Square mean (ω, c)-periodic Limit Stochastic Processes and Some Basic Results

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Abstract: This paper aims to introduce new classes of square mean (ω, c)-periodic limit-type stochastic processes. The main structural characterizations and some qualitative properties of the introduced classes are established, as well as their invariance under the actions of convolution product and the composition principle.

Key-Words: - Stochastic process, Square mean asymptotic (ω, c)-periodicity, Square mean (ω, c)-periodic limit, Convolution product, Composition principle.

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1 Introduction and Preliminaries

As is well known, the class of Bloch periodic functions, which extends the classes of periodic functions and anti-periodic functions plays a fundamental role in mathematics, quantum mechanics, and solid-state physics. For more details about this class of functions, [1], [2], [3] and references cited therein.

In [4], the authors introduced the class of semi-periodic continuous functions and showed that it is equivalent to the class of limit periodic functions considered in [5].

In a recent paper by [6], the class $P_\omega L(\mathbb{R}_+ : E)$ of ω-periodic limit functions, generalizing the space of asymptotically periodic functions, was introduced and studied. Let E be a complex Banach space and $\omega > 0$. A bounded and continuous function $f : \mathbb{R}_+ \longrightarrow E$ is said to be ω-periodic limit if $g(t) = \lim_{n \to +\infty} f(t + n\omega)$ is well defined for each $t \in \mathbb{R}_+$, where $n \in \mathbb{N}$.

The notion of Bloch periodicity has recently been reconsidered by [7], who introduced a new class $P_{\omega,c}(\mathbb{R} : E)$ of (ω, c)-periodic functions (see also [8] and [9]). Let E be a complex Banach space, $\omega > 0$ and $c \in \mathbb{C}\backslash\{0\}$. Using the principal branch of the complex Logarithm, we define $c^{\frac{t}{\omega}} := \exp((\frac{t}{\omega})\log(c))$ and we use the following notations $c^\wedge(t) := c^{\frac{t}{\omega}}$ and $|c|^\wedge(t) := |c^\wedge(t)| = \exp(\frac{t}{\omega}\log(|c|)$. A continuous function $f : \mathbb{R} \longrightarrow E$ is said to be (ω, c)-periodic if and only if $f(t + \omega) = cf(t)$,for all $t \in \mathbb{R}$.

Throughout this paper, we consider a real separable Hilbert space $(H, \|\cdot\|)$ and a filtered complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. Denote by $L^2(\mathbb{P}, H)$ the space of all strongly measurable square integrable H-valued random variables such that $\mathbb{E}\|X\|^2 = \int_\Omega \|X\|^2 d\mathbb{P} < \infty$. It is easy to check that $L^2(\mathbb{P}, H)$ is a Banach space when it is equipped with its natural norm $\|\cdot\|_2$ given by $\|X\|_2 = \left(\mathbb{E}\|X\|^2\right)^{1/2}, \forall X \in L^2(\mathbb{P}, H)$. By $\mathcal{L}(H)$ we denote the space of all bounded linear operators on H which with the usual operator norm $\|\cdot\|_{\mathcal{L}(H)}$ is a Banach space. We recall that a stochastic process $X : \mathbb{R}_+ \longrightarrow L^2(\mathbb{P}, H)$ is continuous (resp. bounded) if for every $s \in \mathbb{R}_+$, $\lim_{t\to s}\mathbb{E}\|X(t) - X(s)\|^2 = 0$ (resp. $\sup_{t\in\mathbb{R}_+}\mathbb{E}\|X(t)\|^2 < \infty$). By $\mathcal{C}\left(\mathbb{R}_+, L^2(\mathbb{P}, H)\right)$ (resp. $\mathcal{CB}\left(\mathbb{R}_+, L^2(\mathbb{P}, H)\right)$) we denote the complex Banach space of all continuous (resp. bounded and continuous) stochastic processes from $\mathbb{R}_+$ into $L^2(\mathbb{P}, H)$, equipped with the norm

$$\|X\|_{2,\infty} = \left(\sup_{t\in\mathbb{R}_+}\left(\mathbb{E}\|X(t)\|^2\right)\right)^{1/2}.$$

A new class of square mean ω-periodic limit stochastic processes, denoted by $P_\omega L\left(\mathbb{R}_+, L^2(\mathbb{P}, H)\right)$ has recently been introduced and investigated by [10], [11]. In [10], the authors proposed the following notion: A stochastic process $X \in \mathcal{CB}\left(\mathbb{R}_+, L^2(\mathbb{P}, H)\right)$ is said to be square mean ω-periodic limit, if there exists $\omega > 0$ such that $\lim_{n\to+\infty}\mathbb{E}\left\|X(t + n\omega) - \widetilde{X}(t)\right\|^2 = 0$, is well defined in $L^2(\mathbb{P}, H)$ for each $t \geq 0$ and for some stochastic process $\widetilde{X} : \mathbb{R}_+ \longrightarrow L^2(\mathbb{P}, H)$. For all $t \geq 0$, the stochastic process $\widetilde{X}$ is called the ω-periodic limit of $X(t + n\omega)$ in $L^2(\mathbb{P}, H)$.

The main purpose of this paper is to extend the above-mentioned research by introducing two new classes of square mean (ω, c)-periodic limit stochastic processes and asymptotically (ω, c)-periodic stochastic processes, thereby expanding their applications to certain differential equations that model various phenomena in fundamental and applied sciences. The paper establishes the main structural characterizations and some qualitative properties of these classes. Furthermore, we will investigate the invariance under the operations of infinite convolution product and the composition principle for these classes of square mean (ω, c)-periodic limit-type stochastic process.

The paper is organized as follows: In Section 2, we introduce the space of square mean (ω, c)-periodic limit processes and the space of square mean asymptotically (ω, c)-periodic processes, and study their fundamental properties, including several basic results. In the remainder of this section, we will state and prove the relationship between the introduced process classes.

Section 3 analyzes the invariance under the actions of the convolution product and examines the composition principle for square mean (ω, c)-periodic limit stochastic processes. The final section of the paper is reserved for the conclusion and some perspectives on the introduced concepts and the obtained results.

2 Square mean (ω, c)-periodic Limit Process

Let $c \in \mathbb{C}\backslash\{0\}$ and $\omega > 0$. We start this section by introducing the following space of (ω, c)-bounded and continuous stochastic processes:

$$\mathcal{CB}_{\omega,c}\left(\mathbb{R}_+, L^2(\mathbb{P}, H)\right) := \left\{ \begin{array}{c} f \in \mathcal{C}\left(\mathbb{R}_+, L^2(\mathbb{P}, H)\right) \\ : \sup\limits_{t\in\mathbb{R}_+} \left(\mathbb{E}\left\|c^\wedge(-t) f(t)\right\|^2\right) < \infty \end{array} \right\}.$$

Proposition 1 *Let $X \in \mathcal{C}\left(\mathbb{R}_+, L^2(\mathbb{P}, H)\right)$. Then $X(\cdot) \in \mathcal{CB}_{\omega,c}\left(\mathbb{R}_+, L^2(\mathbb{P}, H)\right)$ if and only if $c^\wedge(-\cdot)X(\cdot) \in \mathcal{CB}\left(\mathbb{R}_+, L^2(\mathbb{P}, H)\right)$.*

Proof. $(\Longrightarrow)$: Let $X \in \mathcal{CB}_{\omega,c}\left(\mathbb{R}_+, L^2(\mathbb{P}, H)\right)$, by defining $Y(\cdot) = c^\wedge(-\cdot)X(\cdot)$, we have

$$\|Y\|_{2,\infty}^2 = \sup_{t\in\mathbb{R}_+} \mathbb{E}\left\|c^\wedge(-t) X(t)\right\|^2 < \infty,$$

which shows that $Y(\cdot) \in \mathcal{CB}\left(\mathbb{R}_+, L^2(\mathbb{P}, H)\right)$.

$(\Longleftarrow)$: If $c^\wedge(-\cdot)X(\cdot) \in \mathcal{CB}\left(\mathbb{R}_+, L^2(\mathbb{P}, H)\right)$, there exists $Y \in \mathcal{CB}\left(\mathbb{R}_+, L^2(\mathbb{P}, H)\right)$ such that $X(\cdot) = c^\wedge(\cdot)Y(\cdot)$, hence $X \in \mathcal{CB}_{\omega,c}\left(\mathbb{R}_+, L^2(\mathbb{P}, H)\right)$. ■

Theorem 1 *$\mathcal{CB}_{\omega,c}\left(\mathbb{R}_+, L^2(\mathbb{P}, H)\right)$ equipped with the norm*

$$\|X\|_{2,\omega,c} := \left(\sup_{t\in\mathbb{R}_+}\left(\mathbb{E}\left\|c^\wedge(-t)X(t)\right\|^2\right)\right)^{1/2}$$

is a Banach space.

