On One-Parameter Generalization of Jacobsthal Numbers

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Abstract: In this paper, we discuss a one-parameter generalization of Jacobsthal numbers that preserves the recurrence relation with arbitrary initial conditions. We introduce generalized Jacobsthal-Lucas-like numbers, which are simple associations of generalized Jacobsthal numbers. Consequently, we give some new and well-known identities. Furthermore, we propose integral representations of these numbers associated with generalized Jacobsthal and Jacobsthal-Lucas-like numbers. Our results not only generalize the integral representations of the Jacobsthal and Jacobsthal-Lucas numbers but also apply to all one-parameter generalizations of Jacobsthal numbers.

Key-Words: one-parameter Jacobsthal number, generalized Jacobsthal number, generalized Jacobsthal-Lucas number, generalized Jacobsthal-Lucas-like number, Jacobsthal number, Jacobsthal-Lucas number, integral representation.

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1 Introduction

Number sequences have indeed fascinated researchers for decades. Their applications are widespread and span various branches of mathematics and science. Researchers have worked on generalizing these sequences; see in recent years, [1], [2], [3], [4], [5], [6], [7], [8]. Jacobsthal numbers are one of these fascinating generalizations. The *Jacobsthal numbers*, denoted by J_n, are defined by the recurrence relation

$$J_n = J_{n-1} + 2J_{n-2}, \quad n \geq 2,$$

with $J_0 = 0$ and $J_1 = 1$. The *Jacobsthal-Lucas numbers*, denoted by j_n, are defined by the recurrence relation

$$j_n = j_{n-1} + 2j_{n-2}, \quad n \geq 2,$$

with $j_0 = 2$ and $j_1 = 1$. The Jacobsthal and Jacobsthal-Lucas numbers are like the related Fibonacci and Lucas numbers; they are a specific type of Lucas sequence, [9], see more details in [10].

There are some generalizations of the Jacobsthal and Jacobsthal-Lucas numbers defined in different ways; see, for instance, [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22]. In recent years, [19], [20], introduced and studied a one-parameter generalization of Jacobsthal as follows: Let k and n be non-negative integers with $k \geq 2$. The *generalized Jacobsthal numbers*, denoted by $J_{k,n}$, are defined by the recurrence relation

$$J_{k,n} = (k-1)J_{k,n-1} + kJ_{k,n-2}, \quad n \geq 2, \tag{1}$$

with $J_{k,0} = 0$ and $J_{k,1} = 1$. For a one-parameter generalization of Jacobsthal-Lucas numbers, [20], defined *generalized Jacobsthal-Lucas numbers of the first type*, denoted by $j_{k,n}$, by the recurrence relation

$$j_{k,n} = (k-1)j_{k,n-1} + kj_{k,n-2}, \quad n \geq 2, \tag{2}$$

with $j_{k,0} = 2$ and $j_{k,1} = 1$. The author in [19], studied the same recurrence relation (2) with different initial conditions so-called *generalized Jacobsthal-Lucas numbers of the second type*, denoted by $\mathrm{j}_{k,n}$, are defined by the recurrence relation

$$\mathrm{j}_{k,n} = (k-1)\mathrm{j}_{k,n-1} + k\mathrm{j}_{k,n-2}, \quad n \geq 2,$$

with $\mathrm{j}_{k,0} = 2$ and $\mathrm{j}_{k,1} = 2$.

We can see that the generalized Jacobsthal numbers $J_{2,n}$ are the classical Jacobsthal numbers J_n, the generalized Jacobsthal-Lucas numbers of the first type $j_{2,n}$ are the Jacobsthal-Lucas numbers j_n, and the generalized Jacobsthal-Lucas numbers of the second type $\mathrm{j}_{2,n}$ are Jacobsthal-like numbers V_n defined in [18].

In this paper, we study all one-parameter generalizations of Jacobsthal numbers that preserve the recurrence relation (1) with arbitrary initial conditions. We see that there exists one of them, the so-called *generalized Jacobsthal-Lucas-like*, which is a simple association of generalized Jacobsthal numbers. Consequently, we give some new and well-known identities. Furthermore, thanks to the technique of [23], we propose the integral representations of these numbers associated with the generalized Jacobsthal and Jacobsthal-Lucas-like numbers.

2 One-Parameter Jacobsthal Numbers

We introduce a generalization of the Jacobsthal numbers with one positive integer parameter, $k \geq 2$ which is called *one-parameter Jacobsthal numbers*, denoted by $\mathcal{J}_{k,n} = \mathcal{J}_{k,n}(a,b)$, defined by a recurrence relation

$$\mathcal{J}_{k,n} = (k-1)\mathcal{J}_{k,n-1} + k\mathcal{J}_{k,n-2}, \quad n \geq 2, \tag{3}$$

with $\mathcal{J}_{k,0} = a$ and $\mathcal{J}_{k,1} = b$, where a and b are arbitrary non-negative integers such that $a + b \neq 0$. Note that $\mathcal{J}_{k,n}$ correspond to special cases of the Horadam numbers, [24]. The first terms of one-parameter Jacobsthal numbers are:

$$\begin{aligned}
\mathcal{J}_{k,0} &= a\\
\mathcal{J}_{k,1} &= b\\
\mathcal{J}_{k,2} &= (a+b)k - b\\
\mathcal{J}_{k,3} &= (a+b)k^2 - (a+b)k + b\\
\mathcal{J}_{k,4} &= (a+b)k^3 - (a+b)k^2 + (a+b)k - b\\
\mathcal{J}_{k,5} &= (a+b)k^4 - (a+b)k^3 + (a+b)k^2\\
&\quad - (a+b)k + b\\
\mathcal{J}_{k,6} &= (a+b)k^5 - (a+b)k^4 + (a+b)k^3\\
&\quad - (a+b)k^2 + (a+b)k - b.
\end{aligned}$$

