A New Approach for Existence of Strong Trace of Entropy Solution for Degenerate Parabolic Equation

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Abstract: In this note, we propose a new proof of existence of L^∞ strong trace of entropy solution for multidimensional degenerate parabolic-hyperbolic equation in a bounded domain $\Omega \subset \mathbb{R}^\ell$ reached by L^1 convergence. The proof is based on using of concept of quasi solution.

Key-Words: Degenerate parabolic equation; Conservation laws; Boundary value problem; Entropy formulation; Quasi solution; Strong trace.

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1 Introduction

This article is devoted to a new method for the proof of existence of strong trace of entropy solution at the boundary of Ω reached by L^1 convergence for the degenerate parabolic-hyperbolic equation of the type:

$$u_t + \operatorname{div}(f(u) - \nabla\phi(u)) = 0 \text{ in } Q \qquad \text{(E)}$$

where $Q = (0,T)\times\Omega$, and the domain Ω is a bounded part of $\mathbb{R}^\ell$, $\ell \geq 1$ and $\Sigma = (0,T) \times \partial\Omega$. We assume that $\partial\Omega$ is regular (the meaning of this regularity will be specified later). The function $\phi(u)$ is such that if the unknown value u is less than a critical value u_c, i.e., really over an interval of solution values then the equation (E) degenerates to scalar conservation law [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12]. As usual, we only deal with entropy solutions understood in the sense that we select only the physically relevant discontinuous solution [13], [14]. It should also be noted that the boundary condition does not always provide the most natural setting for conservation laws on bounded domains.

Let us take the following linear transport equation: $u_t + au_x = 0$ in $(0,1) \times (0,1)$ with initial condition $u(0,x) = sin(\frac{\alpha}{x})$. Then we have that if $a = 0$, $u(t,x) = u(0,x)$ is obviously a solution and it is not defined on $(0,1) \times \{0\}$. We need a suitable condition on the boundary $(0,T) \times \partial\Omega$. The authors in [15], have first studied this issue for the initial boundary value problem of scalar conservation laws with the assumption $u \in BV(Q)$ and they proposed an appropriate entropy boundary condition.

In [16], the author has extended their result without using the bounded variation of solutions.

The existence of a strong trace at $t = 0$ does not pose a problem. Hence, putting the initial value in the entropy inequality is exactly equivalent to assuming the existence of a strong trace at $t = 0$ i.e. reached by a strong topology (without oscillations), [17].

The question of strong traces arose initially in the context of limit of hyperbolic relaxation towards a scalar conservation law. It involves the introduction of blow-up techniques and the use of the theory of kinetic formulation with as pioneers, [18], which allows using the so-called averaging lemmas. This blow-up method is inherited from techniques widely used for parabolic equations.

The authors in [19], proved existence of strong traces for entropy of $u_t + f(u)_x = 0$ on initial line $t = 0$ under the condition that the flux function $f(u) \in C^1(\mathbb{R})$ is not affine on non-degenerate intervals. Using compensated compactness techniques with a slightly different hypothesis of non-degenerate flux. Those results can be seen as a regularization effect at the boundary induced by the non-degeneracy of flux.

Always for $u_t + f(u)_x = 0$, in [17], the authors proved existence of strong traces for entropy solution on the boundary $\partial\Omega$ of a plane domain $\Omega \subset \mathbb{R}^2$ without non-degeneracy restrictions but under the regularity assumption $f(u) \in C^2(\mathbb{R})$.

In multidimensional scalar conservation laws case, existence of the strong traces for entropy solution was later proved by [20], under the assumptions that the flux $f(u) \in C^3(\mathbb{R}, \mathbb{R}^\ell)$ and satisfies the non-degeneracy condition in the sense that for a $\mathbb{R}^\ell$ vector $\xi \neq 0$ the function $u \to (\xi, f'(u))$ is not constant on sets of positive Lebesgue measure.

In [21], the author proved existence of strong traces for normal components of the entropy fluxes on the boundary of the domain without non-degeneracy conditions on the flux. Besides, with non-degeneracy conditions on the flux, this strong trace is trace of entropy solution. Hence, the author used the technique of $H-$ measure and induction on the spatial dimension and defined the notion of quasi-solution that ensures existence of strong trace of solution. Recall that the concept of $H-$measure was first initiated by [22]. The existence of weak trace for normal component of the flux is know from result by [23].