Proof. Let $(X_n)_n$ be a Cauchy sequence of $\mathcal{CB}_{\omega,c}\left(\mathbb{R}_+, L^2(\mathbb{P}, H)\right)$. From Proposition 1, we can write $X_n(\cdot) = c^\wedge(\cdot)Y_n(\cdot)$ where $(Y_n)_n$ is a sequence of $\mathcal{CB}\left(\mathbb{R}_+, L^2(\mathbb{P}, H)\right)$. For all $m, n \in \mathbb{Z}_+$ we have $\|Y_n - Y_m\|_{2,\infty} = \left\|c^\wedge(-\cdot)\left(Y_n(\cdot) - Y_m(\cdot)\right)\right\|_{2,\infty} = \|X_n - X_m\|_{2,\omega,c}$, which shows that $(Y_n)_n$ is also a Cauchy sequence in the Banach space $\mathcal{CB}\left(\mathbb{R}_+, L^2(\mathbb{P}, H)\right)$. Thus, there exists $Y \in \mathcal{CB}\left(\mathbb{R}_+, L^2(\mathbb{P}, H)\right)$ such that $\lim\limits_{n\to+\infty}\|Y_n - Y\|_{2,\infty} = 0$. Let $X(\cdot) = c^\wedge(\cdot)Y(\cdot)$, then by Proposition 1, we get that $Y \in \mathcal{CB}\left(\mathbb{R}_+, L^2(\mathbb{P}, H)\right)$ and

$$\begin{aligned} &\lim_{n\to+\infty}\|X_n - X\|_{2,\omega,c}^2 \\ = &\lim_{n\to+\infty}\sup_{t\in\mathbb{R}_+}\left(\mathbb{E}\left\|c^\wedge(-t)\left(X_n(t) - X(t)\right)\right\|^2\right) \\ = &\lim_{n\to+\infty}\|Y_n - Y\|_{2,\infty}^2 = 0, \end{aligned}$$

which shows that $\lim\limits_{n\to+\infty} X_n(t) = X(t) = c^\wedge(t)Y(t)$ in $\mathcal{CB}\left(\mathbb{R}_+, L^2(\mathbb{P}, H)\right)$ with respect to $\|\cdot\|_{2,\omega,c}$. Thus $\left(\mathcal{CB}_{\omega,c}\left(\mathbb{R}_+, L^2(\mathbb{P}, H)\right), \|\cdot\|_{2,\omega,c}\right)$ is complete. ■

We are concerned with the following concepts:

Definition 1 *A stochastic process $X \in \mathcal{CB}_{\omega,c}\left(\mathbb{R}_+, L^2(\mathbb{P}, H)\right)$ is said to be square mean (ω, c)-periodic, if there exists $\omega > 0$ such that*

$$\mathbb{E}\left\|c^\wedge(-t)\left(X(t+\omega) - cX(t)\right)\right\|^2 = 0.$$

ω is called the c-period of X. The collection of those square mean (ω, c)-periodic stochastic processes with the same c-period ω will be denoted by $P_{\omega,c}\left(\mathbb{R}_+, L^2(\mathbb{P}, H)\right)$.

When $c = 1$, we write $P_\omega\left(\mathbb{R}_+, L^2(\mathbb{P}, H)\right)$ in spite of $P_{\omega,1}\left(\mathbb{R}_+, L^2(\mathbb{P}, H)\right)$, in this case a such process in $P_\omega\left(\mathbb{R}_+, L^2(\mathbb{P}, H)\right)$ is called square mean ω-periodic stochastic.

The following proposition provides a characterization of the square mean (ω, c)-periodic stochastic processes.

Proposition 2 *Let $X \in \mathcal{CB}_{\omega,c}\left(\mathbb{R}_+, L^2(\mathbb{P}, H)\right)$. Then $X \in P_{\omega,c}\left(\mathbb{R}_+, L^2(\mathbb{P}, H)\right)$ if and only if $X(\cdot) = c^\wedge(\cdot)Y(\cdot)$, where $Y \in P_\omega\left(\mathbb{R}_+, L^2(\mathbb{P}, H)\right)$.*

Proof. $(\Longrightarrow)$: Let $X \in P_{\omega,c}\left(\mathbb{R}_+, L^2(\mathbb{P},H)\right)$, then by defining $Y(t) = c^\wedge(-t)X(t)$ for $t \in \mathbb{R}_+$, we have

$$\begin{aligned}
& \mathbb{E}\left\|Y(t+\omega) - Y(t)\right\|^2 \\
= & \mathbb{E}\left\|c^\wedge(-t-\omega)X(t+\omega) - c^\wedge(-t)X(t)\right\|^2 \\
= & \mathbb{E}\left\|c^\wedge(-t)\left(c^{-1}X(t+\omega) - X(t)\right)\right\|^2 \\
= & |c|^{-2}\,\mathbb{E}\left\|c^\wedge(-t)\left(X(t+\omega) - cX(t)\right)\right\|^2 = 0,
\end{aligned}$$

hence $Y \in P_\omega\left(\mathbb{R}_+, L^2(\mathbb{P},H)\right)$ and $X(\cdot) = c^\wedge(\cdot)Y(\cdot)$.

$(\Longleftarrow)$: If $X(\cdot) = c^\wedge(\cdot)Y(\cdot)$, where $Y \in P_\omega\left(\mathbb{R}_+, L^2(\mathbb{P},H)\right)$ then $\forall t \in \mathbb{R}_+$ we have

$$\begin{aligned}
& \mathbb{E}\left\|c^\wedge(-t)\left(X(t+\omega) - cX(t)\right)\right\|^2 \\
= & \mathbb{E}\left\|c^\wedge(-t)\left(c^\wedge(t+\omega)Y(t+\omega) - cc^\wedge(t)Y(t)\right)\right\|^2 \\
= & \mathbb{E}\left\|cY(t+\omega) - cY(t)\right\|^2 \\
= & |c|^2\,\mathbb{E}\left\|Y(t+\omega) - Y(t)\right\|^2 = 0,
\end{aligned}$$

which shows that $X \in P_{\omega,c}\left(\mathbb{R}_+, L^2(\mathbb{P},H)\right)$. ■

Definition 2 *Let $c \in \mathbb{C}\backslash\{0\}$. A stochastic process $X \in \mathcal{CB}_{\omega,c}\left(\mathbb{R}_+, L^2(\mathbb{P},H)\right)$ is said to be square mean (ω,c)-periodic limit, if there exists $\omega > 0$ such that*

$$\lim_{n\to+\infty}\mathbb{E}\left\|c^\wedge(-t)\left(c^{-n}X(t+n\omega) - \widetilde{X}(t)\right)\right\|^2 = 0$$

is well defined in $L^2(\mathbb{P},H)$ for each $t \geq 0$ and for some stochastic process $\widetilde{X} : \mathbb{R}_+ \longrightarrow L^2(\mathbb{P},H)$.

For all $t \geq 0$, the stochastic process $\widetilde{X}$ is called the (ω,c)-periodic limit of $c^{-n}X(t+n\omega)$ in $L^2(\mathbb{P},H)$. The collection of such square mean (ω,c)-periodic limit stochastic processes is denoted by $P_{\omega,c}L\left(\mathbb{R}_+, L^2(\mathbb{P},H)\right)$.

Remark 1

1. *The process $\widetilde{X}(\cdot)$ is measurable and not necessarily continuous.*
2. *When $c = 1$ (resp. $c = -1$) and $\omega > 0$ arbitrary, the class $P_{\omega,c}L\left(\mathbb{R}_+, L^2(\mathbb{P},H)\right)$ reduces to the class $P_\omega L\left(\mathbb{R}_+, L^2(\mathbb{P},H)\right)$ (resp. $P_{\omega,-1}L\left(\mathbb{R}_+, L^2(\mathbb{P},H)\right)$) of square mean ω-periodic (resp. ω-antiperiodic) limit processes.*

A characterization of square mean (ω,c)-periodic limit stochastic processes is given by the following.

