On Separability of Non-Linear Tri-Harmonic Operators with Matrix Potentials

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Abstract: - In this research, we explore the separability of the non-linear tri-harmonic form operator: in the case of $A[u] = \Delta^3 u(x) + V(x, u(x))u(x), x \in \mathbb{R}^n$, in the space $L_2(\mathbb{R}^n)^l$ with the operator potential $V(x, u(x)) \in L_2(\mathbb{R}^n)^l$ for every $x \in \mathbb{R}^n$, where $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator in $\mathbb{R}^n$. That is the coercive inequality $\|\Delta^3 u\| + \|Vu\| + \|V^{\frac{1}{2}}\Delta^2 u\| + \sum_{i=1}^{n} \|V^{\frac{1}{2}} \frac{\partial^2 u}{\partial x_i^2}\| \le N\|A[u]\|$, holds true.

Key-Words: - Tri-harmonic operators; Coercive inequalities; Separability; Non-linearity; Matrix potentials.

Mathematics Subject Classification: 47F05, 58J99.

Received: July 23, 2024. Revised: November 24, 2024. Accepted: December 15, 2024. Published: February 26, 2025.

1 Introduction

In [1], [2], the authors were the first to establish the fundamental results for the separation of differential expressions.

The study of the separation property for Schrodinger operators on $\mathbb{R}^n$ was examined in [3]. The operator $-\Delta + V$ in $Lp(\mathbb{R}^n)$ is separated if the following condition is satisfied: for all $u \in Lp(\mathbb{R}^n)$ such that $(\Delta + V)u \in Lp(\mathbb{R}^n)$, we have that $\Delta u \in Lp(\mathbb{R}^n)$ and $Vu \in Lp(\mathbb{R}^n)$.

In [4] adequate requirements for separability of the non-linear Schrodinger operator of the form $Su(x) = -\Delta u(x) + V(x, u)u(x)$, was established, where $\Delta u(x) = \sum_{i=1}^{n} \frac{\partial^2 u(x)}{\partial x_i^2}$, with the non-linear matrix potential $V(x, u)$, in the space $L_2(\mathbb{R}^n)^l$.

The Coercive estimates of non-linear Schrodinger and Dirac operators were examined in [5].

In [6], [7], the author investigated non-linear second order differential operators with matrix coefficients and coercive features of a biharmonic operator and the separability of the non-linear biharmonic operator $L[u] = \Delta^2 u(x) + V(x, u(x))u(x)$ in [8].

In [9] the separation for the biharmonic differential operator in Hilbert spaces was deduced, which was expanded in [8], existence and uniqueness are achieved and considered important applications of this topic. Many authors had introduced concepts of separation problem for ordinary and partial differential operators by using many suitable ways. See [10], [11], [12], [13], [14].

The following differential equation is the subject of this study:

$$\Delta^3 u(x) + V(x, u(x))u(x) = f(x),$$
$$u(x) \in W_{2,loc}^4(\mathbb{R}^n)^l, \quad (1)$$

We imply that the function $u(x) \in W_{2,loc}^4(\mathbb{R}^n)^l$ if the vector function $\Phi(x)u(x) \in W_2{}^4(\mathbb{R}^n)^l$ for any function $\Phi(x) \in C_0^\infty(\mathbb{R}^n)$. The space $W_2^4(\mathbb{R}^n)^l$ for l is any natural number is a Hilbert space of all vector functions $u(x) = (u_1(x), \ldots, u_l(x)), x \in \mathbb{R}^n$ that has generalized derivatives $D^\mu u(x), \mu \le 4$ such that $u(x)$ and its derivatives $D^\mu u(x)$ belong to $L_2(\mathbb{R}^n)^l$, and the inner product is defined by

$$\langle u, v \rangle = \langle u, v \rangle |_{L_2(\mathbb{R}^n)^l} + \sum_{|\mu| \le 4} \langle D^\mu u, D^\mu v \rangle |_{L_2(\mathbb{R}^n)^l} = \sum_{i=1}^{l} \int_{\mathbb{R}^n} u_i(x)\, \overline{v_\iota(x)} d(x) + \sum_{|\mu| \le 4} \sum_{i=1}^{l} \int_{\mathbb{R}^n} D^\mu u_i(x) \overline{D^\mu v_\iota(x)} d(x), x \in \mathbb{R}^n,$$ for every $u, v \in W_2^4(\mathbb{R}^n)^l$.

Note that we examine the separation in the space $L_2(\mathbb{R}^n)^l$ for l is any natural number is The Hilbert space of all vector functions $u(x) =$

$(u_1(x),\ldots,u_l(x)), x \in \mathbb{R}^n$, such that $u_i(x) \in L_2(\mathbb{R}^n)$,$1 \le i \le l$, with the inner product defined b

$\langle u,v\rangle = \sum_{i=1}^{l}\langle u_i,v_i\rangle \mid_{L_2(\mathbb{R}^n)^l} + \sum_{i=1}^{l}\int_{\mathbb{R}^n} u_i(x)\,\overline{v_i(x)}d(x), \quad x \in \mathbb{R}^n$,

for every $u,v \in L_2(\mathbb{R}^n)^l$.

2 Main Results

Definition 1. The differential operator and equation (1) are called separated in $L_2(\mathbb{R}^n)^l$, if for all vector functions $u(x) \in L_2(\mathbb{R}^n)^l \cap W_{2,loc}^4(\mathbb{R}^n)^l$ and $f(x) \in L_2(\mathbb{R}^n)^l$ this implies $\Delta^3 u(x), V(x,u(x))u(x) \in L_2(\mathbb{R}^n)^l$.

We let:

$$\langle z^{(1)}, z^{(2)}\rangle = \sum_{j=1}^{l} z_j^{(1)}\overline{z_j^{(2)}},$$

for given $z^i = (z_1{}^{(i)},\ldots,z_l^{(i)}), (i = 1,2)$.

