Exploring the Structure of the Network Marketing Industry through Graph Parameters

CHINMAYEE KUMAR, KUNTALA PATRA
[1]Department of Mathematics
Gauhati University
Guwahati, Assam 781014
INDIA

Abstract: - Network Marketing is a business tool that builds a network of business partners or distributors by directly selling products and services through a word-of-mouth marketing strategy. NM graph is a graphical representation of the network marketing industry. Just like a graph consists of vertices and edges, network marketing consists of distributors/ business partners which represent the vertices of the NM graph, and edges determine the genealogy of the network. In this paper, we explore the structure of an NM graph by graph parameters and see how the graph reflects various properties of a network of people. It is seen that an NM graph is a rooted binary tree. The measure of the influence of a network and the measure of profitability of a network is determined. It is seen that the objective to maximize profit by allocating different stakes to different persons is a typical case of an assignment problem. The genealogy of a network can be determined by studying the adjacency matrix of the NM digraph. Graph parameter like eccentricity represents the strongest person in the network earning maximum profitability. The domination number acts as a measure of the stability of an NM graph.

Key-Words: - Network marketing, graph, tree, domination number, operation research, eccentricity.

Received: July 16, 2024. Revised: November 17, 2024. Accepted: December 8, 2024. Published: January 30, 2025.

1 Introduction

Network Marketing is a booming business of the 21st century. Network Marketing or Multi-Level Marketing is a business tool that builds a consumer network (network of people) called distributors/business partners, that in turn are expected to build a consumer network base thereby expanding the overall organization. This network of distributors enables the flow of profit through the organization of the concerned network. NM companies are also referred to as direct selling companies that use word-of-mouth as a marketing strategy to directly sell a product and earn commission. The recruited participants are called downline distributors and the recruiters are referred to as upline distributors.

An NM graph consists of consumers/distributors which represent the vertices of the NM graph and the genealogy of the network is represented by edges. NM graph is a graphical representation of the network marketing industry that represents the genealogy of a networker and his/her corresponding downline network. It can also be referred to as a collection of networks working independently to expand their own network and thereby expanding the overall organization. This type of hierarchy is not limited to network marketing but is prevalent in many areas. For example, in a university, the head of the institution acts as the central figure, while employees are akin to recruits and manage the student hierarchy. Similarly, a government functions as a hierarchical organization with a president or king at the top, followed by vice presidents, governors, and council members. There are different NM structures functioning in the market. Since studying all of these structures is a difficult job, we have considered a particular case of NM structure.

Because of its diagrammatic representation, graph theory serves as a mathematical model for any system and has a wide range of applications in different fields like physics and chemistry, biology, computer technology, operation research, linguistics, and sociology. Social networks are one of the most commonly seen structures where graphs are used to study social relationships among persons and organizations. Concepts from graph theory and other branches of topology ideas have also been employed to develop statistical models of nervous systems, which the author in [1] and others have applied to the spread of information and other social phenomena. However, there is little consensus among mathematicians regarding terminology, leading social scientists to draw from various mathematical vocabularies or create their own technical terms.

In recent years, the use of social networks has grown exponentially, with platforms like Facebook, Twitter, Instagram, and LinkedIn becoming integral to our daily lives. Social networks have also gained significant importance in the creation and distribution of information. As a result, the volume of information and the need for its analysis have driven the adoption of Big Data and Business Intelligence environments. Through these tools, companies can obtain prescriptive insights, such as those derived from graph theory. In [2], the authors discuss the concept of directed and undirected graphs, along with adjacency matrices, as they relate to WhatsApp groups. In [3], the authors effectively demonstrates the use of graphical parameters in social graphs and highlight the significant role of the domination number. In [4], the authors discusses both the advantages and disadvantages of using social media networks and compares graph theory with social media networks.

Motivated by [5], this paper is an attempt to demonstrate the practical application of graph theory in a network marketing business.

2 Preliminaries

In this section, we present some fundamental definitions related to graph theory for understanding the material in this paper.

Definition 2.1: A ***graph*** consists of a set of objects $V = \{v_1, v_2, \ldots.\}$ called vertices connected to each other through a set $E = \{e_1, e_2, \ldots.\}$ called edges. If *G* represents a graph, then *G* can be defined as an ordered pair of a set of vertices and a set of edges as $G = (V, E)$, [6].