Some particular cases of the previous definition are

(i) $J_{k,n} = \mathcal{J}_{k,n}(0,1)$,

(ii) $j_{k,n} = \mathcal{J}_{k,n}(2,1)$,

(iii) $\mathrm{j}_{k,n} = \mathcal{J}_{k,n}(2,2)$,

(iv) generalized Jacobsthal numbers of the second type, [15], $\mathbb{J}_n = \mathcal{J}_{2,n}(a,b)$,

(v) $J_n = \mathcal{J}_{2,n}(0,1)$,

(vi) $j_n = \mathcal{J}_{2,n}(2,1)$, and

(vii) Jacobsthal-like numbers, [18], $V_n = \mathcal{J}_{2,n}(2,2)$.

The Binet's formulas for the one-parameter Jacobsthal numbers are given in the following theorem.

Theorem 1 (Binet's formulas)**.** *Let k and n be non-negative integers with $k \geq 2$. The one-parameter Jacobsthal numbers $\mathcal{J}_{k,n}$ are given by*

$$\mathcal{J}_{k,n} = \frac{a+b}{k+1}k^n + \frac{ak-b}{k+1}(-1)^n. \tag{4}$$

Proof. The recurrence relation (3) generates a characteristic equation of the form

$$r^2 - (k-1)r - k = 0.$$

Since $\Delta_k = (k+1)^2 > 0$ for $k \geq 2$, we get that two roots are

$$r_1 = k \quad \text{and} \quad r_2 = -1.$$

Therefore, the general term of $\mathcal{J}_{k,n}$ can be expressed in the form:

$$\mathcal{J}_{k,n} = \alpha k^n + \beta(-1)^n$$

for some coefficients α and β. Since $\mathcal{J}_{k,0} = a$ and $\mathcal{J}_{k,1} = b$, we get

$$\alpha + \beta = a \quad \text{and} \quad \alpha k - \beta = b.$$

It can be shown that,

$$\alpha = \frac{a+b}{k+1} \quad \text{and} \quad \beta = \frac{ak-b}{k+1}.$$

Therefore, (4) has been proved. □

If $(a,b) \in \{(0,1),(2,1),(2,2)\}$, then we have the following:

Corollary 2 ([19, Theorem 2.2], [20, Theorem 2.1])**.** *Let k and n be non-negative integers with $k \geq 2$. Then*

$$J_{k,n} = \frac{1}{k+1}\left(k^n - (-1)^n\right), \tag{5}$$

$$j_{k,n} = \frac{3}{k+1}k^n + \frac{2k-1}{k+1}(-1)^n, \tag{6}$$

and

$$\mathrm{j}_{k,n} = \frac{4}{k+1}k^n + \frac{2k-2}{k+1}(-1)^n. \tag{7}$$

If $k = 2$, then we have the following:

Corollary 3 ([15, Theorem 2])**.** *Let n be a non-negative integer. Then*

$$\mathbb{J}_n = \frac{a+b}{3}2^n + \frac{2a-b}{3}(-1)^n.$$

Now, we give a one-parameter Jacobsthal number, the so-called *generalized Jacobsthal-Lucas-like*, which is a simple form of Binet's formula as follows:

Definition 4. Let k and n be non-negative integers with $k \geq 2$. The *generalized Jacobsthal-Lucas-like* numbers, denoted by $\mathcal{L}_{k,n}$, are defined by the recurrence relation

$$\mathcal{L}_{k,n} = (k-1)\mathcal{L}_{k,n-1} + k\mathcal{L}_{k,n-2}, \quad n \geq 2,$$

with $\mathcal{L}_{k,0} = 2$ and $\mathcal{L}_{k,1} = k-1$.

The initial terms of $\{J_{k,n}\}$, $\{j_{k,n}\}$, $\{\mathrm{j}_{k,n}\}$ and $\{\mathcal{L}_{k,n}\}$ presented as in Table 1 (Appendix).

From Theorem 1, the Binet's formulas for the generalized Jacobsthal-Lucas-like numbers $\{\mathcal{L}_{k,n}\}$ are

$$\mathcal{L}_{k,n} = k^n + (-1)^n. \tag{8}$$

It can be seen that (8) is simpler than (6) and (7). Subsequently, it is also known that $\mathcal{L}_{k,n} = \mathcal{J}_{k,n}(2, k-1)$ and $\mathcal{L}_{2,n} = j_n$. Moreover, sequences $\{\mathcal{L}_{3,n}\}$, $\{\mathcal{L}_{4,n}\}$, $\{\mathcal{L}_{5,n}\}$, and $\{\mathcal{L}_{6,n}\}$ are listed in The Online Encyclopaedia of Integer Sequences, [25], under the symbols A102345, A201455, A087404, and A274074, respectively.

Lemma 5. *Let k and n be non-negative integers with $k \geq 2$. Then*

(i) $\mathcal{L}_{k,n} + (k+1)\, J_{k,n} = 2k^n$;

(ii) $\mathcal{L}_{k,n} - (k+1)\, J_{k,n} = 2(-1)^n$;

(iii) $\mathcal{L}_{k,n}^2 - (k+1)^2 J_{k,n}^2 = 4(-k)^n$.