Classical trace results for a parabolic type equation appear in the literature, but the difficulty in the degenerate case lies in the mixture of the two types, parabolic and hyperbolic: [24], [25]. Recall also that a general result of existence of strong trace of solution to degenerate parabolic equation has been proved by [26]. To define traces on the boundary, the author used the framework of the "regular deformable Lipschitz boundary". To state their main result, they introduce a new function, $\chi-$function which comes from the theory of kinetic formulation.

In this paper, we propose another way to obtain the result of strong trace. We proceed as follows: first we set T a regular cut-off function (in the hyperbolic zone) and justify that T is a quasi-solution of the hyperbolic operator $v \longmapsto v_t + \mathrm{div} f(v)$ under non linearity assumption on f and ϕ. In [27], for example, the authors assume that the couple $(f(.), \phi(.))$ is *non-degenerate* in the sense that the functions $\lambda \longmapsto \sum_{i=1}^{\ell} \xi_i f_i(\lambda)$ are not "affine" on the non-degenerate sub intervals where ϕ is constant. After, we impose that $\phi(u^D) \in L^2(0,T,H^1(\Omega))$ (here $u = u^D$ on Σ is the Dirichlet boundary condition) which guarantees the existence of strong trace in the non degenerate parabolic zone. By introducing a bijective function Ψ, we prove global existence of strong trace. Moreover, we propose an application for general boundaries conditions (zeroflux, Robin and Dirichlet).

The paper is organized in five parts. In section 2, we give definition and properties of strong trace. We recall in section 3 the notion of quasi solution. Section 4 is devoted to the proof of existence of strong trace. In the last section, we give some applications for boundary value problems.

2 Definition and Properties of Strong Trace

We assume in this paper that the couple $(f(.), \phi(.))$ is *non-degenerate*. Let us give the definition of strong trace in the L^1 sense and some properties.

Definition 2.1 *Let $\Omega \subset \mathbb{R}^{\ell}$ with Lipschitz boundary. A function $u \in L^\infty(\Omega)$ possess a strong trace $\gamma u \in L^\infty(\partial\Omega)$, at boundary $\partial\Omega$ if for every compact set $K \subset\subset \partial\Omega$*

$$ess \lim_{s\to 0} \int_K |u(\theta(s,\hat{x})) - \gamma u(\hat{x})| d\mathcal{H}^{\ell-1}(\hat{x}) = 0. \quad (1)$$

where $\mathcal{H}^{\ell-1}$ is the $\ell-$dimensional Haussdorf measure and θ is bi-Lipschitz homomorphism such that $\theta(0,.) = Id$.

Remark 2.2 *Some authors state the framework of $\mathcal{C}^1$ regular domains, but it can be generalize a Lipschitz boundary.*

Lemma 2.3 *Let $u \in L^\infty(\Omega)$ (respectively $v \in L^\infty(\Omega)$) such that the strong trace $\gamma u \in L^\infty(\partial\Omega)$ (respectively $\gamma v \in L^\infty(\partial\Omega)$) exists. Then, $\gamma(u+v)$ exists. Moreover $\gamma(u+v) = \gamma(u) + \gamma(v)$.*

Proof. This is a direct consequence of Definition 2.1 since $\gamma(u+v) = \gamma(u)+\gamma(v)$ satisfies the above limit.

Lemma 2.4 *Let $u \in L^\infty(\Omega)$ such that the strong trace $\gamma u \in L^\infty(\partial\Omega)$ exists. For all continuous function $G : \mathbb{R} \longrightarrow \mathbb{R}$ then $\gamma(G(u))$ exists and $\gamma(G(u)) = G(\gamma u)$.*

Proof. Note first that this result is a direct consequence of Definition 2.1 if G is a Lipschitz-continuous function. Then, the lemma holds by using a sequence (G_n) of Lipschitz-continuous functions that converges uniformly to G on $[-\|u\|_\infty, \|u\|_\infty]$.

Lemma 2.5 *Assume that the sequence $(\Psi_h)_h$ is such that :*

$$||\Psi_h||_{L^2(0,T;H^1(\Omega))} \leq \textit{cst and } \Psi_h \to \Psi \textit{ in } L^2(Q).$$

Then $\gamma\Psi_h \to \gamma\Psi$ in $L^2((0,T) \times \partial\Omega)$.