We denote:

$$(u,v) = \int_{\mathbb{R}^n}\langle u(x), v(x)\rangle d(x),$$

if the integral in the right-hand side converges absolutely.

We assume that $V(x,g) \in C^2(\mathbb{R}^n \times \mathbb{C}^l)$, and introduce matrix functions:

$$S(x_1,x_2,\ldots,x_n,\zeta_1,\zeta_2,\ldots,\zeta_l,\eta_1,\eta_2,\ldots,\eta_l) = V^{\frac{1}{2}}(x,g), (x_i \in \mathbb{R}, \zeta_j,\eta_j \in \mathbb{R}),$$

$$T(x_1,x_2,\ldots,x_n,\zeta_1,\zeta_2,\ldots,\zeta_l,\eta_1,\eta_2,\ldots,\eta_l) = S^2(x,g), (x_i \in \mathbb{R}, \zeta_j,\eta_j \in \mathbb{R}),$$

where g is defined by $g = (\zeta_1 + i\eta_1,\ldots,\zeta_l + i\eta_l), V^{\frac{1}{2}}(x,g)$ is the square root of a positive definite Hermitian matrix.

Now we introduce our main result.

Theorem 2. For all $x \in \mathbb{R}^n$, we assume that $g = (\zeta_1 + i\eta_1,\ldots,\zeta_l + i\eta_l), \varphi = (\lambda_1 + i\theta_1,\ldots,\lambda_l + i\theta_l), (\zeta_j,\eta_j,\lambda_j,\theta_j \in \mathbb{R})$ and $u \in W_2{}^1(\mathbb{R}^n)$, the matrix functions $S(x,g)$ and $T(x,g)$ satisfies the conditions:

$$\sum_{i=1}^{n}\left\|S^{-\frac{1}{2}}\frac{\partial^2 S}{\partial x_i{}^2}S^{-\frac{3}{2}}; \mathbb{C}^l\right\|^2 \le \rho^1, \tag{2}$$

$$\sum_{i=1}^{n}\left\|S^{-\frac{1}{2}}\frac{\partial S}{\partial x_i}\frac{\partial u}{\partial x_i}; L_2(\mathbb{R}^n)^l\right\|^2 \le \rho_2\left\|S^{\frac{3}{2}}u; L_2(\mathbb{R}^n)^l\right\|^2, \tag{3}$$

$$\left\|\sum_{j=1}^{l}\lambda_j S^{-\frac{1}{2}}\frac{\partial^2 S}{\partial x_i\partial\zeta_j}g + \theta_j S^{-\frac{1}{2}}\frac{\partial^2 S}{\partial x_i\partial\eta_j}g; \mathbb{C}^l\right\| \le \tau_1\left\|S^{\frac{1}{2}}\varphi; \mathbb{C}^l\right\|, \tag{4}$$

$$\left\|\sum_{j=1}^{l}\lambda_j S^{-\frac{1}{2}}\frac{\partial S}{\partial\zeta_j}g + \theta_j S^{-\frac{1}{2}}\frac{\partial S}{\partial\eta_j}g; \mathbb{C}^l\right\| \le \tau_2\left\|S^{\frac{1}{2}}\varphi; \mathbb{C}^l\right\|, \tag{5}$$

$$\sum_{i=1}^{n}\left\|T^{-\frac{1}{2}}\frac{\partial^2 T}{\partial x_i{}^2}T^{-1}; \mathbb{C}^l\right\|^2 \le \rho_3, \tag{6}$$

$$\sum_{i=1}^{n}\left\|T^{-\frac{1}{2}}\frac{\partial T}{\partial x_i}\frac{\partial u}{\partial x_i}; L_2(\mathbb{R}^n)^l\right\|^2 \le \rho_4\left\|Vu; L_2(\mathbb{R}^n)^l\right\|^2, \tag{7}$$

$$\left\|\sum_{j=1}^{l}\lambda_j S^{-\frac{1}{2}}\frac{\partial^2 T}{\partial x_i\partial\zeta_j}g + \theta_j S^{-\frac{1}{2}}\frac{\partial^2 T}{\partial x_i\partial\eta_j}g; \mathbb{C}^l\right\| \le \tau_3\left\|S\varphi; \mathbb{C}^l\right\|, \tag{8}$$

$$\left\|\sum_{j=1}^{l}\lambda_j S^{-\frac{1}{2}}\frac{\partial T}{\partial\zeta_j}g + \theta_j S^{-\frac{1}{2}}\frac{\partial T}{\partial\eta_j}g; \mathbb{C}^l\right\| \le \tau_4\left\|S\varphi; \mathbb{C}^l\right\|. \tag{9}$$

where $\tau_j (j = \overline{1,4})$ are positive numbers and

$$\rho_1 + 2\rho_2 < 4, \quad \rho_3 + 2\rho_4 < 4. \tag{10}$$

Then the following coercive estimate is true:

$$\left\|\Delta^3 u(x); L_2(\mathbb{R}^n)^l\right\| + \left\|V\big(x,u(x)\big)u(x); L_2(\mathbb{R}^n)^l\right\| + \left\|V^{\frac{1}{2}}\big(x,u(x)\big)\Delta^2 u(x); L_2(\mathbb{R}^n)^l\right\| + \sum_{i=1}^{n}\left\|V^{\frac{1}{2}}\frac{\partial^2 u(x)}{\partial x_i{}^2}; L_2(\mathbb{R}^n)^l\right\| \le N\left\|f(x); L_2(\mathbb{R}^n)^l\right\|. \tag{11}$$

Where N is a positive number independent of $u(x), f(x)$. That is the tri-harmonic operator A is separated in the space $L_2(\mathbb{R}^n)^l$.