Proof. (i) Combining (8) and (5) gives

$$\begin{aligned}&\mathcal{L}_{k,n} + (k+1)\, J_{k,n}\\ &= (k^n + (-1)^n) + (k^n + (-1)^n)\\ &= 2k^n.\end{aligned}$$

(ii) Subtracting (8) and (5) gives

$$\begin{aligned}&\mathcal{L}_{k,n} - (k+1)\, J_{k,n}\\ &= (k^n + (-1)^n) - (k^n - (-1)^n)\\ &= 2(-1)^n.\end{aligned}$$

(iii) It follows from (i) and (ii) that

$$\begin{aligned}&\mathcal{L}_{k,n}^2 - (k+1)^2 J_{k,n}^2\\ &= \mathcal{L}_{k,n}^2 - ((k+1)J_{k,n})^2\\ &= (\mathcal{L}_{k,n} + (k+1)\, J_{k,n})(\mathcal{L}_{k,n} - (k+1)\, J_{k,n})\\ &= (2k^n)(2(-1)^n)\\ &= 4(-k)^n.\end{aligned}$$

This completes the proof. □

Lemma 6. *Let k, m, and n be non-negative integers with $k \geq 2$. Then*

(i) $2J_{k,m+n} = J_{k,m}\mathcal{L}_{k,n} + J_{k,n}\mathcal{L}_{k,m}$;

(ii) $2\mathcal{L}_{k,m+n} = \mathcal{L}_{k,m}\mathcal{L}_{k,n} + (k+1)^2 J_{k,m}J_{k,n}$.

Proof. Using (5) and (8), we obtain,

$$\begin{aligned}&J_{k,m}\mathcal{L}_{k,n} + J_{k,n}\mathcal{L}_{k,m}\\ &= \frac{1}{k+1}\left(k^m - (-1)^m\right)\left(k^n + (-1)^n\right)\\ &\quad + \frac{1}{k+1}\left(k^n - (-1)^n\right)\left(k^m + (-1)^m\right)\\ &= \frac{2}{k+1}\left(k^{m+n} - (-1)^{m+n}\right)\\ &= 2J_{k,m+n}.\end{aligned}$$

and

$$\begin{aligned}&\mathcal{L}_{k,m}\mathcal{L}_{k,n} + (k+1)^2 J_{k,n}J_{k,m}\\ &= \left(k^m + (-1)^m\right)\left(k^n + (-1)^n\right)\\ &\quad + \left(k^n - (-1)^n\right)\left(k^m - (-1)^m\right)\\ &= 2\left(k^{m+n} + (-1)^{m+n}\right)\\ &= 2\mathcal{L}_{k,m+n}.\end{aligned}$$

Hence, (i) and (ii) complete the proof. □

The one-parameter Jacobsthal numbers are associated with generalized Jacobsthal and generalized Jacobsthal-Lucas-like numbers in the following:

Theorem 7. *Let k and n be non-negative integers with $k \geq 2$. Then*

$$\mathcal{J}_{k,n} = \frac{a}{2}\mathcal{L}_{k,n} + \frac{a+2b-ak}{2}\, J_{k,n}.$$

Proof. It follows from (i) and (ii) of Lemma 5 and (4) that

$$\begin{aligned}\mathcal{J}_{k,n} &= \frac{a+b}{k+1}k^n + \frac{ak-b}{k+1}(-1)^n\\ &= \frac{a+b}{k+1}\left(\frac{\mathcal{L}_{k,n} + (k+1)\, J_{k,n}}{2}\right)\\ &\quad + \frac{ak-b}{k+1}\left(\frac{\mathcal{L}_{k,n} - (k+1)\, J_{k,n}}{2}\right)\\ &= \frac{a}{2}\mathcal{L}_{k,n} + \frac{a+2b-ak}{2}\, J_{k,n}.\end{aligned}$$

This completes the proof. □

Theorem 8 (Asymptotic behaviours)**.** *Let k be a positive integer with $k \geq 2$. Then*

$$\lim_{n\to\infty}\frac{\mathcal{J}_{k,n+1}}{\mathcal{J}_{k,n}} = \lim_{n\to\infty}\frac{\mathcal{L}_{k,n+1}}{\mathcal{L}_{k,n}} = k.$$

Proof. By using (4), we have

$$\begin{aligned}&\lim_{n\to\infty}\frac{\mathcal{J}_{k,n+1}}{\mathcal{J}_{k,n}}\\&=\lim_{n\to\infty}\frac{\left(\frac{a+b}{k+1}\right)k^{n+1}+\left(\frac{ak-b}{k+1}\right)(-1)^{n+1}}{\left(\frac{a+b}{k+1}\right)k^{n}+\left(\frac{ak-b}{k+1}\right)(-1)^{n}}\\&=\lim_{n\to\infty}\frac{\left(\frac{a+b}{k+1}\right)k-\left(\frac{ak-b}{k+1}\right)\frac{(-1)^n}{k^n}}{\left(\frac{a+b}{k+1}\right)+\left(\frac{ak-b}{k+1}\right)\frac{(-1)^n}{k^n}}.\end{aligned}$$