Definition 2.2: Two vertices in a graph are said to be ***adjacent*** vertices if there is an edge joining the two vertices; the edge is said to be incident on the vertices. The number of edges incident on a vertex is v is called the ***degree*** of v and is denoted by $\deg(v)$, [6].

Definition 2.3: A ***walk*** is defined as a finite alternating sequence of vertices and edges beginning and ending with vertices, such that each edge is incident with the vertices preceding and following it. An open walk (distinct terminal vertices) in which no vertex appears more than once is called a ***path,*** [6].

Definition 2.4: A graph *G* is said to be ***connected*** if there exists a path between any two distinct vertices in the graph; otherwise, *G* is said to be ***disconnected,*** [6].

Definition 2.5: A ***directed graph*** (or a *digraph*) *G* consists of a set of vertices $V = \{v_1, v_2, \ldots.\}$ and a set of edges $E = \{e_1, e_2, \ldots.\}$ where each vertex is represented by a point and an edge by a line segment with an arrow. If u and v are vertices, then an edge is an unordered pair (u, v), while a directed edge is an ordered pair (u, v), with an arrow directed towards v from u, [6].

Definition 2.6: For a connected graph *G*, the ***distance*** $\boldsymbol{d(u,v)}$ between two of its vertices u and v is the length of the shortest path betweeen them, [6].

Definition 2.7: The ***eccentricity*** $\boldsymbol{E(v)}$ of a vertex v in a graph *G* is the maximum distance between v and all other vertices in *G*. That is,

$$E(v) = \max_{v_i \in G} d(v_i, v_j)$$

A vertex with minimum eccentricity is called the ***center*** of *G* and the eccentricity of the center is called the ***radius*** of *G*, [6].

Definition 2.8: A ***tree*** is a connected graph *G* without any circuit, that is, there is one and only one path between every pair of vertices of *G*. A tree with n vertices have $n - 1$ edges. A ***rooted tree*** is a tree with a special vertex, called the root, that serves as the starting point for all other vertices. A ***rooted binary tree*** is a rooted tree in which every vertex has at most two vertices connected to it a level below, [6].

Definition 2.9: For a graph $G = (V, E)$, a ***dominating set*** is a subset *S* of *V* such that every vertex not in *S* is adjacent to at least one vertex of *S*. The ***domination number*** $\boldsymbol{\sigma(G)}$ is the number of vertices in the smallest dominating set for *G*, [6].

Definition 2.10: The ***adjacency matrix*** $\boldsymbol{A(G)}$ of a graph *G* is the integer matrix with rows and columns indexed by the vertices of *G*, such that the uv-entry of $A(G)$ is equal to the number of edges from u to v which is usually 0 or 1, [6].

Definition 2.11: The ***common neighborhood*** of a graph is a measure of the reliability of a graph. It gives the expected number of vertices to constitute a transitive neighborhood between a randomly chosen pair of vertices that are non-adjacent, [7].

3 NM Graph is a Binary Tree

An NM graph consists of a person who recruits two persons – one left and one right. Both left and right participant again recruits two persons each, who in turn recruit two person each, and so on. So, a typical NM graph looks like (Figure 1).

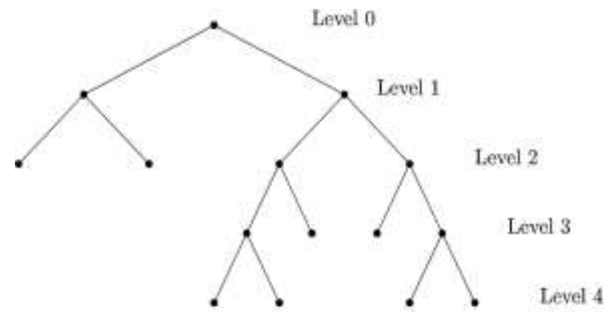

Fig. 1: A 4 level 15 vertex binary tree

Clearly, the above structure is a tree. Moreover, it is a binary tree because there is exactly one vertex of degree 2 and the remaining vertices have degree either 1 or 3. The vertex with degree 2 serves as the root and so NM graph is a rooted binary tree. This root of the NM graph is called the grand upline of the network.