Proposition 3 *Let $X \in \mathcal{CB}_{\omega,c}\left(\mathbb{R}_+, L^2(\mathbb{P},H)\right)$. Then $X \in P_{\omega,c}L\left(\mathbb{R}_+, L^2(\mathbb{P},H)\right)$ if and only if $X(\cdot) = c^\wedge(\cdot)Y(\cdot)$, where $Y \in P_\omega L\left(\mathbb{R}_+, L^2(\mathbb{P},H)\right)$.*

Proof. $(\Longrightarrow)$: Let $X \in P_{\omega,c}L\left(\mathbb{R}_+, L^2(\mathbb{P},H)\right)$, then by defining $Y(t) = c^\wedge(-t)X(t)$ for $t \in \mathbb{R}_+$, we have

$$\begin{aligned}
& \lim_{n\to+\infty}\mathbb{E}\left\|Y(t+n\omega) - Y(t)\right\|^2 \\
= & \lim_{n\to+\infty}\mathbb{E}\left\|c^\wedge(-(t+n\omega))X(t+n\omega) - c^\wedge(-t)X(t)\right\|^2 \\
= & \lim_{n\to+\infty}\mathbb{E}\left\|c^\wedge(-t)\left(c^{-n}X(t+n\omega) - X(t)\right)\right\|^2 \\
= & 0,
\end{aligned}$$

thus $Y \in P_\omega L\left(\mathbb{R}_+, L^2(\mathbb{P},H)\right)$ and $X(\cdot) = c^\wedge(\cdot)Y(\cdot)$.

$(\Longleftarrow)$: If $X(\cdot) = c^\wedge(\cdot)Y(\cdot)$, where $Y \in P_\omega L\left(\mathbb{R}_+, L^2(\mathbb{P},H)\right)$ then there exists $\omega > 0$ such that $\lim_{n\to+\infty}\mathbb{E}\left\|Y(t+n\omega) - \widetilde{Y}(t)\right\|^2 = 0$, is well defined in $L^2(\mathbb{P},H)$ for each $t \geq 0$ and for some stochastic process $\widetilde{Y} : \mathbb{R}_+ \longrightarrow L^2(\mathbb{P},H)$. Therefore, $\forall t \in \mathbb{R}_+$ we have

$$\begin{aligned}
& \lim_{n\to+\infty}\mathbb{E}\left\|c^\wedge(-t)\left(c^{-n}X(t+n\omega) - \widetilde{X}(t)\right)\right\|^2 \\
= & \lim_{n\to+\infty}\mathbb{E}\left\|c^\wedge(-t)\begin{pmatrix} c^{-n}c^\wedge(t+n\omega)Y(t+n\omega) \\ -c^\wedge(t)\widetilde{Y}(t)\end{pmatrix}\right\|^2 \\
= & \lim_{n\to+\infty}\mathbb{E}\left\|Y(t+n\omega) - \widetilde{Y}(t)\right\|^2 \\
= & 0,
\end{aligned}$$

hence $X \in P_{\omega,c}L\left(\mathbb{R}_+, L^2(\mathbb{P},H)\right)$. ■

Example 1 *Consider the process*

$$X(t) = 2^{\frac{t}{2\pi}}\left(\frac{1}{n}Z(t) + \sin t\right),\ n \geq 1,$$

where $Z(t)$ is a sequence of independent and identically distributed random variables such that $Z(t) \sim \mathcal{N}\left(0,\sigma^2\right)$, i.e., normally distributed with mean 0 and variance σ^2. We will show that X is a square mean $(2\pi,2)$-periodic limit process. To establish this, it suffices, using Proposition 3, to show that $Y(t) = 2^{-\frac{t}{2\pi}}X(t)$ is a square mean 2π-periodic limit process. Define $\widetilde{Y}(t) = \sin t$. Then we have

$$Y(t+n2\pi) = \frac{1}{n}Z(t+n2\pi) + \sin t.$$

Therefore,

$$\mathbb{E}\left\|Y(t+n2\pi) - \widetilde{Y}(t)\right\|^2 = \frac{1}{n^2}\mathbb{E}\left\|Z(t+n2\pi)\right\|^2.$$

Since $Z(t)$ are independent and identically distributed random variables, $Z(t+n2\pi)$ are *also independent and identically distributed random variables with the same distribution. Hence*

$$\mathbb{E}\left\|Z\left(t\right)\right\|^{2}=\mathbb{E}\left\|Z\left(t+n2\pi\right)\right\|^{2}=\sigma^{2}.$$

Thus

$$\lim_{n\to+\infty}\mathbb{E}\left\|Y\left(t+nn2\pi\right)-\widetilde{Y}\left(t\right)\right\|^{2}=\lim_{n\to+\infty}\frac{\sigma^{2}}{n^{2}}=0,$$

which shows that Y is a square mean 2π-periodic limit process.

Some properties of square mean (ω,c)-periodic limit stochastic processes are summarized in the following extension of [[10], Theorem 2.4], which is essential for our analysis:

Proposition 4 *We have*

1. $P_{\omega,c}L\left(\mathbb{R}_{+},L^{2}\left(\mathbb{P},H\right)\right)$ *is a linear subspace of* $\mathcal{CB}_{\omega,c}\left(\mathbb{R}_{+},L^{2}\left(\mathbb{P},H\right)\right)$.
2. *If* $X\in P_{\omega,c}L\left(\mathbb{R}_{+},L^{2}\left(\mathbb{P},H\right)\right)$, *then* $\widetilde{X}\in\mathcal{CB}_{\omega,c}\left(\mathbb{R}_{+},L^{2}\left(\mathbb{P},H\right)\right)$ *and* $\exists C>0$ *such that*

$$\left\|\widetilde{X}\right\|_{2,\omega,c}\leq\left\|X\right\|_{2,\omega,c}\leq C.$$

3. *If* $X\in P_{\omega,c}L\left(\mathbb{R}_{+},L^{2}\left(\mathbb{P},H\right)\right)$ *then* $\widetilde{X}\in P_{\omega,c}\left(\mathbb{R}_{+},L^{2}\left(\mathbb{P},H\right)\right)$.
4. *If* $X\in P_{\omega,c}L\left(\mathbb{R}_{+},L^{2}\left(\mathbb{P},H\right)\right)$, *then* $X_{s}\left(\cdot\right)=X\left(\cdot+s\right)\in P_{\omega,c}L\left(\mathbb{R}_{+},L^{2}\left(\mathbb{P},H\right)\right)$, $s\in\mathbb{R}_{+}$.

Proof. The proof of 1 is straightforward. We will only prove 2, 3 and 4. For property 2, the first inequality follows from the fact that

$$\begin{aligned}&\mathbb{E}\left\|c^{\wedge}(-t)\widetilde{X}\left(t\right)\right\|^{2}\\ \leq\;&2\mathbb{E}\left\|c^{\wedge}(-t)\left(\widetilde{X}\left(t\right)-c^{-n}X\left(t+n\omega\right)\right)\right\|^{2}\\ &+2\mathbb{E}\left\|c^{\wedge}(-\left(t+n\omega\right))X\left(t+n\omega\right)\right\|^{2},\end{aligned}$$

and the second inequality is an immediate consequence of the definition and the fact that $P_{\omega,c}L\left(\mathbb{R}_{+},L^{2}\left(\mathbb{P},H\right)\right)\subset\mathcal{CB}_{\omega,c}\left(\mathbb{R}_{+},L^{2}\left(\mathbb{P},H\right)\right)$. Let us show assertion 3, for this suppose that $X\in P_{\omega,c}L\left(\mathbb{R}_{+},L^{2}\left(\mathbb{P},H\right)\right)$, then $\forall\varepsilon>0$, $\exists N\geq0$ sufficiently large, such that $\forall n\geq N$ we have
$\mathbb{E}\left\|c^{\wedge}(-t)\left(c^{-n-1}X\left(t+\left(n+1\right)\omega\right)-\widetilde{X}\left(t\right)\right)\right\|^{2}\leq\frac{\varepsilon}{4|c|^{2}}$,
and
$\mathbb{E}\left\|c^{\wedge}(-\left(t+\omega\right))\left(\begin{array}{c}c^{-n}X\left(t+\left(n+1\right)\omega\right)\\-\widetilde{X}\left(t+\omega\right)\end{array}\right)\right\|^{2}\leq\frac{\varepsilon}{4|c|^{2}}$,

where $\widetilde{X}\left(\cdot\right)$ is the (ω,c)-periodic limit of $c^{-n}X\left(\cdot+n\omega\right)$ in $L^{2}\left(\mathbb{P},H\right)$ when $n\longrightarrow+\infty$. Therefore