Since $k \geq 2$, we have $\left|\frac{-1}{k}\right| < 1$ and so

$$\lim_{n\to\infty}\frac{(-1)^n}{k^n}=0.$$

This implies that

$$\lim_{n\to\infty}\frac{\mathcal{J}_{k,n+1}}{\mathcal{J}_{k,n}}=k.$$

In particular,

$$\lim_{n\to\infty}\frac{\mathcal{L}_{k,n+1}}{\mathcal{L}_{k,n}}=k.$$

This completes the proof. □

If $(a,b) \in \{(0,1),(2,1),(2,2)\}$, then we have the following:

Corollary 9 ([20, Theorem 3.1]). *Let k be a positive integer with $k \geq 2$. Then*

$$\lim_{n\to\infty}\frac{J_{k,n+1}}{J_{k,n}}=\lim_{n\to\infty}\frac{j_{k,n+1}}{j_{k,n}}=\lim_{n\to\infty}\frac{\mathfrak{j}_{k,n+1}}{\mathfrak{j}_{k,n}}=k.$$

If $k = 2$, then we have the following:

Corollary 10 ([15, Theorem 1]).

$$\lim_{n\to\infty}\frac{\mathbb{J}_{n+1}}{\mathbb{J}_n}=2.$$

Theorem 11 (Catalan's identities). *Let k, n, and r be non-negative integers with $k \geq 2$ and $n \geq r$. Then*

$$\mathcal{J}_{k,n-r}\mathcal{J}_{k,n+r}-\mathcal{J}_{k,n}^2=(a+b)(ak-b)(-k)^{n-r}J_{k,r}^2.$$

Proof. Let $\alpha = \frac{a+b}{k+1}$ and $\beta = \frac{ak-b}{k+1}$. By using (4), we have

$$\begin{aligned}&\mathcal{J}_{k,n-r}\mathcal{J}_{k,n+r}\\&=\left(\alpha k^{n-r}+\beta(-1)^{n-r}\right)\left(\alpha k^{n+r}+\beta(-1)^{n+r}\right)\\&=\alpha^2k^{2n}+\alpha\beta(-k)^{n-r}\left(k^{2r}+(-1)^{2r}\right)+\beta^2\end{aligned}$$

and

$$\begin{aligned}\mathcal{J}_{k,n}^2&=(\alpha k^n+\beta(-1)^n)^2\\&=\alpha^2k^{2n}+2\alpha\beta(-k)^{n-r}(-k)^r+\beta^2.\end{aligned}$$

Then

$$\begin{aligned}&\mathcal{J}_{k,n-r}\mathcal{J}_{k,n+r}-\mathcal{J}_{k,n}^2\\&=\alpha\beta(-k)^{n-r}\left(k^{2r}-2(-k)^r+(-1)^{2r}\right)\\&=(a+b)(ak-b)(-k)^{n-r}\left[\frac{1}{k+1}\left(k^r-(-1)^r\right)\right]^2\\&=(a+b)(ak-b)(-k)^{n-r}\mathcal{J}_{k,r}^2.\end{aligned}$$

This completes the proof. □

If $(a,b) \in \{(2,k-1),(0,1),(2,1),(2,2)\}$, then we have the following:

Corollary 12. *Let k, n, and r be non-negative integers with $k \geq 2$ and $n \geq r$. Then*

(i) $\mathcal{L}_{k,n-r}\mathcal{L}_{k,n+r}-\mathcal{L}_{k,n}^2=(k+1)^2(-k)^{n-r}J_{k,r}^2$;

(ii) $J_{k,n-r}J_{k,n+r}-J_{k,n}^2=(-1)^{n-r+1}k^{n-r}J_{k,r}^2$;

(iii) $j_{k,n-r}j_{k,n+r}-j_{k,n}^2=(6k-3)(-k)^{n-r}J_{k,r}^2$;

(iv) $\mathfrak{j}_{k,n-r}\mathfrak{j}_{k,n+r}-\mathfrak{j}_{k,n}^2=8(k-1)(-k)^{n-r}J_{k,r}^2$.

Remark 13. As in Corollary 12, we get the following:

1. (ii) and (iii) are presented in [20, Theorems 3.2 and 3.3];
2. (ii) and (iv) are the corrections of [19, Theorem 4.1]. More precisely, there are errors by using $J_{k,n}=\left(\frac{1}{k+1}\right)(k^n+(-1)^n)$ and miscalculating the last two equations of the proof of [19, Theorem 4.1].

If $k = 2$, then we have the following:

Corollary 14 ([15]). *Let n and r be non-negative integers with $n \geq r$. Then*

$$\mathbb{J}_{n-r}\mathbb{J}_{n+r}-\mathbb{J}_n^2=(a+b)(2a-b)(-2)^{n-r}J_r^2.$$

Note that $r = 1$ in Theorem 11, the Catalan's identities give Cassini's identities for the one-parameter Jacobsthal numbers as follows:

Theorem 15 (Cassini's identities). *Let k and n be non-negative integers with $k \geq 2$. Then*

$$\mathcal{J}_{k,n-1}\mathcal{J}_{k,n+1}-\mathcal{J}_{k,n}^2=(a+b)(ak-b)(-k)^{n-1}.$$

If $(a,b) \in \{(2,k-1),(0,1),(2,1),(2,2)\}$, then we have the following:

Corollary 16 ([19, Theorem 4.2], [20, Corollaries 3.1 and 3.3])**.** *Let k and n be non-negative integers with $k \geq 2$. Then*

(i) $\mathcal{L}_{k,n-1}\mathcal{L}_{k,n+1} - \mathcal{L}_{k,n}^2 = (k+1)^2(-k)^{n-1}$;

(ii) $J_{k,n-1}J_{k,n+1} - J_{k,n}^2 = (-1)^n k^{n-1}$;

(iii) $j_{k,n-1}j_{k,n+1} - j_{k,n}^2 = (6k-3)(-k)^{n-1}$;

(iv) $\mathrm{j}_{k,n-r}\mathrm{j}_{k,n+r} - \mathrm{j}_{k,n}^2 = 8(k-1)(-k)^{n-1}$.