3 Auxiliary Lemmas

Lemma 3. Assume that in equation (1) the vector functions:

$$f(x) \in L_2(\mathbb{R}^n)^l, u(x) \in L_2(\mathbb{R}^n)^l \cap W^4_{2,loc}(\mathbb{R}^n)^l$$

Then the vector functions:

$$V^{\frac{1}{2}}(x,u(x))u(x), \frac{\partial^3 u}{\partial x_i{}^3} \in L_2(\mathbb{R}^n)^l, \ \ i = 1,2,\dots,n.$$

Proof. Let $\Psi(x) \in C_0^\infty(\mathbb{R}^n)$ be a fixed nonnegative function equalling one as $|x| < 1$. For each positive number ε we let $\Psi_\varepsilon(x) = \Psi(\varepsilon x)$. In view the identities:

$$\Delta\, u = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i{}^2} \ \ and \ \ (f, \Psi_\varepsilon u) = (\Delta^3 u, \Psi_\varepsilon u) + (V(x,u)u, \Psi_\varepsilon u).$$

Hence,

$$(\Delta^3 u, \Psi_\varepsilon u) = (\Delta\Delta^2 u, \Psi_\varepsilon u) = \sum_{i=1}^n (\frac{\partial^2}{\partial x_i{}^2}\Delta^2 u, \Psi_\varepsilon u) = -\sum_{i=1}^n (\frac{\partial}{\partial x_i}\Delta^2 u, \frac{\partial}{\partial x_i}\Psi_\varepsilon u).$$

So we have

$$(f, \Psi_\varepsilon u) \ = \ (\Delta^2 u, \sum_{i=1}^n \frac{\partial^2 \Psi_\varepsilon}{\partial x_i{}^2} u) + 2(\Delta^2 u, \sum_{i=1}^n \frac{\partial \Psi_\varepsilon}{\partial x_i}\frac{\partial u}{\partial x_i}) \ + (\Delta^2 u, \sum_{i=1}^n \Psi_\varepsilon \frac{\partial^2 u}{\partial x_i{}^2}) + (Vu, \Psi_\varepsilon u), \quad (12)$$

where $(.,.)$ denotes the scalar product in the space $L_2(\mathbb{R}^n)^l$. Then passing to the limit as $\varepsilon \to 0$, we find

$$(f,u) = (\Delta^2 u, \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i{}^2}) + (Vu, u),$$

$$Re(f,u) \geq -(\Delta^2 u, \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i{}^2}) + (Vu,u),$$

We have:

$$|f,u| \geq \sum_{i,j,k=1}^n (\frac{\partial}{\partial x_j}\frac{\partial^2 u}{\partial x_k{}^2}, \frac{\partial}{\partial x_j}\frac{\partial^2 u}{\partial x_i{}^2}) + (Vu,u),$$

which completes the proof.

Lemma 4, Assume that conditions $(2)-(5)$ hold and let a vector function $u(x)$ in the class $L_2(\mathbb{R}^n)^l \cap W^4_{2,loc}(\mathbb{R}^n)^l$ solves equation (1) with the function $f(x) \in L_2(\mathbb{R}^n)^l$.Then the vector functions $S^{\frac{3}{2}}(x,u(x))u(x), S^{\frac{1}{2}}(x,u(x))\Delta^2 u, S^{\frac{1}{2}}(x,u(x))\frac{\partial^2 u}{\partial x_i{}^2}$, $i = 1,\dots,n$ belong to the space $L_2(\mathbb{R}^n)^l$.

Proof. Let $\Psi(x)$ be the same as in proof of lemma 3. It is clear that

$$(f, \Psi_\varepsilon S u) = (\Delta^3 u, \Psi_\varepsilon S u) + (V(x,u)u, \Psi_\varepsilon S u).$$

Employing the identity:

$$\frac{\partial(\Psi_\varepsilon S u)}{\partial x_i} = \frac{\partial \Psi_\varepsilon}{\partial x_i} S u + \Psi_\varepsilon \frac{\partial S}{\partial x_i} u + \sum_{p=1}^l \Psi_\varepsilon\, Re \frac{\partial u_p}{\partial x_i}\frac{\partial S}{\partial \zeta_p} u + \sum_{p=1}^l \Psi_\varepsilon Im \frac{\partial u_p}{\partial x_i}\frac{\partial S}{\partial \eta_p} u + \Psi_\varepsilon S \frac{\partial u}{\partial x_i},$$

hence,

$$(\Delta^3 u, \Psi_\varepsilon S u) = (\Delta\Delta^2 u, \Psi_\varepsilon S u) = \sum_{i=1}^n \left(\frac{\partial^2}{\partial x_i{}^2}\Delta^2 u, \Psi_\varepsilon S u\right) = -\sum_{i=1}^n (\frac{\partial}{\partial x_i}\Delta^2 u, \frac{\partial}{\partial x_i}(\Psi_\varepsilon S u)),$$

after some transformations we get:

$$(\Delta^3 u, \Psi_\varepsilon S u) \ = \ \left(\Delta^2 u, \sum_{i=1}^n \frac{\partial^2 \Psi_\varepsilon}{\partial x_i{}^2} S u\right) + 2\left(\Delta^2 u, \sum_{i=1}^n \frac{\partial \Psi_\varepsilon}{\partial x_i}\frac{\partial S}{\partial x_i} u\right) + 2\left(\Delta^2 u, \sum_{i=1}^n \frac{\partial \Psi_\varepsilon}{\partial x_i} S \frac{\partial u}{\partial x_i}\right) + (\Delta^2 u, \sum_{i=1}^n \Psi_\varepsilon \frac{\partial^2 S}{\partial x_i{}^2} u) + 2(\Delta^2 u, \sum_{i=1}^n \Psi_\varepsilon \frac{\partial S}{\partial x_i}\frac{\partial u}{\partial x_i}) + (\Delta^2 u, \sum_{i=1}^n \Psi_\varepsilon S \frac{\partial^2 u}{\partial x_i{}^2}) + 2(\Delta^2 u, \sum_{i=1}^n \frac{\partial \Psi_\varepsilon}{\partial x_i}\sum_{p=1}^l (Re \frac{\partial u_p}{\partial x_i}\frac{\partial S}{\partial \zeta_p} + Im \frac{\partial u_p}{\partial x_i}\frac{\partial S}{\partial \eta_p}) u) \ + 2(\Delta^2 u, \sum_{i=1}^n \Psi_\varepsilon \sum_{p=1}^l (Re \frac{\partial u_p}{\partial x_i}\frac{\partial^2 S}{\partial x_i \partial \zeta_p} + Im \frac{\partial u_p}{\partial x_i}\frac{\partial^2 S}{\partial x_i \partial \eta_p}) u) \ + 2(\Delta^2 u, \sum_{i=1}^n \Psi_\varepsilon \sum_{p=1}^l (Re \frac{\partial u_p}{\partial x_i}\frac{\partial S}{\partial \zeta_p} + Im \frac{\partial u_p}{\partial x_i}\frac{\partial S}{\partial \eta_p})\frac{\partial u}{\partial x_i}) \ + (\Delta^2 u, \sum_{i=1}^n \Psi_\varepsilon \sum_{p=1}^l (Re \frac{\partial^2 u_p}{\partial x_i{}^2}\frac{\partial S}{\partial \zeta_p} + Im \frac{\partial^2 u_p}{\partial x_i{}^2}\frac{\partial S}{\partial \eta_p}) u).$$

We obtain:

$$(f, \Psi_\varepsilon S u) \ = \ C_1^\varepsilon(u) + 2C_2^\varepsilon(u) + 2C_3^\varepsilon(u) + C_4^\varepsilon(u) + 2C_5^\varepsilon(u) + C_6^\varepsilon(u) + 2C_7^\varepsilon(u) + 2C_8^\varepsilon(u) + 2C_9^\varepsilon(u) + C_{10}^\varepsilon(u) + (Vu, \Psi_\varepsilon S u). \quad (13)$$

Where:

$$C_1^\varepsilon(u) = (\Delta^2 u, \sum_{i=1}^n \frac{\partial^2 \Psi_\varepsilon}{\partial x_i{}^2} S u), \ \ C_2^\varepsilon(u) = (\Delta^2 u, \sum_{i=1}^n \frac{\partial \Psi_\varepsilon}{\partial x_i}\frac{\partial S}{\partial x_i} u),$$

$$C_3^\varepsilon(u) = (\Delta^2 u, \sum_{i=1}^n \frac{\partial \Psi_\varepsilon}{\partial x_i} S \frac{\partial u}{\partial x_i}), \ \ C_4^\varepsilon(u) = (\Delta^2 u, \sum_{i=1}^n \Psi_\varepsilon \frac{\partial^2 S}{\partial x_i{}^2} u),$$

$$C_5^\varepsilon(u) = (\Delta^2 u, \sum_{i=1}^n \Psi_\varepsilon \frac{\partial S}{\partial x_i}\frac{\partial u}{\partial x_i}), \ \ C_6^\varepsilon(u) = (\Delta^2 u, \sum_{i=1}^n \Psi_\varepsilon S \frac{\partial^2 u}{\partial x_i{}^2}),$$

$C_7^\varepsilon(u) = (\Delta^2 u, \sum_{i=1}^n \frac{\partial \Psi_\varepsilon}{\partial x_i} \sum_{p=1}^l (Re \frac{\partial u_p}{\partial x_i} \frac{\partial S}{\partial \zeta_p} + Im \frac{\partial u_p}{\partial x_i} \frac{\partial S}{\partial \eta_p}) u),$

$C_8^\varepsilon(u) = (\Delta^2 u, \sum_{i=1}^n \Psi_\varepsilon \sum_{p=1}^l (Re \frac{\partial u_p}{\partial x_i} \frac{\partial^2 S}{\partial x_i \partial \zeta_p} + Im \frac{\partial u_p}{\partial x_i} \frac{\partial^2 S}{\partial x_i \partial \eta_p}) u),$

$C_9^\varepsilon(u) = (\Delta^2 u, \sum_{i=1}^n \Psi_\varepsilon \sum_{p=1}^l (Re \frac{\partial u_p}{\partial x_i} \frac{\partial S}{\partial \zeta_p} + Im \frac{\partial u_p}{\partial x_i} \frac{\partial S}{\partial \eta_p}) \frac{\partial u}{\partial x_i}),$

$C_{10}^\varepsilon(u) = (\Delta^2 u, \sum_{i=1}^n \Psi_\varepsilon \sum_{p=1}^l (Re \frac{\partial^2 u_p}{\partial x_i^2} \frac{\partial S}{\partial \zeta_p} + Im \frac{\partial^2 u_p}{\partial x_i^2} \frac{\partial S}{\partial \eta_p}) u).$

Estimating the functionals, by lemma 3 we find that the functionals:
$C_1^\varepsilon(u), C_2^\varepsilon(u), C_3^\varepsilon(u), C_7^\varepsilon(u) tend\ to\ zero\ as\ \varepsilon \to 0.$

Hence for the other functionals:
$|C_4^\varepsilon(u)| = |S^{\frac{1}{2}} \Psi_\varepsilon^{\frac{1}{2}} \Delta^2 u, \sum_{i=1}^n \Psi_\varepsilon^{\frac{1}{2}} S^{-\frac{1}{2}} \frac{\partial^2 S}{\partial x_i^2} S^{-\frac{3}{2}} S^{\frac{1}{2}} S u)|,$
for any $\alpha > 0$ and $y_1, y_2 \in \mathbb{R}$, we have
$|y_1||y_2| \le \frac{\alpha}{2}|y_1|^2 + \frac{1}{2\alpha}|y_2|^2.$