4 Number of Vertices in a NM Graph

A NM graph is a rooted binary tree and we start at the root, i.e., the vertex with degree 2. There is only one vertex at level 0, at most 2 vertices at level 1, at most 4 vertices at level 2 and so on. Therefore, the maximum number of vertices up to the level k follows the GP series:

$$2^0, 2^1, 2^2, 2^3, \dots\dots, 2^k$$

And the maximum number of vertices / size of the network up to level k is given by:

$$2^0 + 2^1 + 2^2 + 2^3 + \cdots + 2^k$$

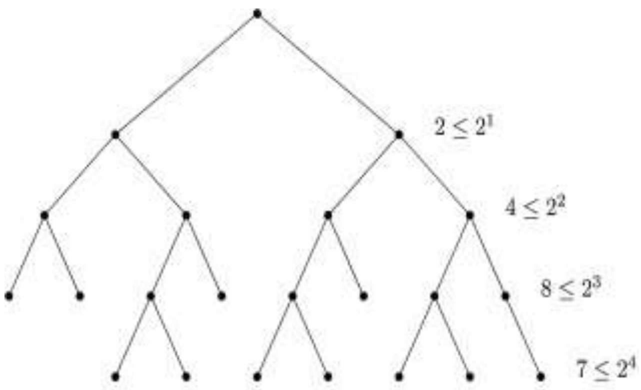

Fig. 2: A 4-level NM graph

If n is the number of vertices at level k, then clearly $n \leq 2^k$ (Figure 2).

5 Dependent and Independent Network

When discussing dependence and independence in the context of an NM graph, we are actually referring to the connection to profitability within the network. The profitability of one networker may or may not be influenced by others in the network. When the profitability of a person or network is affected by another person or network within the same system, we say the two networks are dependent; otherwise, they are independent. The parent network is always dependent on the sub-networks. The benefit of the parent network may or may not benefit his sub-networks, but the benefit of sub-networks always contributes to the profitability of the parent network. Consider the following NM graph (Figure 3).

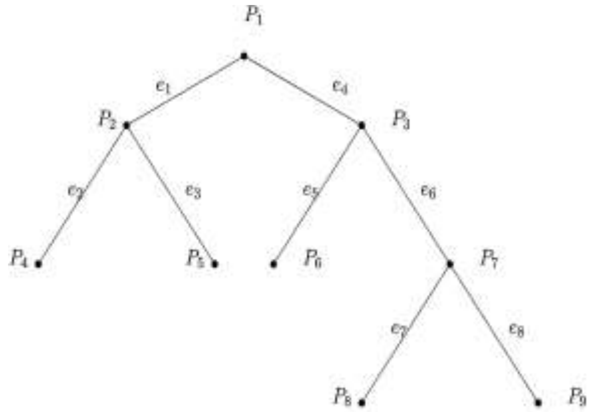

Fig. 3: A 3-level NM graph

Here, the network with root P_2 is independent of the network with root P_3 because the profit generated from the former network is not shared by the latter network. Whereas the network with root P_1 is not independent of the network of P_6, P_1 being the parent network is always dependent on his sub-networks for profit to be generated.

In a graphical context, this can be put in the following way:

Two networks are independent if their respective path toward the nearest common parent network do not have a common edge.

In the above structure, consider the network with roots P_5 and P_6. The nearest common parent network for both of them is root P_1. The path from P_5 to P_1 is $P_5 e_3 P_2 e_1 P_1$ and that of P_6 to P_1 is $P_6 e_5 P_3 e_4 P_1$. The two paths do not have a common edge and so they are independent. But the networks with roots P_3 and P_9, with P_1 as the nearest common parent network, have a common edge e_4. Hence, the two networks are dependent.

6 Strength of a Network: Strong and Weak Network

The strength of a network is measured by the consistent flow of profit in the network. This profit flow occurs only when both sides of the network are active, meaning continuous transactions are happening on both ends. Suppose, a person P_1 recruits P_2 and P_3. P_1 may not stop at that and go on recruiting P_4, P_5, P_6 and so on. Since P_1 can place only two persons below him, he places P_4, P_5, P_6 below P_2 and P_3. Suppose the graph structure looks like (Figure 4):

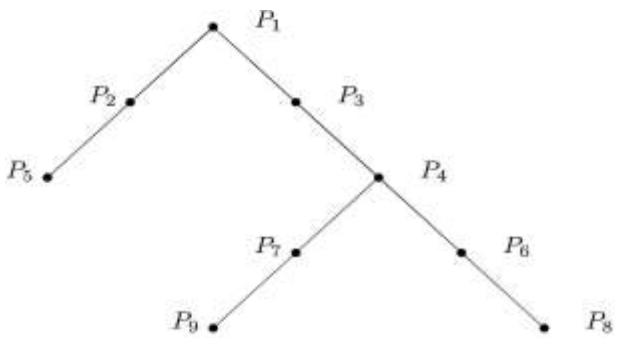

Fig. 4: A 4-level 9 vertex NM graph

Now, P_5 refuses to work and so thus P_1. P_4 being an active member, goes on recruiting people, so thus his downlines. In such a scenario, the main root P_1 stops receiving any profit, while root P_4 of his sub-network continues earning profit. In such cases, we say that the network with root P_1 is weak and the network with node P_4 is strong. It may be noted here that a sub-network may be stronger than the main network. The sub-network is a subgraph of the NM graph.