If $k = 2$, then we have the following:

Corollary 17 ([15])**.** *Let n be a non-negative integer. Then*

$$\mathbb{J}_{n-1}\mathbb{J}_{n+1} - \mathbb{J}_n^2 = (a+b)(2a-b)(-2)^{n-1}.$$

Theorem 18 (Generating functions)**.** *Let k be a positive integer with ≥ 2. The generating function for the one-parameter Jacobsthal numbers $\mathcal{J}_{k,n}$ is*

$$\sum_{n=0}^{\infty} \mathcal{J}_{k,n}x^n = \frac{a + (a+b-ak)x}{1-(k-1)x-kx^2}.$$

Proof. Let $\mathcal{J}_k(x) = \sum_{n=0}^{\infty} \mathcal{J}_{k,n}x^n$. Using recurrence (3) and the initial conditions $\mathcal{J}_{k,0} = a$ and $\mathcal{J}_{k,1} = b$, we have

$$\begin{aligned}
&\mathcal{J}_k(x)\\
&= \mathcal{J}_{k,0} + \mathcal{J}_{k,1}x + \sum_{n=2}^{\infty}\mathcal{J}_{k,n}x^n\\
&= a + bx + \sum_{n=2}^{\infty}((k-1)\mathcal{J}_{k,n-1} + k\mathcal{J}_{k,n-2})x^n\\
&= a + bx + (k-1)\sum_{n=2}^{\infty}\mathcal{J}_{k,n-1}x^n + k\sum_{n=2}^{\infty}\mathcal{J}_{k,n-2}x^n\\
&= a + bx + (k-1)x\sum_{n=2}^{\infty}\mathcal{J}_{k,n-1}x^{n-1}\\
&\quad + kx^2\sum_{n=2}^{\infty}\mathcal{J}_{k,n-2}x^{n-2}\\
&= a + bx + (k-1)x\sum_{n=1}^{\infty}\mathcal{J}_{k,n}x^n + kx^2\sum_{n=0}^{\infty}\mathcal{J}_{k,n}x^n\\
&= a + bx - (k-1)x\mathcal{J}_{k,0} + (k-1)x\sum_{n=0}^{\infty}\mathcal{J}_{k,n}x^n\\
&\quad + kx^2\sum_{n=0}^{\infty}\mathcal{J}_{k,n}x^n\\
&= a + bx - a(k-1)x + \mathcal{J}_{k,n}(x) + kx^2\mathcal{J}_{k,n}(x).
\end{aligned}$$

It follows that

$$(1-(k-1)x-kx^2)\mathcal{J}_{k,n}(x) = a + (a+b-ak)x$$

and so

$$\mathcal{J}_k(x) = \frac{a+(a+b-ak)x}{1-(k-1)x-kx^2}.$$

This completes the proof. □

If $(a,b) \in \{(2,k-1),(0,1),(2,1),(2,2)\}$, then we have the following:.

Corollary 19 ([19, Theorem 2.1], [20, Theorems 3.7 and 3.8])**.** *Let k be a positive integer with ≥ 2. Then*

(i) $\displaystyle\sum_{n=0}^{\infty}\mathcal{L}_{k,n}x^n = \frac{2+(3-k)x}{1-(k-1)x-kx^2}$;

(ii) $\displaystyle\sum_{n=0}^{\infty}J_{k,n}x^n = \frac{x}{1-(k-1)x-kx^2}$;

(iii) $\displaystyle\sum_{n=0}^{\infty}j_{k,n}x^n = \frac{2+(3-2k)x}{1-(k-1)x-kx^2}$;

(iv) $\displaystyle\sum_{n=0}^{\infty}\mathrm{j}_{k,n}x^n = \frac{2+(4-2k)x}{1-(k-1)x-kx^2}$.

If $k = 2$, then we have the following:

Corollary 20. *The generating function for $\mathbb{J}_n$ is*

$$\sum_{n=0}^{\infty}\mathbb{J}_n x^n = \frac{a+(b-a)x}{1-x-2x^2}.$$

At the end of this section, we give the combinatorial formula for the generalized Jacobsthal numbers as follows:

Lemma 21. *Let k and n be non-negative integers with $k \geq 2$. Then*

$$J_{k,n} = \frac{1}{2^{n-1}}\sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n}{2i+1}(k-1)^{n-2i-1}(k+1)^i$$

and

$$\mathcal{L}_{k,n} = \frac{1}{2^{n-1}}\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{2i}(k-1)^{n-2i-1}(k+1)^i.$$

Proof. By using (5) and (8), we have

$$J_{k,n} = \frac{1}{k+1}\left(k^n - (-1)^n\right)$$

$$
\begin{aligned}
&= \frac{1}{k+1}\left(\frac{(k-1)+(k+1)}{2}\right)^n \\
&\quad - \frac{1}{k+1}\left(\frac{(k-1)-(k+1)}{2}\right)^n
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{L}_{k,n} &= k^n + (-1)^n \\
&= \left(\frac{(k-1)+(k+1)}{2}\right)^n \\
&\quad + \left(\frac{(k-1)-(k+1)}{2}\right)^n.
\end{aligned}
$$

It follows from the nth powers that

$$J_{k,n} = \frac{1}{2^{n-1}} \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2i+1} (k-1)^{n-2i-1}(k+1)^i$$

and

$$\mathcal{L}_{k,n} = \frac{1}{2^{n-1}} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i} (k-1)^{n-2i-1}(k+1)^i.$$

This completes the proof. □

Using Theorem 7 and Lemma 21, we obtain the following results.