$$\begin{aligned}&\mathbb{E}\left\|c^{\wedge}(-t)\left(\widetilde{X}\left(t+\omega\right)-c\widetilde{X}\left(t\right)\right)\right\|^{2}\\ =\;&\mathbb{E}\left\|c^{\wedge}(-t)\widetilde{X}\left(t+\omega\right)-cc^{\wedge}(-t)\widetilde{X}\left(t\right)\right\|^{2}\\ =\;&\mathbb{E}\left\|cc^{\wedge}(-\left(t+\omega\right))\widetilde{X}\left(t+\omega\right)-cc^{\wedge}(-t)\widetilde{X}\left(t\right)\right\|^{2}\\ =\;&\left|c\right|^{2}\mathbb{E}\left\|c^{\wedge}(-\left(t+\omega\right))\widetilde{X}\left(t+\omega\right)-c^{\wedge}(-t)\widetilde{X}\left(t\right)\right\|^{2}\\ \leq\;&2\left|c\right|^{2}\mathbb{E}\left\|\begin{array}{c}c^{\wedge}(-\left(t+\omega\right))\\\left(c^{-n}X\left(t+\left(n+1\right)\omega\right)-\widetilde{X}\left(t+\omega\right)\right)\end{array}\right\|^{2}\\ &+2\left|c\right|^{2}\mathbb{E}\left\|\begin{array}{c}c^{\wedge}(-\left(t+\omega\right))c^{-n}X\left(t+\left(n+1\right)\omega\right)\\-c^{\wedge}(-t)\widetilde{X}\left(t\right)\end{array}\right\|^{2}\\ =\;&2\left|c\right|^{2}\mathbb{E}\left\|\begin{array}{c}c^{\wedge}(-\left(t+\omega\right))\\\left(c^{-n}X\left(t+\left(n+1\right)\omega\right)-\widetilde{X}\left(t+\omega\right)\right)\end{array}\right\|^{2}\\ &+2|c|^{2}\mathbb{E}\left\|c^{\wedge}(-t)\left(c^{-n-1}X\left(t+\left(n+1\right)\omega\right)-\widetilde{X}\left(t\right)\right)\right\|^{2}\\ \leq\;&2\left|c\right|^{2}\frac{\varepsilon}{4\left|c\right|^{2}}+2\left|c\right|^{2}\frac{\varepsilon}{4\left|c\right|^{2}}=\varepsilon,\end{aligned}$$

which shows that $\widetilde{X}\in P_{\omega,c}\left(\mathbb{R}_{+},L^{2}\left(\mathbb{P},H\right)\right)$.

To show property 4, let $X\in P_{\omega,c}L\left(\mathbb{R}_{+},L^{2}\left(\mathbb{P},H\right)\right)$, then $\forall\varepsilon>0,\exists N\geq0,\forall n\geq N$, $\mathbb{E}\left\|c^{\wedge}(-t)\left(c^{-n}X\left(t+n\omega\right)-\widetilde{X}\left(t\right)\right)\right\|^{2}\leq\frac{\varepsilon}{|c^{\wedge}(s)|^{2}}$, where $\widetilde{X}\left(\cdot\right)$ is the (ω,c)-periodic limit of $c^{-n}X\left(\cdot+n\omega\right)$ in $L^{2}\left(\mathbb{P},H\right)$ when $n\longrightarrow+\infty$. We must show that for any $s\in\mathbb{R}_{+}$, the process $\widetilde{X_{s}}\left(\cdot\right)$ is the (ω,c)-periodic limit of $c^{-n}X_{s}\left(\cdot+n\omega\right)$ in $L^{2}\left(\mathbb{P},H\right)$. We have

$$\begin{aligned}&\mathbb{E}\left\|c^{\wedge}(-t)\left(c^{-n}X_{s}\left(t+n\omega\right)-\widetilde{X_{s}}\left(t\right)\right)\right\|^{2}\\ =\;&\mathbb{E}\left\|c^{\wedge}(-t)c^{-n}X\left(t+s+n\omega\right)-c^{\wedge}(-t)\widetilde{X}\left(t+s\right)\right\|^{2}\\ =\;&\mathbb{E}\left\|c^{\wedge}(s)\left[\begin{array}{c}c^{\wedge}(-\left(t+s\right))\\\left(c^{-n}X\left(t+s+n\omega\right)-\widetilde{X}\left(t+s\right)\right)\end{array}\right]\right\|^{2}\\ =\;&\mathbb{E}\left\|\begin{array}{c}c^{\wedge}(s)c^{\wedge}(-\left(t+s\right))c^{-n}X\left(t+s+n\omega\right)\\-c^{\wedge}(-t)\widetilde{X}\left(t+s\right)\end{array}\right\|^{2}\\ =\;&\left|c^{\wedge}(s)\right|^{2}\mathbb{E}\left\|\begin{array}{c}c^{\wedge}(-\left(t+s\right))\\\left(c^{-n}X\left(t+s+n\omega\right)-\widetilde{X}\left(t+s\right)\right)\end{array}\right\|^{2}\\ \leq\;&\varepsilon,\end{aligned}$$

hence $X_{s}\left(\cdot\right)\in P_{\omega,c}L\left(\mathbb{R}_{+},L^{2}\left(\mathbb{P},H\right)\right)$. ■

Corollary 1 *Assertion 3 implies that*

$\mathbb{E}\left\|c^\wedge(-t)\left(\widetilde{X}(t+m\omega)-c\widetilde{X}(t)\right)\right\|^2 = 0$, *for all* $m\in\mathbb{N}$.

We continue by stating and proving the following result.

Theorem 2 $P_{\omega,c}L\left(\mathbb{R}_+,L^2(\mathbb{P},H)\right)$ *equipped with the norm* $\|\cdot\|_{2,\omega,c}$ *is a Banach space.*

Proof. From 1-Proposition 4 it is clear that $P_{\omega,c}L\left(\mathbb{R}_+,L^2(\mathbb{P},H)\right)$ is a linear space. On the other hand, it is easy to verify that $\|\cdot\|_{2,\omega,c}$ defines a norm on $P_{\omega,c}L\left(\mathbb{R}_+,L^2(\mathbb{P},H)\right)$, it remains to show that $P_{\omega,c}L\left(\mathbb{R}_+,L^2(\mathbb{P},H)\right)$ is complete with respect to the norm $\|\cdot\|_{2,\omega,c}$. To this end, it suffices to show that $PL_{\omega,c}\left(\mathbb{R}_+,L^2(\mathbb{P},H)\right)$ is a closed subspace of $\mathcal{CB}_{\omega,c}\left(\mathbb{R}_+,L^2(\mathbb{P},H)\right)$. Indeed, let $(X_p)_{p\geq 0}$ be a sequence of $P_{\omega,c}L\left(\mathbb{R}_+,L^2(\mathbb{P},H)\right)$, such that $\lim_{p\to+\infty}X_p(t)=X(t)$ uniformly in $t\in\mathbb{R}_+$, i.e., $\forall\varepsilon>0$, $\exists s\geq 0$ sufficiently large, such that $\forall p\geq s$

$$\mathbb{E}\left\|c^\wedge(-t)\left(X_p(t)-X(t)\right)\right\|^2\leq\frac{\varepsilon}{9},\ \forall t\in\mathbb{R}_+. \tag{2.1}$$

Since $(X_p)_{p\geq 0}\in P_{\omega,c}L\left(\mathbb{R}_+,L^2(\mathbb{P},H)\right)$, then

$$\lim_{n\to+\infty}\mathbb{E}\left\|c^\wedge(-t)\left(c^{-n}X_p(t+n\omega)-\widetilde{X}_p(t)\right)\right\|^2=0,$$

for each $p\geq 0$ and each $t\in\mathbb{R}_+$ and for some stochastic process $\widetilde{X}_p:\mathbb{R}_+\longrightarrow L^2(\mathbb{P},H)$, i.e., $\forall\varepsilon>0$, $\exists N\geq 0$ sufficiently large, such that $\forall n\geq N$

$$\mathbb{E}\left\|c^\wedge(-t)\left(c^{-n}X_p(t+n\omega)-\widetilde{X}_p(t)\right)\right\|^2\leq\frac{\varepsilon}{9}. \tag{2.2}$$

On the other hand, since $(X_p(t))_{p\geq 0}$ is a Cauchy sequence in $L^2(\mathbb{P},H)$, then $(c^\wedge(-(t+n\omega))X_p(t+n\omega))_{p\geq 0}$ is too, i.e., $\forall\varepsilon>0$, $\exists r\geq 0$, such that $\forall p,q\geq r$

$$\left\|\begin{array}{c}c^\wedge(-(t+n\omega))\\(X_p(t+n\omega)-X_q(t+n\omega))\end{array}\right\|^2\leq\frac{|c|^{2n}}{9}\varepsilon,\ \forall n\geq 0 \tag{2.3}$$