To summarize, the strength of a network in comparison to another network (either a sub-network or another independent network) can be measured by the following factors:

1. The number of minimum active recruits at each level is 2, provided both the networks share the same number of levels. The greater the number of recruits at each level, the stronger the network becomes.
2. If the number of recruits at each level is the same for the two networks, then the network with a greater number of levels is the stronger one among the two.

In conclusion, having more vertices in one network (graph) compared to another (subgraph or different graph) does not necessarily mean that the first network is stronger than the second.

7 Measure of Profitability

A networker earns a check/profit if there is a transaction of goods/services at both ends, i.e., left and right. A network might get stronger on one side, while the other side may be weak. In such cases, the flow of profit remains limited or sometimes even nil. Consider the NM graph (Figure 5):

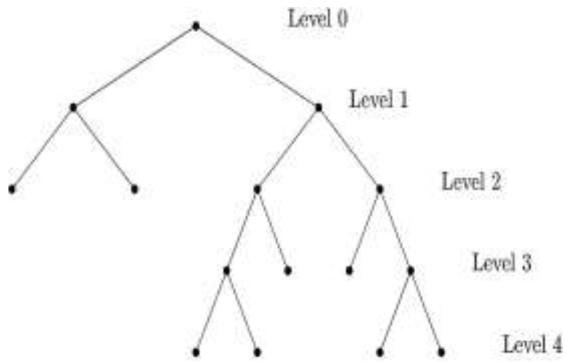

Fig. 5: A 4-level 15 vertex NM graph

In the above structure, level 1 consists of two vertices which is the maximum possible number of vertices at that level. Hence profit will be generated. At level 2, the transaction happens both on the left as well as the right side, so profit is generated for the main root. At level 3, only the right side of the network is functioning while the left side is shut down. So, there shall be no flow of profit towards the root. However, if we consider the sub-network at level 2, there is still a flow of profit as both the left and right leg of the root are active.

Consider an NM structure where a person earns profit or commission if there is a transaction of at least x amount on both sides (left and right), with each recruited member making a transaction of x amount or more. It is important to note that the commission earned by a person is directly proportional to the total transactions occurring within their network. Let us consider the following two networks (Figure 6 and Figure 7).

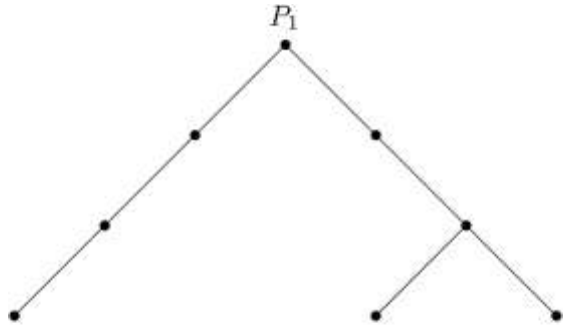

Fig. 6: A 3-level NM graph showing 3 levels of active participation under P_1

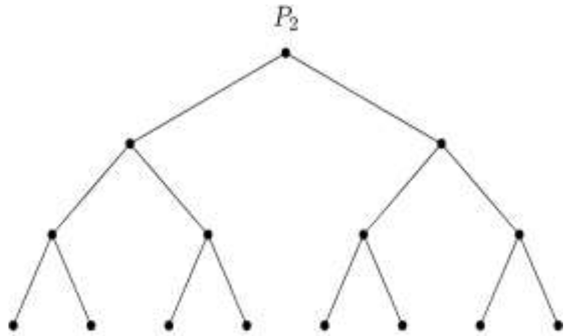

Fig. 7: A 3-level NM graph showing 3 levels of active participation under P_2, with more vertices(recruits) as compared to P_1

In the first scenario (Figure 6), the amount generated at level 1 is 2x, and the same applies to the second network (Figure 7). However, at level 2, the first network generates 2x, while the second network generates 4x. This clearly shows that the second networker earns more profit than the first one, even though each level has a minimum of two recruits and the number of levels is the same in both cases.