Consequently:
$|C_4^\varepsilon(u)| \le \frac{\alpha_1}{2} \|(S^{\frac{1}{2}} \Psi_\varepsilon^{\frac{1}{2}} \Delta^2 u\|^2 + \frac{\rho_1}{2\alpha_1} \|S^{\frac{1}{2}} \Psi_\varepsilon^{\frac{1}{2}} u\|^2,$
$|C_5^\varepsilon(u)| = |(S^{\frac{1}{2}} \Psi_\varepsilon^{\frac{1}{2}} \Delta^2 u, \sum_{i=1}^n \Psi_\varepsilon^{\frac{1}{2}} S^{-\frac{1}{2}} \frac{\partial S}{\partial x_i} \frac{\partial u}{\partial x_i})| \le \frac{\alpha_2}{2} \|(S^{\frac{1}{2}} \Psi_\varepsilon^{\frac{1}{2}} \Delta^2 u\|^2 + \frac{\rho_2}{2\alpha_2} \|(S^{\frac{3}{2}} \Psi_\varepsilon^{\frac{1}{2}} u\|^2,$
$|C_6^\varepsilon(u)| = |(S^{\frac{1}{2}} \Psi_\varepsilon^{\frac{1}{2}} \Delta^2 u, \sum_{i=1}^n \Psi_\varepsilon^{\frac{1}{2}} S^{\frac{1}{2}} \frac{\partial^2 u}{\partial x_i^2}| \le \frac{\alpha_3}{2} \|(S^{\frac{1}{2}} \Psi_\varepsilon^{\frac{1}{2}} \Delta^2 u\|^2 + \frac{1}{2\alpha_3} \sum_{i=1}^n \|\Psi_\varepsilon^{\frac{1}{2}} S^{\frac{1}{2}} \frac{\partial^2 u}{\partial x_i^2}\|^2,$
$|C_8^\varepsilon(u)| = |S^{\frac{1}{2}} \Psi_\varepsilon^{\frac{1}{2}} \Delta^2 u, \sum_{i=1}^n \Psi_\varepsilon^{\frac{1}{2}} \sum_{p=1}^l \left(Re S^{-\frac{1}{2}} \frac{\partial u_p}{\partial x_i} \frac{\partial^2 S}{\partial x_i \partial \zeta_p} + Im S^{-\frac{1}{2}} \frac{\partial u_p}{\partial x_i} \frac{\partial^2 S}{\partial x_i \partial \eta_p} \right) u)| \le \tau_1 \|S^{\frac{1}{2}} \Psi_\varepsilon^{\frac{1}{2}} \Delta^2 u\|^2,$
$|C_9^\varepsilon(u)| = |S^{\frac{1}{2}} \Psi_\varepsilon^{\frac{1}{2}} \Delta^2 u, \sum_{i=1}^n \Psi_\varepsilon^{\frac{1}{2}} \sum_{p=1}^l (Re S^{-\frac{1}{2}} \frac{\partial u_p}{\partial x_i} \frac{\partial S}{\partial \zeta_p} + Im S^{-\frac{1}{2}} \frac{\partial u_p}{\partial x_i} \frac{\partial S}{\partial \eta_p}) \frac{\partial u}{\partial x_i})| \le \tau_2 \|S^{\frac{1}{2}} \Psi_\varepsilon^{\frac{1}{2}} \Delta^2 u\|^2,$
$|C_{10}^\varepsilon(u)| = |S^{\frac{1}{2}} \Psi_\varepsilon^{\frac{1}{2}} \Delta^2 u, \sum_{i=1}^n \Psi_\varepsilon^{\frac{1}{2}} \sum_{p=1}^l (Re S^{-\frac{1}{2}} \frac{\partial^2 u_p}{\partial x_i^2} \frac{\partial S}{\partial \zeta_p} + Im S^{-\frac{1}{2}} \frac{\partial^2 u_p}{\partial x_i^2} \frac{\partial S}{\partial \eta_p}) u)| \le \tau_2 \|S^{\frac{1}{2}} \Psi_\varepsilon^{\frac{1}{2}} \Delta^2 u\|^2.$

Then we obtain the following estimates:

$|C_4^\varepsilon(u)| \le \frac{\alpha_1}{2} (\Delta^2 u, \Psi_\varepsilon S \Delta^2 u) + \frac{\rho_1}{2\alpha_1} (Vu, \Psi_\varepsilon S u),$
$|C_5^\varepsilon(u)| \le \frac{\alpha_2}{2} (\Delta^2 u, \Psi_\varepsilon S \Delta^2 u) + \frac{\rho_2}{2\alpha_2} (Vu, \Psi_\varepsilon S u),$
$|C_6^\varepsilon(u)| \le \frac{\alpha_3}{2} (\Delta^2 u, \Psi_\varepsilon S \Delta^2 u) + \frac{1}{2\alpha_3} \sum_{i=1}^n (\frac{\partial^2 u}{\partial x_i^2}, \Psi_\varepsilon S \frac{\partial^2 u}{\partial x_i^2}),$
$|C_8^\varepsilon(u)| \le \tau_1 (\Delta^2 u, \Psi_\varepsilon S \Delta^2 u),$ $|C_9^\varepsilon(u)| \le \tau_2 (\Delta^2 u, \Psi_\varepsilon S \Delta^2 u),$ $|C_{10}^\varepsilon(u)| \le \tau_2 (\Delta^2 u, \Psi_\varepsilon S \Delta^2 u).$

Where ρ_1, ρ_2, τ_1 and τ_2 are the constants in conditions (2)-(5) and $\alpha_1, \alpha_2, \alpha_3$ are arbitrary positive numbers, we employed inequality (5) in estimating the functionals $C_9^\varepsilon(u)$ and $C_{10}^\varepsilon(u)$: in the case
$g = (g_1, g_2, \dots, g_l) = u(x) = (u_1(x), u_2(x), \dots, u_l(x)),$
and
$g = (g_1, g_2, \dots, g_l) = \frac{\partial u}{\partial x_i} = (\frac{\partial u_1}{\partial x_i}, \frac{\partial u_2}{\partial x_i}, \dots, \frac{\partial u_l}{\partial x_i}).$