Theorem 22 (Combinatorial formulas). *Let k and n be non-negative integers with $k \geq 2$. Then*

$$
\begin{aligned}
\mathcal{J}_{k,n} &= \frac{a}{2^n} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i} (k-1)^{n-2i-1}(k+1)^i \\
&+ \frac{a+2b-ak}{2^n} \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2i+1} (k-1)^{n-2i-1}(k+1)^i.
\end{aligned}
$$

If $(a, b) \in \{(2, 1), (2, 2)\}$, then we have the following:

Corollary 23. *Let k and n be non-negative integers with $k \geq 2$. Then*

$$
\begin{aligned}
j_{k,n} &= \frac{1}{2^{n-1}} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i} (k-1)^{n-2i-1}(k+1)^i \\
&+ \frac{2-k}{2^{n-1}} \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2i+1} (k-1)^{n-2i-1}(k+1)^i
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{j}_{k,n} &= \frac{1}{2^{n-1}} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i} (k-1)^{n-2i-1}(k+1)^i \\
&+ \frac{3-k}{2^{n-1}} \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2i+1} (k-1)^{n-2i-1}(k+1)^i.
\end{aligned}
$$

If $k = 2$, then we have the following:

Corollary 24. *Let n be non-negative integers. Then*

$$\mathbb{J}_n = \frac{a}{2^n} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i} 3^i + \frac{2b-a}{2^n} \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2i+1} 3^i.$$

3 Integral Representations

Several ways are available to represent the special numbers, one of which is an integral representation; see, for example, [23], [26], [27], [28], [29], [30], [31], [32], [33], [34], [35], [36], [37], [38], [39], [40].

In this section, we obtain new integral representations for the one-parameter Jacobsthal numbers. We start with the integral representation for the generalized Jacobsthal number $J_{k,\ell n}$ based on two numbers $J_{k,\ell}$ and $\mathcal{L}_{k,\ell}$.

Theorem 25. *Let k, ℓ, and n be non-negative integers with $k \geq 2$. The generalized Jacobsthal numbers $J_{k,\ell n}$ are represented by*

$$J_{k,\ell n} = \frac{nJ_{k,\ell}}{2^n} \int_{-1}^{1} (\mathcal{L}_{k,\ell} + (k+1)\, J_{k,\ell} x)^{n-1} dx. \quad (9)$$

Proof. For $n = 0$ or $\ell = 0$, we have done. Let us assume that $\ell, n > 0$. Let $u(x) = \mathcal{L}_{k,\ell} + (k+1)J_{k,\ell}x$. Then $du = (k+1)J_{k,\ell}dx$ and so

$$\int_{u(-1)}^{u(1)} u^{n-1} du = \left[(\mathcal{L}_{k,\ell} + (k+1)\, J_{k,\ell} x)^n\right]_{-1}^{1}.$$

Using integration by substitution leads to

$$
\begin{aligned}
&\int_{-1}^{1} (\mathcal{L}_{k,\ell} + (k+1)\, J_{k,\ell} x)^{n-1} dx \\
&= \frac{1}{(k+1)J_{k,\ell}} \int_{u(-1)}^{u(1)} u^{n-1} du \\
&= \frac{1}{n(k+1)\, J_{k,\ell}} (\mathcal{L}_{k,\ell} + (k+1)\, J_{k,\ell})^n \\
&\quad - \frac{1}{n(k+1)\, J_{k,\ell}} (\mathcal{L}_{k,\ell} - (k+1)\, J_{k,\ell})^n.
\end{aligned}
$$

From (i) and (ii) of Lemma 5 with n replaced with ℓ, we get

$$
\begin{aligned}
&\int_{-1}^{1} (\mathcal{L}_{k,\ell} + (k+1)\, J_{k,\ell} x)^{n-1} dx \\
&= \frac{1}{n(k+1)\, J_{k,\ell}} \left[\left(2k^\ell\right)^n - \left(2(-1)^\ell\right)^n\right] \\
&= \frac{2^n}{nJ_{k,\ell}} \left[\frac{1}{k+1}\left(k^{\ell n} - (-1)^{\ell n}\right)\right].
\end{aligned}
$$

It follows from (5) with the replacement n by ℓn that

$$\int_{-1}^{1}(\mathcal{L}_{k,\ell}+(k+1)\,J_{k,\ell}x)^{n-1}dx=\frac{2^n}{nJ_{k,\ell}}J_{k,\ell n}.$$

Then (9) has been proved. □

Setting $k = 2$ in Theorems 25, we have the following corollaries.

Corollary 26 ([38], Theorem 3.1)**.** *Let ℓ and n be non-negative integers. The Jacobsthal numbers $J_{\ell n}$ are represented by*

$$J_{\ell n}=\frac{nJ_\ell}{2^n}\int_{-1}^{1}(j_\ell+3J_\ell x)^{n-1}dx.$$

Next, we obtain integral representations for the generalized Jacobsthal-Lucas-like numbers $\mathcal{L}_{k,\ell n}$ based on the two numbers $J_{k,\ell}$ and $\mathcal{L}_{k,\ell}$.