For the proof see, [28].

3 Notion of Quasi-solution

We denote by $M(Q)$ the set of Radon measures on Q, *i.e.* the dual space of $\mathcal{C}(Q)$:

$$\nu \in M(Q) \text{ if } \forall K \subset\subset Q, \ \ |\nu|(K) < \infty$$

and by $M_{b,\partial\Omega}$ the set of Radon measures finite up to the boundary of Ω *i.e*

$$\nu \in M_{b,\partial\Omega} \text{ if } \sup_{\omega \subset\subset \Omega} |\nu|((t,s)\times\omega) < \infty,$$
$$\forall 0 < t < s < T.$$

Notice that for $\nu \in M_{b,\partial\Omega}$, $\sup_{0<t<s<T.} |\nu|((t,s)\times\omega)$ can be infinite.

Now, consider a measure $\mu \in M_{b,\partial\Omega}$ and the hyperbolic equation:

$$v_t + \text{div}\tilde{f}(v) = -\mu \text{ in } Q =]0,T[\times\Omega. \quad (2)$$

We shall state the notion of quasi-solution for the operator $v \longmapsto v_t + \text{div}\tilde{f}(v)$, the author proves, for each k in some dense set of $\mathbb{R}$, the existence of a strong trace of the normal component of Kruzhkov's entropy vector flux $\mathcal{F}(v,k) := sign(v-k)(\tilde{f}(v)-\tilde{f}(k)).\eta$ which can be written by $\mathcal{F}(\tilde{v},k) := sign(\tilde{v}-k)(\tilde{f}(\tilde{v})-\tilde{f}(k)).\eta$, for any quasi-solution v. Moreover, in the case where $\tilde{f}$ is not constant in a non-degenerate interval, $\tilde{v} \in L^\infty(\Sigma)$ is unique and it is the strong trace of this quasi-solution v.

Definition 3.1 *A bounded measurable function $v \in [0, ||u||_\infty]$ is called quasi-solution of*

$$v_t + div\tilde{f}(v) = 0 \quad (3)$$

if for $k \in [0, ||u||_\infty]$

$$\partial|v-k| + div\bigg(\mathcal{F}(u,k)\bigg) = -\mu_k \text{ in } D'(Q), \quad (4)$$

where $\mu_k \in M_{b,\partial\Omega}(Q)$.

Remark 3.2 *i) We precise that a function satisfying* (4) *is not a priori a weak solution of* (2)*. In fact from* (4) *with $k = 0$ it follows that*

$$v_t + div\tilde{f}(v) = -\mu \text{ in } D'(Q)$$

with $\mu \in M_{b,\partial\Omega}(Q)$.

ii) The class of quasi-solutions includes entropy solutions as well as entropy subsolutions and entropy supersolutions. Remark that if v is an entropy sub- and super-solution, then

$$\partial_t(v-k)^+ + div\bigg(\mathcal{F}^+(v,a)\bigg) = -\mu_k^+ \text{ in } D'(Q), \quad (5)$$

$$\partial_t(v-k)^- + div\bigg(\mathcal{F}^-(v,a)\bigg) = -\mu_k^- \text{ in } D'(Q), \quad (6)$$

with $\mathcal{F}^\pm := sign^\pm(v-k)(\tilde{f}(v)-\tilde{f}(k)), \mu_k^\pm \in M(\Omega)$ not necessarily in $M_{b,\partial\Omega}$.

iii) Notice that due to (5) *and* (6)*, if v is quasi-solution then for each $a, b \in \mathbb{R}$*

$$T_{[a,b]}(v)_t + div\tilde{f}(T_{[a,b]}(v)) = -\mu_{a,b} \text{ in } D'(Q), \quad (7)$$

where $\mu_{a,b} \in M_{b,\partial\Omega}(Q)$ and is a cut-off function defined as:

$$T_{[a,b]}(r) = \begin{cases} a & if\, r < a, \\ r & if\, a \le r \le b \\ b & if\, r > b. \end{cases} \quad (8)$$