So, from (2.2) and (2.3), $\forall\varepsilon>0$, $\exists N,r\geq 0$ sufficiently large, such that $\forall n\geq N$ and $\forall p,q\geq r$

$$\begin{aligned}
&\mathbb{E}\left\|c^\wedge(-t)\left(\widetilde{X}_p(t)-\widetilde{X}_q(t)\right)\right\|^2\\
\leq\ &3\mathbb{E}\left\|c^\wedge(-t)\left(c^{-n}X_p(t+n\omega)-\widetilde{X}_p(t)\right)\right\|^2\\
&+3\mathbb{E}\left\|c^\wedge(-(t+n\omega))\left(X_p(t+n\omega)-X_q(t+n\omega)\right)\right\|^2\\
&+3\mathbb{E}\left\|c^\wedge(-t)\left(c^{-n}X_q(t+n\omega)-\widetilde{X}_q(t)\right)\right\|^2\\
=\ &3\mathbb{E}\left\|c^\wedge(-t)\left(c^{-n}X_p(t+n\omega)-\widetilde{X}_p(t)\right)\right\|^2\\
&+3|c|^{-2n}\mathbb{E}\left\|c^\wedge(-t)\left(X_p(t+n\omega)-X_q(t+n\omega)\right)\right\|^2\\
&+3\mathbb{E}\left\|c^\wedge(-t)\left(c^{-n}X_q(t+n\omega)-\widetilde{X}_q(t)\right)\right\|^2\\
\leq\ &3\left(\frac{\varepsilon}{9}+\frac{\varepsilon}{9}+\frac{\varepsilon}{9}\right)=\varepsilon,
\end{aligned}$$

hence $\left(\widetilde{X}_p(t)\right)_{p\geq 0}$ is a Cauchy sequence in $L^2(\mathbb{P},H)$ which is complete, thus by noting $\widetilde{X}(t)$ the pointwise limit of $\widetilde{X}_p(t)$ we have

$$\lim_{p\to+\infty}\mathbb{E}\left\|c^\wedge(-t)\left(\widetilde{X}_p(t)-\widetilde{X}(t)\right)\right\|^2=0,$$

i.e., $\forall\varepsilon>0$, $\exists s'\geq 0$ sufficiently large, such that $\forall p\geq s'$

$$\mathbb{E}\left\|c^\wedge(-t)\left(\widetilde{X}_p(t)-\widetilde{X}(t)\right)\right\|^2\leq\frac{\varepsilon}{9} \tag{2.4}$$

From (2.1), (2.2), and (2.4), $\forall\varepsilon>0$, $\exists N,s,s'\geq 0$ sufficiently large, such that $\forall n\geq N$, $\forall p\geq\max(s,s')$, we have

$$\begin{aligned}
&\mathbb{E}\left\|c^\wedge(-t)\left(c^{-n}X(t+n\omega)-\widetilde{X}(t)\right)\right\|^2\\
\leq\ &3\mathbb{E}\left\|c^\wedge(-t)\left(c^{-n}X(t+n\omega)-c^{-n}X_p(t+n\omega)\right)\right\|^2\\
&+3\mathbb{E}\left\|c^\wedge(-t)\left(c^{-n}X_p(t+n\omega)-\widetilde{X}_p(t)\right)\right\|^2\\
&+3\mathbb{E}\left\|c^\wedge(-t)\left(\widetilde{X}_p(t)-\widetilde{X}(t)\right)\right\|^2\\
\leq\ &3\frac{\varepsilon}{9}+3\frac{\varepsilon}{9}+3\frac{\varepsilon}{9}=\varepsilon,
\end{aligned}$$

which shows that $X\in P_{\omega,c}L\left(\mathbb{R}_+,L^2(\mathbb{P},H)\right)$. Hence $P_{\omega,c}L\left(\mathbb{R}_+,L^2(\mathbb{P},H)\right)$ is a closed subspace of $\mathcal{CB}_{\omega,c}\left(\mathbb{R}_+,L^2(\mathbb{P},H)\right)$. ■

By $\mathcal{C}_0\left(\mathbb{R}_+,L^2(\mathbb{P},H)\right)$ we denote the space of all stochastic processes $X\in\mathcal{CB}\left(\mathbb{R}_+,L^2(\mathbb{P},H)\right)$ vanishing at infinity, i.e., $\lim_{t\to+\infty}\mathbb{E}\|X(t)\|^2=0$.

Definition 3 *The space of stochastic processes*

(ω, c)-vanishing at infinity is denoted and defined by

$$\mathcal{C}_{0,\omega,c}\left(\mathbb{R}_+, L^2(\mathbb{P}, H)\right) := \left\{ \begin{array}{c} X \in \mathcal{CB}_{\omega,c}\left(\mathbb{R}_+, L^2(\mathbb{P}, H)\right) : \\ \lim\limits_{t\to+\infty} \mathbb{E}\left\|c\hat{}(-t)X(t)\right\|^2 = 0 \end{array} \right\}.$$

It is easy to prove the following characterization of $\mathcal{C}_{0,\omega,c}\left(\mathbb{R}_+, L^2(\mathbb{P}, H)\right)$.

Proposition 5 *Let $X \in \mathcal{CB}_{\omega,c}\left(\mathbb{R}_+, L^2(\mathbb{P}, H)\right)$. Then $X \in \mathcal{C}_{0,\omega,c}\left(\mathbb{R}_+, L^2(\mathbb{P}, H)\right)$ if and only if $X(\cdot) = c\hat{}(\cdot)Y(\cdot)$, where $Y \in \mathcal{C}_0\left(\mathbb{R}_+, L^2(\mathbb{P}, H)\right)$.*

The concept of square mean asymptotic (ω, c)-periodicity of stochastic processes is given as follows.

Definition 4 *Let $c \in \mathbb{C}\backslash\{0\}$. A process $X \in \mathcal{CB}_{\omega,c}\left(\mathbb{R}_+, L^2(\mathbb{P}, H)\right)$ is said to be square mean asymptotically (ω, c)-periodic if $X = Y + Z$, where $Y \in P_{\omega,c}\left(\mathbb{R}_+, L^2(\mathbb{P}, H)\right)$ and $Z \in \mathcal{C}_{0,\omega,c}\left(\mathbb{R}_+, L^2(\mathbb{P}, H)\right)$.*

The space of all square mean asymptotically (ω, c)-periodic stochastic processes is denoted by $AP_{\omega,c}\left(\mathbb{R}_+, L^2(\mathbb{P}, H)\right)$.

In the case $c = 1$, the authors have proved, in [10], that $AP_\omega\left(\mathbb{R}_+, L^2(\mathbb{P}, H)\right)$ is a closed subspace of $P_\omega L\left(\mathbb{R}_+, L^2(\mathbb{P}, H)\right)$. The same statement holds in the general case $c \in \mathbb{C}\backslash\{0\}$, as the next result shows:

Proposition 6 *$AP_{\omega,c}\left(\mathbb{R}_+, L^2(\mathbb{P}, H)\right)$ is a closed subspace of $P_{\omega,c}L\left(\mathbb{R}_+, L^2(\mathbb{P}, H)\right)$.*

Proof. First of all, let us show that $AP_{\omega,c}\left(\mathbb{R}_+, L^2(\mathbb{P}, H)\right) \subset P_{\omega,c}L\left(\mathbb{R}_+, L^2(\mathbb{P}, H)\right)$. To this end, let $X \in AP_{\omega,c}\left(\mathbb{R}_+, L^2(\mathbb{P}, H)\right)$, then $\forall n \geq 1, \forall t \geq 0$ we have

$$\begin{aligned}
&\mathbb{E}\left\|c\hat{}(-t)\left(c^{-n}X(t+n\omega) - Y(t)\right)\right\|^2 \\
&= \mathbb{E}\left\|c\hat{}(-t)\begin{pmatrix} c^{-n}Y(t+n\omega) \\ +c^{-n}Z(t+n\omega) \\ -Y(t) \end{pmatrix}\right\|^2 \\
&= \mathbb{E}\left\| \begin{array}{c} c\hat{}(-(t+n\omega))Y(t+n\omega) \\ +c\hat{}(-(t+n\omega))Z(t+n\omega) \\ -c\hat{}(-t)Y(t) \end{array} \right\|^2 \\
&\leq 2\mathbb{E}\left\|c\hat{}(-t)\begin{pmatrix} c^{-n}X(t+n\omega) \\ -c^{-n}Y(t+n\omega) \end{pmatrix}\right\|^2 \\
&+2\mathbb{E}\left\|c\hat{}(-t)\left(c^{-n}Y(t+n\omega) - Y(t)\right)\right\|^2 \\
&= 2\mathbb{E}\left\|c\hat{}(-(t+n\omega))\begin{pmatrix} X(t+n\omega) \\ -Y(t+n\omega) \end{pmatrix}\right\|^2 \\
&+2\mathbb{E}\left\|c\hat{}(-t)\left(c^{-n}Y(t+n\omega) - Y(t)\right)\right\|^2 \\
&= 2\mathbb{E}\left\|c\hat{}(-(t+n\omega))Z(t+n\omega)\right\|^2,
\end{aligned}$$

which implies that $\lim\limits_{n\to+\infty} \mathbb{E}\left\|c\hat{}(-t)\left(c^{-n}X(t+n\omega) - Y(t)\right)\right\|^2 = 0$, $\forall t \geq 0$, hence $X \in P_{\omega,c}L\left(\mathbb{R}_+, L^2(\mathbb{P}, H)\right)$.