Theorem 27. *Let k, ℓ, and n be non-negative integers with $k \geq 2$. The generalized Jacobsthal-Lucas-like numbers $\mathcal{L}_{k,\ell n}$ are represented by*

$$\mathcal{L}_{k,\ell n}=\frac{1}{2^n}\int_{-1}^{1}(\mathcal{L}_{k,\ell}+(k+1)J_{k,\ell}x)^{n-1} \times(\mathcal{L}_{k,\ell}+(n+1)(k+1)J_{k,\ell}x)dx. \quad (10)$$

Proof. For $n = 0$ or $\ell = 0$, it is easy to see that (10) holds. We assume now that $\ell, n > 0$. We will solve (10) using integration by parts. Let u and v be such that

$$u(x)=\mathcal{L}_{k,\ell}+(n+1)(k+1)J_{k,\ell}x$$

and

$$dv=(\mathcal{L}_{k,\ell}+(k+1)J_{k,\ell}x)^{n-1}dx.$$

Then $du = (n+1)(k+1)J_{k,\ell}dx$ and so

$$v=\int(\mathcal{L}_{k,\ell}+\Delta_k\,J_{k,\ell}x)^{n-1}dx =\frac{1}{n(k+1)J_{k,\ell}}\left(\mathcal{L}_{k,\ell}+(k+1)J_{k,\ell}x\right)^n.$$

It follows that

$$\begin{aligned}I=&\frac{1}{2^n}\int_{-1}^{1}(\mathcal{L}_{k,\ell}+(n+1)(k+1)J_{k,\ell}x)\\&\times(\mathcal{L}_{k,\ell}+(k+1)J_{k,\ell}x)^{n-1}dx\\=&\frac{1}{2^n n(k+1)J_{k,\ell}}\left(\mathcal{L}_{k,\ell}+(n+1)(k+1)J_{k,\ell}x\right)\\&\times\left(\mathcal{L}_{k,\ell}+(k+1)J_{k,\ell}x\right)^n\Big|_{-1}^{1}\\&-\frac{n+1}{n2^n}\int_{-1}^{1}(\mathcal{L}_{k,\ell}+(k+1)J_{k,\ell}x)^n dx. \quad (11)\end{aligned}$$

Replacing n by $n+1$ in (9) becomes

$$J_{k,\ell n+\ell}=\frac{(n+1)J_{k,\ell}}{2^{n+1}}\int_{-1}^{1}(\mathcal{L}_{k,\ell}+(k+1)J_{k,\ell}x)^n dx$$

and so

$$\frac{2J_{k,\ell n+\ell}}{nJ_{k,\ell}}=\frac{n+1}{n2^n}\int_{-1}^{1}(\mathcal{L}_{k,\ell}+(k+1)J_{k,\ell}x)^n dx.$$

This together with (11) gives

$$\begin{aligned}I=&\frac{1}{n2^n(k+1)J_{k,\ell}}\times\\&\Big[\left(\mathcal{L}_{k,\ell}+(k+1)J_{k,\ell}\right)^n\left(\mathcal{L}_{k,\ell}+(n+1)(k+1)J_{k,\ell}\right)\\&-\left(\mathcal{L}_{k,\ell}-(k+1)J_{k,\ell}\right)^n\left(\mathcal{L}_{k,\ell}-(n+1)(k+1)J_{k,\ell}\right)\Big]\\&-\frac{2J_{k,\ell n+\ell}}{nJ_{k,\ell}}.\end{aligned}$$

Using (i) and (ii) of Lemma 5, and (i) of Lemma 6, it follows that

$$\begin{aligned}I=&\frac{1}{2^n n(k+1)J_{k,\ell}}\\&\times\Big[2^n k^{\ell n}\left(\mathcal{L}_{k,\ell}+(n+1)(k+1)J_{k,\ell}\right)\\&-2^n(-1)^{\ell n}\left(\mathcal{L}_{k,\ell}-(n+1)(k+1)J_{k,\ell}\right)\Big]\\&-\frac{1}{n\,J_{k,\ell}}\left(J_{k,\ell n}\mathcal{L}_{k,\ell}+J_{k,\ell}\mathcal{L}_{k,\ell n}\right)\\=&\frac{1}{n\,J_{k,\ell}}\left[\frac{1}{k+1}\left(k^{\ell n}-(-1)^{\ell n}\right)\right]\mathcal{L}_{k,\ell}\\&+\frac{n+1}{n}\left(k^{\ell n}+(-1)^{\ell n}\right)-\frac{J_{k,\ell n}\mathcal{L}_{k,\ell}}{n\,J_{k,\ell}}-\frac{\mathcal{L}_{k,\ell n}}{n}\\=&\frac{n+1}{n}\mathcal{L}_{k,\ell n}-\frac{\mathcal{L}_{k,\ell n}}{n}\\=&\mathcal{L}_{k,\ell n},\end{aligned}$$

which completes the proof. □

Setting $k = 2$ in Theorem 27, we have the following corollary.

