2 \cdot$$

Therefore from (1), the Laplace operator Δ^{II} of S is given as follows:

$$\Delta^{II} = -\frac{\sqrt{q}}{\sqrt{4ab}}\left[\frac{\sqrt{b}}{\sqrt{a}}\frac{\partial^2}{\partial u^2}+\frac{\sqrt{a}}{\sqrt{b}}\frac{\partial^2}{\partial v^2}\right]. \tag{19}$$

Let (y_1, y_2, y_3) be the components of the vector $\boldsymbol{y}$. Applying (19) for the functions y_1 and y_2, we find:

$$\Delta^{II} y_1 = \Delta^{II}(u) = 0, \tag{20}$$

$$\Delta^{II} y_2 = \Delta^{II}(v) = 0. \tag{21}$$

$$\Delta^{II} y_3 = \Delta^{II}(au^2 + bv^2) = -2\sqrt{q}. \tag{22}$$

Let $A = [a_{ij}]$. Applying relation (2) for the position vector (17) we find:

$$\Delta^{II} y_1 = a_{11}y_1 + a_{12}y_2 + a_{13}y_3, \tag{23}$$

$$\Delta^{II} y_2 = a_{21}y + a_{22}y_2 + a_{23}y_3, \tag{24}$$

$$\Delta^{II} y_3 = a_{31}y_1 + a_{32}y_2 + a_{33}y_3. \tag{25}$$

Taking into account (20) and (23) we have:

$$a_{11}u + a_{12}v + a_{13}(au^2 + bv^2) = 0,$$

then

$$a_{11} = 0\,, a_{12} = 0, \text{ and } a_{13} = 0.$$

From (21) and (24) we have

$$a_{21}u + a_{22}v + a_{23}(au^2 + bv^2) = 0,$$

then

$$a_{21} = 0\,, a_{22} = 0, \text{ and } a_{23} = 0.$$

Finally, from (22) and (25) we get:

$$-2\sqrt{q} = a_{31}u + a_{32}v + a_{33}(au^2 + bv^2). \tag{26}$$

Putting $u = 0$ in (26) and taking both sides of the equation to the power 2, we get:

$$\left(-2\sqrt{1+4b^2v^2}\right)^2 = (a_{32}v + a_{33}bv^2)^2$$

or

$$\frac{b^2}{4}{a_{33}}^2v^4 + ba_{32}a_{33}v^3 + {a_{32}}^2v^2$$
$$-16b^2v^2 - 4 = 0$$

The above equation is a polynomial in v with constant coefficients so it can be easily verified that $a_{32} = a_{33} = 0$. So, equation (26) reduces to:

$$-2\sqrt{q} = a_{31}u$$

or

$$4(1 + 4a^2u^2 + 4b^2v^2) = {a_{31}}^2u^2.$$

This equation cannot be held true. So, we prove:

Theorem 3. There are no quadric surfaces of the second kind $z = ax^2 + by^2,\ \ a,b \in \mathbb{R},\ \ a,b > 0$ satisfying the relation $\Delta^{II}\boldsymbol{y} = A\boldsymbol{y}$.

4 Conclusion

This research article was divided into three sections, where after the introduction, the needed definitions and relations regarding this interesting field of study were given. Then a formula for the Laplace operator corresponding to the first fundamental form I was proved once for the position vector and another for the Gauss map of a surface S by using the calculation of tensors theory . Finally, we classify the quadric surfaces S satisfying the relation $\Delta\boldsymbol{y} = A\boldsymbol{y}$, for a real square matrix A of order 3. An interesting study can be drawn, if this type of study can be applied to other classes of surfaces that have not been investigated yet such as spiral surfaces, or tubular surfaces.

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Declaration of Generative AI and AI-assisted Technologies in the Writing Process
During the preparation of this work, the authors used (AI) in order to improve the readability and language of the manuscript. After using this tool, the authors reviewed and edited the content as needed and take full responsibility for the content of the publication.

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Conflict of Interest

The authors have no conflicts of interest to declare.