Now suppose that $(X_n)_{n\geq1}$ is a sequence of $AP_{\omega,c}\left(\mathbb{R}_+, L^2(\mathbb{P}, H)\right)$ such that $\lim\limits_{n\to+\infty} X_n = X$. We must show that $X \in AP_{\omega,c}\left(\mathbb{R}_+, L^2(\mathbb{P}, H)\right)$. Since $X_{nn\geq1} \subset AP_{\omega,c}\left(\mathbb{R}_+, L^2(\mathbb{P}, H)\right)$, then $\forall n \geq 1$, there exist $Y_n \in P_{\omega,c}\left(\mathbb{R}_+, L^2(\mathbb{P}, H)\right)$ and $Z_n \in \mathcal{C}_{0,\omega,c}\left(\mathbb{R}_+, L^2(\mathbb{P}, H)\right)$ such that $X_n = Y_n + Z_n$. It is clear that $(Y_n)_{n\geq0}$ and $(Z_n)_{n\geq0}$ converge, otherwise $(X_n)_{n\geq0}$ does not converge. Consequently, there exist Y and Z such that $\lim\limits_{n\to+\infty} Y_n = Y$, $\lim\limits_{n\to+\infty} Z_n = Z$, and $X = Y + Z$. Furthermore, $Y \in P_{\omega,c}\left(\mathbb{R}_+, L^2(\mathbb{P}, H)\right)$ and $Z \in \mathcal{C}_{0,\omega,c}\left(\mathbb{R}_+, L^2(\mathbb{P}, H)\right)$. Indeed, since $Y_n \in P_{\omega,c}\left(\mathbb{R}_+, L^2(\mathbb{P}, H)\right)$ then

$$\mathbb{E}\left\|c\hat{}(-t)\left(Y_n(t+\omega) - cY_n(t)\right)\right\|^2 = 0,$$

and as $\lim\limits_{n\to+\infty} Y_n = Y$, then $\forall\varepsilon > 0$, $\exists N$ sufficiently large, such that $\forall n \geq N$

$$\mathbb{E}\left\|c\hat{}(-t)\left(Y_n(t) - Y(t)\right)\right\|^2 \leq \frac{\varepsilon}{6|c|^2}$$

and

$$\mathbb{E}\left\|c\hat{}(-t)\left(Y_n(t+\omega) - Y(t+\omega)\right)\right\|^2 \leq \frac{\varepsilon}{6}.$$

Hence

$$\begin{aligned}
&\mathbb{E}\left\|c\hat{}(-t)\left(Y(t+\omega) - cY(t)\right)\right\|^2 \\
\leq\ & 3\mathbb{E}\left\|c\hat{}(-t)\left(Y(t+\omega) - Y_n(t+\omega)\right)\right\|^2 \\
&+3\mathbb{E}\left\|c\hat{}(-t)\left(Y_n(t+\omega) - cY_n(t)\right)\right\|^2 \\
&+3|c|^2\mathbb{E}\left\|c\hat{}(-t)\left(Y_n(t) - Y(t)\right)\right\|^2 \\
=\ & 3\mathbb{E}\left\|c\hat{}(-t)\left(Y(t+\omega) - Y_n(t+\omega)\right)\right\|^2 \\
&+3|c|^2\mathbb{E}\left\|c\hat{}(-t)\left(Y_n(t) - Y(t)\right)\right\|^2 \\
\leq\ & \varepsilon,
\end{aligned}$$

which shows that $\mathbb{E}\left\|c\hat{}(-t)\left(Y(t+\omega) - cY(t)\right)\right\|^2 = 0$, thus $Y \in P_{\omega,c}\left(\mathbb{R}_+, L^2(\mathbb{P}, H)\right)$. On the other hand, as $Z_n \in \mathcal{C}_{0,\omega,c}\left(\mathbb{R}_+, L^2(\mathbb{P}, H)\right)$ and $\lim\limits_{n\to+\infty} Z_n = Z$, then $\forall\varepsilon > 0$, $\exists T > 0$, such that $\forall t \geq T$, $\mathbb{E}\left\|c\hat{}(-t)Z_n(t)\right\|^2 \leq \frac{\varepsilon}{4}$,and $\forall\varepsilon > 0$, $\exists N \geq 0$ sufficiently large, such that $\forall n \geq N$; $\mathbb{E}\left\|c\hat{}(-t)\left(Z_n(t) - Z(t)\right)\right\|^2 \leq \frac{\varepsilon}{4}$. Consequently, $\forall\varepsilon > 0$, $\exists N$ sufficiently large, $\exists T > 0$, such that

$\forall n \geq N$ and $\forall t \geq T$, we have

$$\begin{aligned} & \mathbb{E}\left\|c^\wedge(-t)Z(t)\right\|^2 \\ \leq\ & 2\mathbb{E}\left\|c^\wedge(-t)\left(Z(t)-Z_n(t)\right)\right\|^2 \\ & +2\mathbb{E}\left\|c^\wedge(-t)Z_n(t)\right\|^2 \\ \leq\ & \varepsilon, \end{aligned}$$

hence $Z \in \mathcal{C}_{0,\omega,c}\left(\mathbb{R}_+, L^2(\mathbb{P},H)\right)$. This simply completes the proof. ■

Keeping in mind Theorem 2, the last Proposition and their proofs, we can simply deduce the following result.

Theorem 3 $\left(AP_{\omega,c}\left(\mathbb{R}_+, L^2(\mathbb{P},H)\right), \|\cdot\|_{2,\omega,c}\right)$ *is a Banach space.*

3 Invariance under the Action of the Convolution Product and the Composition Principle

We start this section by examining the invariance of the square mean (ω,c)-periodic limit properties under the infinite convolution product

$$S(t) := \int_{-\infty}^{t} R(t-s)\,X(s)\,ds,$$

where $(R(t))_{t\geq 0}$ is a strongly continuous operator family of $\mathcal{L}(H)$ satisfies certain assumptions.

Theorem 4 *Let* $(R(t))_{t\geq 0} \subset \mathcal{L}(H)$ *be strongly continuous linear operator such that* $\|c^\wedge(-t)R(t)\|^2 \leq \varphi(t)$, *where* $\varphi \in L^1(\mathbb{R}_+)$. *If* $X \in P_{\omega,c}L\left(\mathbb{R}_+, L^2(\mathbb{P},H)\right)$, *then the process*

$$S(t) = \int_{-\infty}^{t} R(t-s)\,X(s)\,ds,$$

is a well defined element of $P_{\omega,c}L\left(\mathbb{R}_+, L^2(\mathbb{P},H)\right)$.

Proof. Let us first show that $S \in \mathcal{CB}_{\omega,c}\left(\mathbb{R}_+, L^2(\mathbb{P},H)\right)$. Given $X \in P_{\omega,c}L\left(\mathbb{R}_+, L^2(\mathbb{P},H)\right)$, by Proposition 3, there exists a process $Y \in P_{\omega}L\left(\mathbb{R}_+, L^2(\mathbb{P},H)\right)$ such that $X(t) = c^\wedge(t)Y(t)$. Substituting this into $S(t)$, we have

$$\begin{aligned} S(t) &= \int_{-\infty}^{t} c^\wedge(s)R(t-s)\,Y(s)\,ds \\ &= \int_{0}^{+\infty} c^\wedge(t-r)R(r)\,Y(t-r)\,dr \\ &= c^\wedge(t)\int_{0}^{+\infty} c^\wedge(-r)R(r)\,Y(t-r)\,dr. \end{aligned}$$

By defining $Z(t) = \int_0^{+\infty} c^\wedge(-r)R(r)\,Y(t-r)\,dr$, for $t \in \mathbb{R}_+$, we get $S(t) = c^\wedge(t)Z(t)$. According to Proposition 1, to show that $S \in \mathcal{CB}_{\omega,c}\left(\mathbb{R}_+, L^2(\mathbb{P},H)\right)$, it suffices to prove that $Z \in \mathcal{CB}\left(\mathbb{R}_+, L^2(\mathbb{P},H)\right)$. Applying the Cauchy-Schwarz inequality and using the assumption $\|c^\wedge(-t)R(t)\| \leq \varphi(t)$, we get

$$\begin{aligned} \mathbb{E}\|Z(t)\|^2 &= \mathbb{E}\left\|\int_0^{+\infty} c^\wedge(-r)R(r)\,Y(t-r)\,dr\right\|^2 \\ &\leq \mathbb{E}\left(\int_0^{+\infty}\left\|c^\wedge(-r)R(r)\,Y(t-r)\right\|dr\right)^2 \\ &\leq \mathbb{E}\left(\int_0^{+\infty}\varphi(r)\left\|Y(t-r)\right\|dr\right)^2 \\ &\leq \int_0^{+\infty}\varphi(r)\,dr\int_0^{+\infty}\varphi(r)\,\mathbb{E}\left\|Y(t-r)\right\|^2 dr \\ &\leq \|\varphi\|_{L^1}^2\,\|Y\|_{L^\infty}^2. \end{aligned}$$

Hence $\sup_{t\in\mathbb{R}_+}\mathbb{E}\|Z(t)\|^2 < \infty$, which shows that Z is bounded in $L^2(\mathbb{P},H)$.