Corollary 28 ([38], Theorem 3.2)**.** *Let ℓ and n be non-negative integers. The Jacobsthal-Lucas numbers $j_{\ell n}$ are represented by*

$$j_{\ell n}=\frac{1}{2^n}\int_{-1}^{1}(j_\ell+3(n+1)J_\ell x)(j_\ell+3J_\ell x)^{n-1}dx.$$

Finally, new integral representations for the one-parameter Jacobsthal numbers associated with the generalized Jacobsthal and generalized Jacobsthal-Lucas-like numbers are presented as follows:

Theorem 29. *Let k, ℓ, and n be non-negative integers with $k \geq 2$. The one-parameter Jacobsthal numbers $\mathcal{J}_{k,\ell n}$ are represented by*

$$\mathcal{J}_{k,\ell n} = \frac{1}{2^{n+1}} \int_{-1}^{1} (\mathcal{L}_{k,\ell} + (k+1)J_{k,\ell}x)^{n-1} \times \begin{pmatrix} a\mathcal{L}_{k,\ell} + (a+2b-ak)nJ_{k,\ell} \\ +a(n+1)(k+1)J_{k,\ell}x \end{pmatrix} dx.$$

Proof. From Theorem 7, we obtain

$$\mathcal{J}_{k,\ell n} = \frac{a}{2}\mathcal{L}_{k,\ell n} + \frac{a+2b-ak}{2} J_{k,\ell n}. \qquad (12)$$

Applying the integral representations of $J_{k,\ell n}$ and $\mathcal{L}_{k,\ell n}$ from Theorems 25 and 27 to (12), this completes the proof. □

Remark 30. As in Theorems 7 and 29, we have the following results.

1. If $a = 0$, then $\mathcal{J}_{k,n} = bJ_{k,n}$ and

$$\mathcal{J}_{k,\ell n} = \frac{bnJ_{k,\ell}}{2^n} \int_{-1}^{1} (\mathcal{L}_{k,\ell} + (k+1)J_{k,\ell}x)^{n-1} dx.$$

2. If $ak = a + 2b$, then $\mathcal{J}_{k,n} = \frac{a}{2}\mathcal{L}_{k,n}$ and

$$\mathcal{J}_{k,\ell n} = \frac{a}{2^{n+1}} \int_{-1}^{1} (\mathcal{L}_{k,\ell} + (k+1)J_{k,\ell}x)^{n-1} \times (\mathcal{L}_{k,\ell} + (n+1)(k+1)J_{k,\ell}x)dx.$$

3. When $a \neq 0$, $\mathcal{J}_{k,\ell}$ and $J_{k,\ell}$ are known, we can replace $\mathcal{L}_{k,\ell}$ by using

$$\mathcal{L}_{k,\ell} = \frac{2}{a}\left(\mathcal{J}_{k,\ell} - \frac{a+2b-ak}{2} J_{k,n}\right).$$

4. When $ak \neq a + 2b$, $\mathcal{J}_{k,\ell}$ and $\mathcal{L}_{k,\ell}$ are known, we can replace $J_{k,\ell}$ by using

$$J_{k,\ell} = \frac{2}{a+2b-ak}\left(\mathcal{J}_{k,\ell} - \frac{a}{2}\mathcal{L}_{k,n}\right).$$

Setting $k = 2$ in Theorem 29, we have the following corollary.

Corollary 31. *Let ℓ and n be non-negative integers. The generalized Jacobsthal numbers $\mathbb{J}_{\ell n}$ are represented by*

$$\mathbb{J}_{\ell n} = \frac{1}{2^{n+1}} \int_{-1}^{1} (j_\ell + 3J_\ell x)^{n-1} \times (aj_\ell + (2b-a)nJ_\ell + 3a(n+1)J_\ell x)dx.$$

Remark 32. As in Corollary 31, the integral representations of Jacobsthal-like numbers V_n are deduced on setting $(a, b) = (2, 2)$. More precisely,

$$V_{\ell n} = \frac{1}{2^n} \int_{-1}^{1} (j_\ell + 3J_\ell x)^{n-1} \times (j_\ell + nJ_\ell + 3(n+1)J_\ell x)dx.$$

4 Conclusions

In this paper, we study a one-parameter generalization of Jacobsthal numbers that preserves the recurrence relation with the arbitrary initial conditions. We introduce a one-parameter Jacobsthal number, so-called *generalized Jacobsthal-Lucas-like,* which is a simple association of generalized Jacobsthal numbers. We also give some new and well-known identities. Furthermore, thanks to the technique of [23], we propose the integral representations of these numbers associated with the generalized Jacobsthal and Jacobsthal-Lucas-like numbers. Our results not only generalize the integral representations of the Jacobsthal and Jacobsthal-Lucas numbers but also apply to all one-parameter Jacobsthal numbers.

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APPENDIX

The initial terms of $\{J_{k,n}\}$, $\{j_{k,n}\}$, $\{\mathfrak{j}_{k,n}\}$ and $\{\mathcal{L}_{k,n}\}$ presented in Table 1 as follows:

Table 1. Comparison of initial terms of $\{J_{k,n}\}$, $\{j_{k,n}\}$, $\{\mathfrak{j}_{k,n}\}$ and $\{\mathcal{L}_{k,n}\}$

n	0	1	2	3	4	5
$J_{k,n}$	0	1	$k-1$	k^2-k+1	k^3+k^2-k+1	$k^4-k^3+k^2-k+1$
$j_{k,n}$	2	1	$3k-1$	$3k^2-3k+1$	$3k^3+3k^2-3k+1$	$3k^4-3k^3+3k^2-3k+1$
$\mathfrak{j}_{k,n}$	2	2	$4k-2$	$4k^2-4k+2$	$4k^3+4k^2-4k+2$	$4k^4-4k^3+4k^2-4k+2$
$\mathcal{L}_{k,n}$	2	$k-1$	k^2+1	k^3-1	k^4+1	k^5-1