To summarize, there will be a continuous flow of profit in a network if there are at least 2 active persons at each level. Profit is maximum if the number of active persons at level k is 2^k.

Suppose at level k, the number of recruits be t and $t \in [2, 2^k]$, t being a positive integer. Then at level k,

$$Profitability \propto t$$

i.e.,

$$Profit = kt$$

k being a constant.

8 Assignment Problem

An assignment problem is a special type of linear programming problem where the objective is to minimize the cost or time of completing a number of jobs by a number of persons. More on this topic can be found in [8] and [9].

A person when he starts a NM business, receives some stakes that generate money over the course of time. Suppose a networker has 4 stakes and each stake can generate a potential profitability as follows (Table 1):.

Table 1. Stakes and Profitability of a Networker

Stake	**1**	**2**	**3**	**4**
Potential Profitability	**x**	**x**	**x**	**x**

The networker can earn additional profitability by placing the stakes on his downline members as suited. Now the goal is to maximize profit. So, the networker thoughtfully places the stakes on his downlines that have built a strong network of his own. Suppose there are 6 downline members and the size of their respective networks are as follows (Table 2).

Table 2. Members and size of their respective network

Downline member	**A**	**B**	**C**	**D**	**E**	**F**
Size of network	y_1	y_2	y_3	y_4	y_5	y_6

This is a typical case of an assignment problem where the objective is to find the optimum allocation of a number of tasks to an equal number of facilities. In our case, the problem is an unbalanced one since the number of stakes and the number of downlines is different. Thus, the problem is to first determine the effectiveness matrix, convert the maximization problem to a minimization problem, and then find the solution to the minimization problem by Hungarian Method, [8].

9 Digraph and Adjacency Matrix

An NM graph is also a digraph when the connections between the persons/vertices are represented by directed edges. Consider the NM graph (Figure 8):

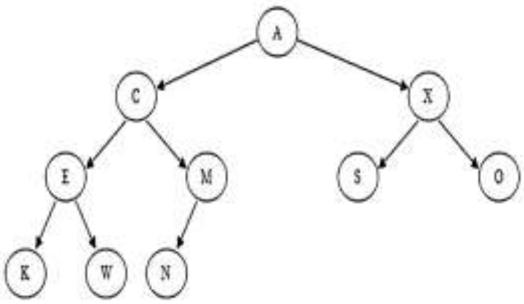

Fig. 8: A NM digraph

The directed edge from A to C means A is upline to C and so on.

We are already familiar with the contribution of linear algebra to the study of graph structures. One such aspect of linear algebra is the study of the adjacency matrix of a graph. The adjacency matrix $X = [x_{ij}]$ of a digraph G of order n is a $(0, 1)$ matrix of order n and defined as:

$$x_{ij} = 1, \text{ if there is an edge directed from } i^{th} \text{ vertex to } j^{th}$$
$$= 0, \text{ otherwise}$$

The adjacency matrix of the above digraph is:

$$X = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The following observations can be made from the adjacency matrix of a digraph:

The sum of each row equals the out-degree of the corresponding vertex, and the sum of each column equals the in-degree of the corresponding vertex, [10]. This clearly brings out the genealogy of the network.

10 Eccentricity

Let us look at some of the NM structures Figure 9, Figure 10, Figure 11 and Figure 12. The vertices are labeled as per their respective eccentricities.

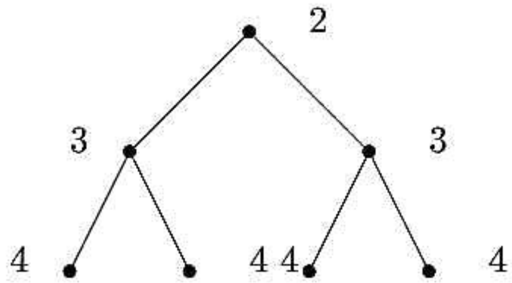

Fig. 9: A 2-level NM graph with 7 vertices labeled as per their respective eccentricities