Let $s \in \mathbb{R}$. Applying the Cauchy-Schwarz inequality and the assumption $\|c^\wedge(-t)R(t)\| \leq \varphi(t)$, we have

$$\begin{aligned} & \mathbb{E}\|Z(t)-Z(s)\|^2 \\ & = \mathbb{E}\left\|\int_0^{+\infty} c^\wedge(-r)R(r)\left(Y(t-r)-Y(s-r)\right)dr\right\|^2 \\ & \leq \mathbb{E}\left(\int_0^{+\infty}\left\|c^\wedge(-r)R(r)\left(Y(t-r)-Y(s-r)\right)\right\|dr\right)^2 \\ & \leq \mathbb{E}\left(\int_0^{+\infty}\varphi(r)\left\|Y(t-r)-Y(s-r)\right\|dr\right)^2 \end{aligned}$$

$$\leq \left(\int_0^{+\infty} \varphi(r)\,dr\right)\left(\int_0^{+\infty} \varphi(r)\,\mathbb{E}\left\|Y(t-r)-Y(s-r)\right\|^2 dr\right)$$
$$\leq \|\varphi\|_{L^1}\int_0^{+\infty} \varphi(r)\,\mathbb{E}\left\|Y(t-r)-Y(s-r)\right\|^2 dr.$$

On the other hand, we have

$$\int_0^{+\infty} \varphi(r)\,\mathbb{E}\left\|Y(t-r)-Y(s-r)\right\|^2 dr \leq 4\|\varphi\|_{L^1}\|Y\|^2_{L^\infty}.$$

Using the fact that $Y \in \mathcal{CB}\left(\mathbb{R}_+, L^2(\mathbb{P},H)\right)$ and applying the dominated convergence theorem, we obtain

$$\lim_{t\longrightarrow s}\int_0^{+\infty} \varphi(r)\,\mathbb{E}\left\|Y(t-r)-Y(s-r)\right\|^2 dr = 0.$$

Thus, $\lim_{t\longrightarrow s}\mathbb{E}\left\|Z(t)-Z(s)\right\|^2 = 0$, which shows the continuity of Z in $L^2(\mathbb{P},H)$. Therefore, $Z \in \mathcal{CB}\left(\mathbb{R}_+, L^2(\mathbb{P},H)\right)$.

Let us now show that S is square mean (ω,c)-periodic limit. Since $X \in P_{\omega,c}L\left(\mathbb{R}_+, L^2(\mathbb{P},H)\right)$,

$$\lim_{n\to+\infty}\mathbb{E}\left\|c\^{}(-t)\left(c^{-n}X(t+n\omega)-\widetilde{X}(t)\right)\right\|^2 = 0,$$

is well defined in $L^2(\mathbb{P},H)$ for each $t \geq 0$ and for some stochastic process $\widetilde{X}: \mathbb{R}_+ \longrightarrow L^2(\mathbb{P},H)$. Define $\widetilde{S}(t) = \int_{-\infty}^{t} R(t-s)\,\widetilde{X}(s)\,ds$. Then

$$\mathbb{E}\left\|c\^{}(-t)\left(c^{-n}S(t+n\omega)-\widetilde{S}(t)\right)\right\|^2$$
$$= \mathbb{E}\left\|\begin{pmatrix} c\^{}(-t) \\ c^{-n}\int_{-\infty}^{t+n\omega} R(t+n\omega-s)\,X(s)\,ds \\ -\int_{-\infty}^{t} R(t-s)\,\widetilde{X}(s)\,ds\end{pmatrix}\right\|^2$$
$$= \mathbb{E}\left\|\begin{pmatrix} c\^{}(-t) \\ c^{-n}\int_{-\infty}^{t} R(t-s)\,X(s+n\omega)\,ds \\ -\int_{-\infty}^{t} R(t-s)\,\widetilde{X}(s)\,ds\end{pmatrix}\right\|^2$$
$$= \mathbb{E}\left\| c\^{}(-t) \left[\int_{-\infty}^{t} R(t-s)\begin{pmatrix} c^{-n}X(s+n\omega) \\ -\widetilde{X}(s)\end{pmatrix} ds\right]\right\|^2$$
$$= \mathbb{E}\left\|c\^{}(-t)\left[\int_0^{+\infty} R(s)\begin{pmatrix} c^{-n}X(t-s+n\omega) \\ -\widetilde{X}(t-s)\end{pmatrix} ds\right]\right\|^2$$
$$\leq \mathbb{E}\left[\int_0^{+\infty}\left\|\begin{matrix} c\^{}(-t)R(s) \\ \left(c^{-n}X(t-s+n\omega)-\widetilde{X}(t-s)\right)\end{matrix}\right\| ds\right]^2$$
$$\leq \mathbb{E}\left[\int_0^{+\infty}\left\|\begin{matrix} c\^{}(-s)R(s)\,c\^{}(-(t-s)) \\ \left(c^{-n}X(t-s+n\omega)-\widetilde{X}(t-s)\right)\end{matrix}\right\| ds\right]^2$$
$$\leq \mathbb{E}\left[\int_0^{+\infty}\varphi(s)\left\|\begin{pmatrix} c\^{}(-(t-s)) \\ c^{-n}X(t-s+n\omega) \\ -\widetilde{X}(t-s)\end{pmatrix}\right\| ds\right]^2.$$

Applying the Cauchy-Schwarz inequality

$$\mathbb{E}\left\|c\^{}(-t)\left(c^{-n}S(t+n\omega)-\widetilde{S}(t)\right)\right\|^2$$
$$\leq \left(\int_0^{+\infty}\varphi(s)\right)^2$$
$$\times\left(\int_0^{+\infty}\mathbb{E}\left\|\begin{matrix} c\^{}(-(t-s)) \\ \left(c^{-n}X(t-s+n\omega)-\widetilde{X}(t-s)\right)\end{matrix}\right\|^2 ds\right)$$
$$\leq \|\varphi\|^2_{L^1}\int_0^{+\infty}\mathbb{E}\left\|\begin{pmatrix} c\^{}(-(t-s)) \\ c^{-n}X(t-s+n\omega) \\ -\widetilde{X}(t-s)\end{pmatrix}\right\|^2 ds.$$

From 4-Proposition 4 and the dominated convergence theorem, it follows that

$$\lim_{n\longrightarrow+\infty}\mathbb{E}\left\|c\^{}(-t)\left(c^{-n}S(t+n\omega)-\widetilde{S}(t)\right)\right\|^2 = 0.$$

Thus, $S \in P_{\omega,c}L\left(\mathbb{R}_+, L^2(\mathbb{P},H)\right)$. ■

Remark 2 *The above result shows that in complex systems where operators and stochastic processes interact, it is essential to preserve certain properties through operations such as convolution. The impact of a strongly continuous linear operator R on a square mean (ω,c)-periodic limit process X can significantly influence the resulting process. To ensure that the infinite convolution $\int_{-\infty}^{t} R(t-s)\,X(s)\,ds$ retains the square mean (ω,c)-periodicity and limiting features of X, it is crucial to manage the growth of $R(t)$ with specific conditions. This approach not only maintains the integrability, boundedness, and periodicity of the resulting process but also enhances our understanding of its long-term behavior and stability.*

In order to examine the composition principle for square mean (ω,c)-periodic limit stochastic processes, we need to generalize Definition 2. Let us first introduce the following space

$$\mathcal{CB}_{\omega,c}\left(\mathbb{R}_+\times L^2(\mathbb{P},H), L^2(\mathbb{P},H)\right)$$
$$:= \left\{\begin{matrix} f \in \mathcal{C}\left(\mathbb{R}_+\times L^2(\mathbb{P},H), L^2(\mathbb{P},H)\right): \\ \sup_{t\in\mathbb{R}_+}\mathbb{E}\left\|c\^{}(-t)f(t,X(t))\right\|^2 < \infty \\ \text{uniformly for } X \in L^2(\mathbb{P},H)\end{matrix}\right\}.$$

Definition 5 *Let $c \in \mathbb{C}\backslash\{0\}$. A stochastic process $f \in \mathcal{CB}_{\omega,c}\left(\mathbb{R}_+ \times L^2(\mathbb{P},H), L^2(\mathbb{P},H)\right)$ is said to be square mean (ω,c)-periodic limit in $t \in \mathbb{R}_+$ uniformly in $X \subset K$ where K is a bounded subset of $L^2(\mathbb{P},H)$, if there exists $\omega > 0$ such that*

$$\lim_{n\to+\infty}\mathbb{E}\left\|c^\wedge(-t)\left[c^{-n}f(t+n\omega,X(t))-\widetilde{f}(t,X(t))\right]\right\|^2=0,$$

is well defined in $L^2(\mathbb{P},H)$ for each $t \geq 0$ and for some stochastic process $\widetilde{f} : \mathbb{R}_+ \times L^2(\mathbb{P},H) \longrightarrow L^2(\mathbb{P},H)$.