Based on the obtained estimates, by identity (13) we get:
$|f, \Psi_\varepsilon S u| \ge -|C_1^\varepsilon(u)| - 2|C_2^\varepsilon(u)| - 2|C_3^\varepsilon(u)| + \frac{\alpha_1}{2} (\Delta^2 u, \Psi_\varepsilon S \Delta^2 u) - \frac{\rho_1}{2\alpha_1} (Vu, \Psi_\varepsilon S u) + \alpha_2 (\Delta^2 u, \Psi_\varepsilon S \Delta^2 u) - \frac{\rho_2}{\alpha_2} (Vu, \Psi_\varepsilon S u) + \frac{\alpha_3}{2} (\Delta^2 u, \Psi_\varepsilon S \Delta^2 u) + \frac{1}{2\alpha_3} \sum_{i=1}^n (\frac{\partial^2 u}{\partial x_i^2}, \Psi_\varepsilon S \frac{\partial^2 u}{\partial x_i^2}) - 2|C_7^\varepsilon(u)| + 2\tau_1 (\Delta^2 u, \Psi_\varepsilon S \Delta^2 u) + 2\tau_2 (\Delta^2 u, \Psi_\varepsilon S \Delta^2 u) + \tau_2 (\Delta^2 u, \Psi_\varepsilon S \Delta^2 u) + (Vu, \Psi_\varepsilon S u).$

Passing to the limit as $\varepsilon \to 0$, we have:
$|(f, Su)| \ge [1 - \frac{\rho_1}{2\alpha_1} - \frac{\rho_2}{\alpha_2}](Vu, Su) + [\frac{\alpha_1}{2} + \alpha_2 + \frac{\alpha_3}{2} + 2\tau_1 + 3\tau_2](\Delta^2 u, S \Delta^2 u) + \frac{1}{2\alpha_3} \sum_{i=1}^n (\frac{\partial^2 u}{\partial x_i^2}, S \frac{\partial^2 u}{\partial x_i^2}).$

Applying Cauchy-Schwarz inequality, then we obtain the inequality:

$$\|f; L_2(\mathbb{R}^n)^l\| \|Su; L_2(\mathbb{R}^n)^l\| \ge |f, Su| \ge \left[1 - \frac{\rho_1}{2\alpha_1} - \frac{\rho_2}{\alpha_2}\right](Vu, Su) + \left[\frac{\alpha_1}{2} + \alpha_2 + \frac{\alpha_3}{2} + 2\tau_1 + 3\tau_2\right](\Delta^2 u, S \Delta^2 u) + \frac{1}{2\alpha_3} \sum_{i=1}^n \left(\frac{\partial^2 u}{\partial x_i^2}, S \frac{\partial^2 u}{\partial x_i^2}\right). \quad (14)$$

Hence $Su \in L_2(\mathbb{R}^n)^l$ by lemma 3, and choose $\alpha_1, \alpha_2, \alpha_3$ satisfy the conditions:
$\frac{\rho_1}{2\alpha_1} + \frac{\rho_2}{\alpha_2} < 1,$ $\frac{\alpha_1}{2} + \alpha_2 + \frac{\alpha_3}{2} + 2\tau_1 + 3\tau_2 > 0,$ $\frac{1}{2\alpha_3} > 0.$

From inequality (14) it follows that the vector functions:

$$S^{\frac{3}{2}}(x,u(x))u(x), S^{\frac{1}{2}}(x,u(x))\Delta^2 u, S^{\frac{1}{2}}(x,u(x))\frac{\partial^2 u}{\partial {x_i}^2} \in L_2(\mathbb{R}^n)^l,\ \ i=1,\dots,n.$$

The proof is complete.

4 Proof of Theorem 2

Employing the identity:

$$(f,\Psi_\varepsilon Vu) = (\Delta^3 u, \Psi_\varepsilon Vu) + (V(x,u)u, \Psi_\varepsilon Vu).$$

And

$$\frac{\partial(\Psi_\varepsilon Tu)}{\partial x_i} = \frac{\partial \Psi_\varepsilon}{\partial x_i}Tu + \Psi_\varepsilon \frac{\partial T}{\partial x_i}u + \sum_{p=1}^{l}\Psi_\varepsilon\, Re\frac{\partial u_p}{\partial x_i}\frac{\partial T}{\partial \zeta_p}u + \sum_{p=1}^{l}\Psi_\varepsilon Im\frac{\partial u_p}{\partial x_i}\frac{\partial T}{\partial \eta_p}u + \Psi_\varepsilon T\frac{\partial u}{\partial x_i},$$

where $T(x,g)=V(x,g)$, hence after some transformations we have

$$(f,\Psi_\varepsilon Vu) = E_1^\varepsilon(u) + 2E_2^\varepsilon(u) + 2E_3^\varepsilon(u) + E_4^\varepsilon(u) + 2E_5^\varepsilon(u) + E_6^\varepsilon(u) + 2E_7^\varepsilon(u) + 2E_8^\varepsilon(u) + 2E_9^\varepsilon(u) + E_{10}^\varepsilon(u) + (Vu,\Psi_\varepsilon Vu). \quad (15)$$