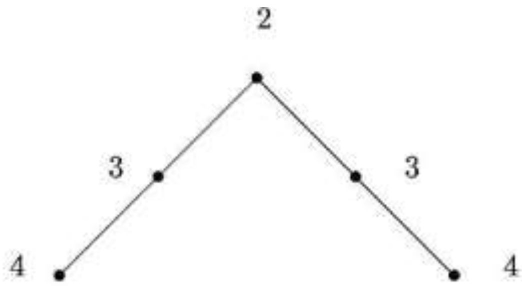

Fig. 10: A 2-level NM graph with 5 vertices labeled as per their respective eccentricities

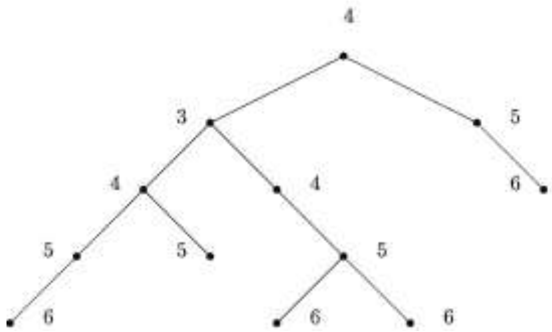

Fig. 11: A 4-level NM graph with 12 vertices labeled as per their respective eccentricities

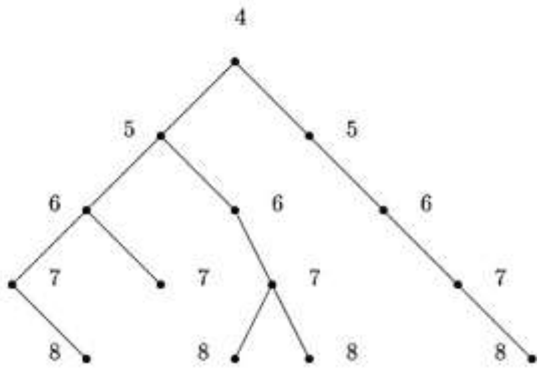

Fig. 12: A 4-level NM graph with 14 vertices labeled as per their respective eccentricities

If we look at these structures, we can clearly take note of the person earning maximum profitability in each network and those persons represent the vertices with minimum eccentricity. In the first scenario (Figure 9), the parent network with an eccentricity of 2 has two levels of active participation beneath it. Meanwhile, networks with eccentricities of 3 and 4 have only one and zero levels of participation, respectively. The same pattern is observed in the second scenario (Figure 10). Similarly, in the third scenario (Figure 11), the parent network with an eccentricity of 4 has just two levels of active participation, whereas the sub-network with an eccentricity of 3 at level two has three levels of active participation. As a result, in this case, the sub-network with an eccentricity of 3 is generating more profit than the parent network.

More formally, in terms of the NM graph, the center represents the vertex with the strongest network base or in other words the person earning maximum profitability. The radius represents the number of levels the strongest member maintains.

11 Domination Number

If we wish to associate ourselves with a network marketing entity, we would like the structure to be in the most stable form. There are a number of parameters to measure the stability of a network and one of these is the common neighbourhood measure of a graph. In [7], the authors discuss measures that evaluate the neighborhoods of all pairs of vertices in any connected graph. They state that when comparing two graph models, the more stable one is the graph with the larger common neighborhood. The authors also prove that for connected graphs G_1 and G_2, if the dominating number of G_1 is greater than that of G_2, then the common neighborhood of G_1 is smaller than that of G_2. Let us take two NM structures as shown below (Figure 13 and Figure 14):