By $P_{\omega,c}L\left(\mathbb{R}_+ \times L^2(\mathbb{P},H), L^2(\mathbb{P},H)\right)$ we denote the space of these square mean (ω,c)-periodic limit stochastic processes.

If $f \in \mathcal{CB}_{\omega,c}\left(\mathbb{R}_+ \times L^2(\mathbb{P},H), L^2(\mathbb{P},H)\right)$ and $\phi \in P_{\omega,c}L\left(\mathbb{R}_+ \times L^2(\mathbb{P},H)\right)$, then the Nemytskii's operator $\mathcal{N}(\phi)(\cdot) = f(\cdot,\phi(\cdot))$ is invariant on $P_{\omega,c}L\left(\mathbb{R}_+ \times L^2(\mathbb{P},H)\right)$. More precisely, the following result gives the composition principle for square mean (ω,c)-periodic limit stochastic processes.

Theorem 5 *Let $c \in \mathbb{C}\backslash\{0\}$ such that $|c| \geq 1$ and let $f \in P_{\omega,c}L\left(\mathbb{R}_+ \times L^2(\mathbb{P},H), L^2(\mathbb{P},H)\right)$. Suppose that f is Lipschitzian in the following sense:*

$$\begin{aligned}\exists L_f &> 0 : \mathbb{E}\|f(t,Y(t)) - f(t,Z(t))\|^2 \\ &\leq L_f\mathbb{E}\|Y(t) - Z(t)\|^2, \\ \forall t &\geq 0, \forall Y,Z \in L^2(\mathbb{P},H).\end{aligned}$$

Then for each stochastic process $\phi \in P_{\omega,c}L\left(\mathbb{R}_+, L^2(\mathbb{P},H)\right)$, the stochastic process $t \longmapsto F(t) = f(t,\phi(t))$ is square mean (ω,c)-periodic limit.

Proof. Since ϕ is a square mean (ω,c)-periodic limit process, there exists $\omega > 0$ such that

$$\lim_{n\to+\infty}\mathbb{E}\left\|c^\wedge(-t)\left[c^{-n}\phi(t+n\omega) - \widetilde{\phi}(t)\right]\right\|^2 = 0 \tag{3.1}$$

for each $t \geq 0$ and some stochastic process $\widetilde{\phi} : \mathbb{R}_+ \longrightarrow L^2(\mathbb{P},H)$. By 2-Proposition 4, we can choose a bounded subset K of $L^2(\mathbb{P},H)$ such that $\phi(t), \widetilde{\phi}(t) \in K$ for all $t \in \mathbb{R}_+$. Thus $t \longmapsto F(t,\phi(t))$ is an element of $\mathcal{CB}_{\omega,c}\left(\mathbb{R}_+ \times L^2(\mathbb{P},H), L^2(\mathbb{P},H)\right)$. On the other hand, since $f \in P_{\omega,c}L\left(\mathbb{R}_+ \times L^2(\mathbb{P},H), L^2(\mathbb{P},H)\right)$, then

$$\lim_{n\to+\infty}\mathbb{E}\left\|c^\wedge(-t)\left[\begin{array}{c} c^{-n}f(t+n\omega,X) \\ -\widetilde{f}(t,X)\end{array}\right]\right\|^2 = 0 \tag{3.2}$$

for each $t \in \mathbb{R}_+$ and each $X \in K$, and for some stochastic process $\widetilde{f} : \mathbb{R}_+ \times L^2(\mathbb{P},H) \longrightarrow L^2(\mathbb{P},H)$. Define the process

$$\begin{array}{rccc}\widetilde{F}: & \mathbb{R}_+ & \longrightarrow & L^2(\mathbb{P},H) \\ & t & \longmapsto & \widetilde{f}\left(t,\widetilde{\phi}(t)\right)\end{array}.$$

According to the Lipschitz condition of f, we have

$$\begin{aligned}&\mathbb{E}\left\|c^\wedge(-t)\left[c^{-n}F(t+n\omega) - \widetilde{F}(t)\right]\right\|^2 \\ =\ &\mathbb{E}\left\|c^\wedge(-t)\left[\begin{array}{c} c^{-n}f(t+n\omega,\phi(t+n\omega)) \\ -\widetilde{f}\left(t,\widetilde{\phi}(t)\right)\end{array}\right]\right\|^2 \\ =\ &\mathbb{E}\left\|c^\wedge(-t)\left[\begin{array}{c} c^{-n}f(t+n\omega,\phi(t+n\omega)) \\ -c^{-n}f\left(t+n\omega,\widetilde{\phi}(t)\right) \\ +c^{-n}f\left(t+n\omega,\widetilde{\phi}(t)\right) \\ -\widetilde{f}\left(t,\widetilde{\phi}(t)\right)\end{array}\right]\right\|^2 \\ \leq\ &3L_f\mathbb{E}\left\|c^\wedge(-t)\left(c^{-n}\phi(t+n\omega) - \widetilde{\phi}(t)\right)\right\|^2 \\ &+3\left|c^{-n}-1\right|^2 L_f\mathbb{E}\left\|c^\wedge(-t)\widetilde{\phi}(t)\right\|^2 \\ &+3\mathbb{E}\left\|c^\wedge(-t)\left(\begin{array}{c} c^{-n}f\left(t+n\omega,\widetilde{\phi}(t)\right) \\ -\widetilde{f}\left(t,\widetilde{\phi}(t)\right)\end{array}\right)\right\|^2.\end{aligned}$$

From (3.1) and (3.2), combined with the facts that $\mathbb{E}\left\|c^\wedge(-t)\widetilde{\phi}(t)\right\|^2 < \infty$ and $\lim_{n\to+\infty}|c^{-n}-1|^2 = 0$ when $|c| > 1$, we deduce that

$$\lim_{n\to+\infty}\mathbb{E}\left\|c^\wedge(-t)\left[c^{-n}F(t+n\omega) - \widetilde{F}(t)\right]\right\|^2 = 0,$$

which is well defined for $\widetilde{F}(t) = \widetilde{f}\left(t,\widetilde{\phi}(t)\right)$. Hence $F \in P_{\omega,c}L\left(\mathbb{R}_+ \times L^2(\mathbb{P},H), L^2(\mathbb{P},H)\right)$. ■

Remark 3 *When $c \in \mathbb{C}\backslash\{0\}$ and $|c| = 1$, the previous result holds if and only if $c = e^{i\theta}$ where θ is a rational multiple of π. More precisely, c must be of the form $c = e^{i\frac{2k\pi}{h}}$, where k and h are integers, with $h > 0$, and $0 \leq k < h$. In this case, we also have $\lim_{n\to+\infty}|c^{-n}-1|^2 = 0$, and thus the result remains valid.*

4 Conclusion

In connection with the recent concepts of (ω,c)-periodic functions introduced by [7], [9], and square mean ω-periodic limit processes introduced by [10], [11], we have introduced and systematically analyzed two new classes of (ω, c)-periodic type stochastic processes. These include square mean (ω, c)-periodic limit stochastic processes and asymptotically (ω, c)-periodic stochastic processes. We have clarified various structural properties, characterizations, and several significant results related to these introduced classes of processes.

It is worth noting that the above theoretical results can be applied in the qualitative analysis of square mean asymptotically (ω, c)-periodic solutions for the following semilinear stochastic differential equation

$dX(t) = AX(t)\,dt + f(t, X(t))\,dt + g(t, X(t))\,dB(t), t \in \mathbb{R}_+, X(0) = X_0$,

where $A : D(A) \subset L^2(\mathbb{P}, H) \longrightarrow L^2(\mathbb{P}, H)$ is a closed linear operator, $f, g : \mathbb{R}_+ \times L^2(\mathbb{P}, H) \longrightarrow L^2(\mathbb{P}, H)$ are jointly Lipschitz continuous and bounded functions, $(B(t))_{t\in\mathbb{R}_+}$ is a two-sided standard one-dimensional Brownian motion with values in H and $\mathcal{F}_t$-adapted and $X_0 \in L^2(\mathbb{P}, H)$. This is the subject of our forthcoming research work.

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