Where:

$$E_1^\varepsilon(u) = (\Delta^2 u, \sum_{i=1}^{n}\frac{\partial^2\Psi_\varepsilon}{\partial {x_i}^2}Tu),\ \ E_2^\varepsilon(u) = (\Delta^2 u, \sum_{i=1}^{n}\frac{\partial\Psi_\varepsilon}{\partial x_i}\frac{\partial T}{\partial x_i}u),$$

$$E_3^\varepsilon(u) = (\Delta^2 u, \sum_{i=1}^{n}\frac{\partial\Psi_\varepsilon}{\partial x_i}T\frac{\partial u}{\partial x_i}),\ \ E_4^\varepsilon(u) = (\Delta^2 u, \sum_{i=1}^{n}\Psi_\varepsilon\frac{\partial^2 T}{\partial {x_i}^2}u),$$

$$E_5^\varepsilon(u) = (\Delta^2 u, \sum_{i=1}^{n}\Psi_\varepsilon\frac{\partial T}{\partial x_i}\frac{\partial u}{\partial x_i}),\ \ E_6^\varepsilon(u) = (\Delta^2 u, \sum_{i=1}^{n}\Psi_\varepsilon T\frac{\partial^2 u}{\partial {x_i}^2}),$$

$$E_7^\varepsilon(u) = (\Delta^2 u, \sum_{i=1}^{n}\frac{\partial\Psi_\varepsilon}{\partial x_i}\sum_{p=1}^{l}(Re\frac{\partial u_p}{\partial x_i}\frac{\partial T}{\partial \zeta_p} + Im\frac{\partial u_p}{\partial x_i}\frac{\partial T}{\partial \eta_p})\,u),$$

$$E_8^\varepsilon(u) = (\Delta^2 u, \sum_{i=1}^{n}\Psi_\varepsilon\sum_{p=1}^{l}(Re\frac{\partial u_p}{\partial x_i}\frac{\partial^2 T}{\partial x_i\partial \zeta_p} + Im\frac{\partial u_p}{\partial x_i}\frac{\partial^2 T}{\partial x_i\partial \eta_p})\,u),$$

$$E_9^\varepsilon(u) = (\Delta^2 u, \sum_{i=1}^{n}\Psi_\varepsilon\sum_{p=1}^{l}(Re\frac{\partial u_p}{\partial x_i}\frac{\partial T}{\partial \zeta_p} + Im\frac{\partial u_p}{\partial x_i}\frac{\partial T}{\partial \eta_p})\frac{\partial u}{\partial x_i}),$$

$$E_{10}^\varepsilon(u) = (\Delta^2 u, \sum_{i=1}^{n}\Psi_\varepsilon\sum_{p=1}^{l}(Re\frac{\partial^2 u_p}{\partial {x_i}^2}\frac{\partial T}{\partial \zeta_p} + Im\frac{\partial^2 u_p}{\partial {x_i}^2}\frac{\partial T}{\partial \eta_p})\,u).$$

The functionals:
$E_1^\varepsilon(u), E_2^\varepsilon(u), E_3^\varepsilon(u), E_7^\varepsilon(u)$ tend to zero as $\varepsilon \to 0$.

And for the other functionals, we obtain the following estimates:

$$|E_4^\varepsilon(u)| \le \frac{\alpha_4}{2}(\Delta^2 u, \Psi_\varepsilon T\Delta^2 u) + \frac{\rho_3}{2\alpha_4}(Vu,\Psi_\varepsilon Vu),$$

$$|E_5^\varepsilon(u)| \le \frac{\alpha_5}{2}(\Delta^2 u, \Psi_\varepsilon T\Delta^2 u) + \frac{\rho_4}{2\alpha_5}(Vu,\Psi_\varepsilon Vu),$$

$$|E_6^\varepsilon(u)| \le \frac{\alpha_6}{2}(\Delta^2 u, \Psi_\varepsilon T\Delta^2 u) + \frac{1}{2\alpha_6}\sum_{i=1}^{n}(\frac{\partial^2 u}{\partial {x_i}^2}, \Psi_\varepsilon T\frac{\partial^2 u}{\partial {x_i}^2}),$$

$$|E_8^\varepsilon(u)| \le \tau_3(\Delta^2 u, \Psi_\varepsilon T\Delta^2 u),\quad |E_9^\varepsilon(u)| \le \tau_4(\Delta^2 u, \Psi_\varepsilon T\Delta^2 u),\ \ |E_{10}^\varepsilon(u)| \le \tau_4(\Delta^2 u, \Psi_\varepsilon T\Delta^2 u).$$

Where ρ_3, ρ_4, τ_3 and τ_4 are the constants in conditions (6)-(10) and $\alpha_4, \alpha_5, \alpha_6$ are arbitrary positive numbers, we employed inequality (9) in estimating the functionals $E_9^\varepsilon(u)$ and $E_{10}^\varepsilon(u)$: in the case

$$g = (g_1, g_2, \dots, g_l) = u(x) = (u_1(x), u_2(x), \dots, u_l(x)),$$

and

$$g = (g_1, g_2, \dots, g_l) = \frac{\partial u}{\partial x_i} = (\frac{\partial u_1}{\partial x_i}, \frac{\partial u_2}{\partial x_i}, \dots, \frac{\partial u_l}{\partial x_i}).$$

Based on the obtained estimates, by identity (15) we get:

$$|f,\Psi_\varepsilon Vu| \ge \frac{\alpha_4}{2}(\Delta^2 u, \Psi_\varepsilon V\Delta^2 u) - \frac{\rho_3}{2\alpha_4}(Vu,\Psi_\varepsilon Vu) + \alpha_5(\Delta^2 u, \Psi_\varepsilon V\Delta^2 u) - \frac{\rho_4}{\alpha_5}(Vu,\Psi_\varepsilon Vu) + \frac{\alpha_6}{2}(\Delta^2 u, \Psi_\varepsilon V\Delta^2 u) + \frac{1}{2\alpha_6}\sum_{i=1}^{n}(\frac{\partial^2 u}{\partial {x_i}^2}, \Psi_\varepsilon V\frac{\partial^2 u}{\partial {x_i}^2}) + 2\tau_3(\Delta^2 u, \Psi_\varepsilon V\Delta^2 u) + 2\tau_4(\Delta^2 u, \Psi_\varepsilon V\Delta^2 u) + (Vu,\Psi_\varepsilon Vu).$$