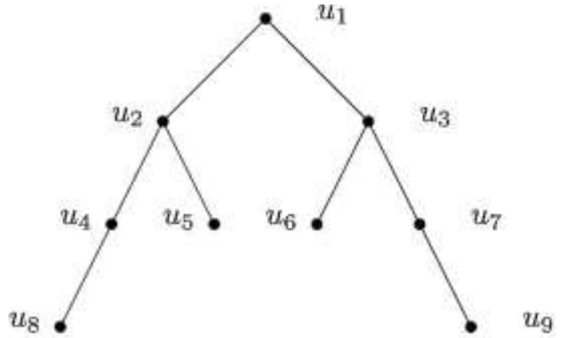

Fig. 13: A 3-level 9 vertex NM graph with domination number 4

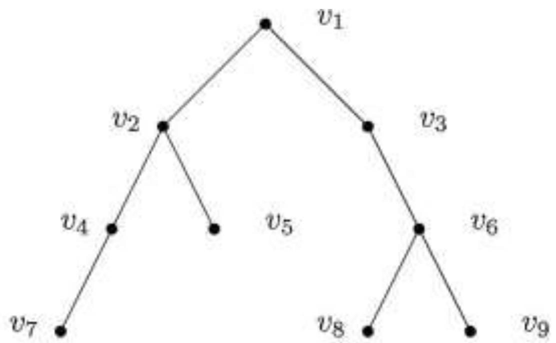

Fig. 14: A 3-level 9 vertex NM graph with domination number 3

The minimum dominating set for the first structure (Figure 13) is $S_1 = \{u_2, u_3, u_8, u_9\}$ and that of the other structure (Figure 14) is $S_2 = \{v_2, v_6, v_7\}$. Therefore, the domination number of the two structures are 4 and 3 respectively. So, we can say that the second NM structure is more stable out of the two.

12 Conclusion

In this paper, we establish a connection between graph theory and network marketing structures. We observe that a network marketing structure, referred to as an NM graph, can be represented as a rooted binary tree. We introduce terms such as dependent and independent networks, as well as strong and weak networks. Two networks are considered independent if the paths from their respective roots to their nearest common parent network share no common edge. The strength of a network is assessed by the consistent flow of profit within it, and profitability is directly proportional to the number of recruits at each level, assuming there are at least two recruits per level. The goal of maximizing profit by assigning different stakes to different individuals within an NM structure is a classic example of an assignment problem. The genealogy of a network can be determined by analyzing the adjacency matrix of the NM digraph. Graph parameters, such as eccentricity, indicate the most influential person in the network who earns the maximum profit, while the domination number serves as a measure of the NM graph's stability.

13 Limitations and Future Scope

Our study focuses on a single network marketing entity, but there are numerous such structures operating in today's world, each with its own unique approach. In our model, we assume that a person can place only two individuals directly under the—

one on the right and one on the left—resulting in profit generation. If they recruit more than two people, the additional recruits must be placed under those already recruited. However, this isn't the case for all network marketing structures. Some allow a person to directly recruit and place more than two individuals under them. Additionally, our study is limited to a small network, whereas in reality, these networks can be vast, making it challenging to calculate mathematical or graphical parameters. It would be interesting to explore how these parameters change or remain consistent when applied to other network marketing structures in the market. There may be better alternatives, and we are open to suggestions.

References:

[1] P. V. Marsden, The Sociology of James S. Coleman, *Annual Review of Sociology*, Vol.31, 2005, pp.1-24. https://doi.org/10.1146/annurev.soc.31.041304.122209.

[2] K. B. Vidhyadharani, M. Ramya, Graph Theory in Social Media Related to WhatsApp, *International Journal for Research in Applied Science and Engineering Technology*, Vol .10, No.11, 2022, pp. 1194-1199. https://doi.org/10.22214/ijraset.2022.47566.

[3] K. J. Barman, K. Patra, Exploring the Structure of Social Media Network with the Concepts of Graph Theory, *International Journal of Engineering and Advanced Technology*, Vol.9, No.4, 2020, pp.1136-1141. http://www.doi.org/10.35940/ijeat.D8306.049420.

[4] B. Mondal, K. De, A Comparison of Graph Theory and Social Media Network, *International Journal of Current Advanced Research*, Vol.7, No.2(A), 2018, pp.9649-9652.

[5] P. J. Carrington, J. Scott, S. Wasserman, *Models and methods in social network analysis*, Cambridge, New York: Cambridge University Press , 2005.

[6] N. Deo, *Graph Theory with Applications to Engineering and Computer Science*, Prentice-Hall, India, 1979.

[7] P. Dundar, A. Aytac, E. Kilie, Common Neighbourhood of a Graph, *Boletim da Sociedade Paranaense de Matematica*, Vol.35, No.7, 2017, pp.23-32. http://dx.doi.org/10.5269/bspm.v35i1.22464.

[8] R. K. Gupta, *Krishna's Linear Programming*, KRISHNA Prakashan Media (P) Ltd. 1, 2009.

[9] M. S. Bazaraa, J. J. Jarvis, H. D. Sherali, *Linear Programming and Network Flows*, Wiley-India Edition, 2008.

[10] F. Harary, *Graph Theory*, Narosa Publishing House, 2001.

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The authors equally contributed to the present research, at all stages from the formulation of the problem to the final findings and solution.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

No funding was received for conducting this study.

Conflict of Interest

The authors have no conflicts of interest to declare.