Passing to the limit as $\varepsilon \to 0$, we have:

$$|(f,Vu)| \ge [1 - \frac{\rho_3}{2\alpha_4} - \frac{\rho_4}{\alpha_5}](Vu,Vu) + [\frac{\alpha_4}{2} + \alpha_5 + \frac{\alpha_6}{2} + 2\tau_3 + 3\tau_4](\Delta^2 u, V\Delta^2 u) + \frac{1}{2\alpha_6}\sum_{i=1}^{n}(\frac{\partial^2 u}{\partial {x_i}^2}, V\frac{\partial^2 u}{\partial {x_i}^2}).$$

Applying Cauchy-Schwarz inequality, then we obtain the inequality:

$$\left\|f; L_2(\mathbb{R}^n)^l\right\|\left\|Vu; L_2(\mathbb{R}^n)^l\right\| \ge |f,Vu| \ge [1 - \frac{\rho_3}{2\alpha_4} - \frac{\rho_4}{\alpha_5}](Vu,Vu) + [\frac{\alpha_4}{2} + \alpha_5 + \frac{\alpha_6}{2} + 2\tau_3 + 3\tau_4](\Delta^2 u, V\Delta^2 u) + \frac{1}{2\alpha_6}\sum_{i=1}^{n}(\frac{\partial^2 u}{\partial {x_i}^2}, V\frac{\partial^2 u}{\partial {x_i}^2}). \quad (16)$$

Choose $\alpha_4, \alpha_5, \alpha_6$ satisfy the conditions:

$$\frac{\rho_3}{2\alpha_4} + \frac{\rho_4}{\alpha_5} < 1,\ \frac{\alpha_4}{2} + \alpha_5 + \frac{\alpha_6}{2} + 2\tau_3 + 3\tau_4 > 0,\ \ \frac{1}{2\alpha_6} > 0.$$

Equation (16) can be written as:

$$[1-(\frac{\rho_3}{2\alpha_4}+\frac{\rho_4}{\alpha_5})]\|Vu\|^2+[\frac{\alpha_4}{2}+\alpha_5+\frac{\alpha_6}{2}+2\tau_3+3\tau_4]\|V^{\frac{1}{2}}\Delta^2u\|^2+\frac{1}{2\alpha_6}\sum_{i=1}^{n}\|V^{\frac{1}{2}}\frac{\partial^2u}{\partial x_i{}^2}\|^2\le\|f\|\|Vu\|.$$

Then

$$\|Vu\|\le[1-(\tfrac{\rho_3}{2\alpha_4}+\tfrac{\rho_4}{\alpha_5})]^{-1}\|f\|,$$

$$\left\|V^{\frac{1}{2}}\Delta^2u\right\|\le\left[\frac{\alpha_4}{2}+\alpha_5+\frac{\alpha_6}{2}+2\tau_3+3\tau_4\right]^{-\frac{1}{2}}\left[1-\left(\frac{\rho_3}{2\alpha_4}+\frac{\rho_4}{\alpha_5}\right)\right]^{-\frac{1}{2}}\|f\|,$$

and

$$\sum_{i=1}^{n}\|V^{\frac{1}{2}}\frac{\partial^2u}{\partial x_i{}^2}\|\le n[\tfrac{1}{2\alpha_6}]^{-\frac{1}{2}}\left[1-\left(\tfrac{\rho_3}{2\alpha_4}+\tfrac{\rho_4}{\alpha_5}\right)\right]^{-\frac{1}{2}}\|f\|,$$

on the other hand, we have:

$$f=\Delta^3u+Vu\Rightarrow\|\Delta^3u\|\le\|f\|+\|Vu\|,$$

then

$$\|\Delta^3u\|\le[1+\left[1-\left(\tfrac{\rho_3}{2\alpha_4}+\tfrac{\rho_4}{\alpha_5}\right)\right]^{-1}]\|f\|.$$

hence

$$\|\Delta^3u\|+\|Vu\|+\|V^{\frac{1}{2}}\Delta^2u\|+\sum_{i=1}^{n}\|V^{\frac{1}{2}}\frac{\partial^2u}{\partial x_i{}^2}\|\le N\|f\|,$$

where

$$N=1+2\left[1-\left(\tfrac{\rho_3}{2\alpha_4}+\tfrac{\rho_4}{\alpha_5}\right)\right]^{-1}+[\left[\tfrac{\alpha_4}{2}+\alpha_5+\tfrac{\alpha_6}{2}+2\tau_3+3\tau_4\right]^{-\frac{1}{2}}+n[\tfrac{1}{2\alpha_6}]^{-\frac{1}{2}}]\left[1-\left(\tfrac{\rho_3}{2\alpha_4}+\tfrac{\rho_4}{\alpha_5}\right)\right]^{-\frac{1}{2}},$$

where N is a constant independent on $u(x), f(x)$.

Now the coercive inequality (11) is valid. The proof of theorem 2 is complete.

5 Conclusions

The purpose of this research is to study the separation problem for the non-linear tri-harmonic operator $A[u]=\Delta^3u(x)+V(x,u(x))u(x), x\in\mathbb{R}^n$, under certain conditions and a specific space $L_2(\mathbb{R}^n)^l$, also to deduce the compounds of the operator at the same space. We got the coercive inequality $\|\Delta^3u\|+\|Vu\|+\|V^{\frac{1}{2}}\Delta^2u\|+\sum_{i=1}^{n}\|V^{\frac{1}{2}}\frac{\partial^2u}{\partial x_i{}^2}\|\le N\|A[u]\|$, which is rewarded the separable of the operator A.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The authors equally contributed in the present research, at all stages from the formulation of the problem to the final findings and solution.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

No funding was received for conducting this study.

Conflict of Interest

The authors have no conflicts of interest to declare.