Theorem 3.3 *Suppose a function $u(t,x)$ is a quasi solution of* (3) *and (f,ϕ) non degenerate. Then there exists a function $\gamma u \in L^\infty(\Sigma)$ such that γu is the strong trace of entropy solution u of at the boundary Σ.*

To establish that a given function v is a quasi-solution for the operator $v \longmapsto v_t + \text{div}\tilde{f}(v)$, we will often use the following version of the argument:

Lemma 3.4 *Let $\mathcal{F} \in L^\infty(Q)$ and assume that $div\mathcal{F} = \mu + \nu$, where μ is a measure finite up to the boundary and ν is a positive measure. Then ν is also finite up to the boundary. In particular, $\mathcal{F}$ is a divergence measure fields on Q.*

Remark 3.5 *Divergence-measure fields are extended vector fields, including vector fields in L^p and vector-valued Radon measures, whose divergences are Radon measures.*

Proof of Lemma 3.4. Consider $\{\xi_\delta\}_{\delta>0}$ a boundary layer sequence *i.e.* ξ_δ is a sequence of $\mathcal{C}^1(\Omega) \cap \mathcal{C}(\overline{\Omega})$ such that $\lim_{\delta\to 0} \xi_\delta = 1$ pointwise in Ω, $0 \le \xi_\delta \le 1$, $||\nabla\xi_\delta||_{L^1} < cst$ and $\xi_\delta = 0$ on $\partial\Omega$. Then, thanks to Lebesgue's theorem, we have

$$\begin{aligned}\nu(Q) &= \lim_{\delta\to 0} \langle \mu, \xi_\delta \rangle - \int_0^T\int_\Omega \mathcal{F}.\nabla\xi_\delta dxdt \\ &\le |\mu|(Q) + ||\mathcal{F}||_\infty ||\nabla\xi_\delta||_{L^1} \le C.\end{aligned}$$

4 Existence of Strong Trace

Consider the following equation obtained by approximating $\phi(u)$ by $\phi_\epsilon(u^\epsilon) = \phi(u^\epsilon) + \epsilon Id(u^\epsilon)$ for each $\epsilon > 0$.

$$u_t^\epsilon - \text{div}(f(u^\epsilon) - \nabla\phi_\epsilon(u^\epsilon)) = 0 \text{ in } Q = (0,T)\times\Omega. \quad (\text{E}_\epsilon)$$

The main result in this paper is the following theorem:

Theorem 4.1 *(Existence of strong trace for vanishing viscosity limits) Assume that $u = \lim_{\epsilon\to 0} u^\epsilon$ (in the a.e. sense) and the following estimates hold*

$$||u^\epsilon||_{L^\infty} \le C; \quad (9)$$

$$||\nabla\phi(u^\epsilon)||_{L^2(Q)} \le C \quad (10)$$

$$||\epsilon \nabla u^\epsilon||_{L^2(Q)} \leq C. \quad (11)$$

Then, u is local entropy solution and there exists a strong trace $\gamma u \in L^\infty(\Sigma)$ on the boundary $\Sigma = (0,T) \times \partial\Omega$.

Proof. Consider $T \in C^2(\mathbb{R})$ such that: $T(x) = 0$ if $x \leq 0$, increasing on $[0, u_c]$ and $T(x) = u_c$ if $x \geq u_c$ and denote by $v = T(u)$. Taking in (E$_\epsilon$), $T'(u^\epsilon)\xi(t)\psi(x)$ as a test function, we find

$$\begin{aligned}
&\int_0^T \left\langle u_t^\epsilon, T'(u^\epsilon)\psi(x)\right\rangle \xi(t)dt \\
&- \int_0^T\int_\Omega f(u^\epsilon)\psi(x)\xi(t).\nabla T'(u^\epsilon)dxdt \\
&- \int_0^T\int_\Omega f(u^\epsilon)T'(u^\epsilon)\xi(t).\nabla\psi(x)dxdt \\
&+ \int_0^T\int_\Omega \nabla\phi_\epsilon(u^\epsilon)T'(u^\epsilon)\xi(t).\nabla\psi(x)dxdt \\
&+ \int_0^T\int_\Omega \nabla\phi_\epsilon(u^\epsilon)\psi\xi.\nabla T'(u^\epsilon)dxdt = 0. \quad (12)
\end{aligned}$$

By using chain rule [29], the first integral of (12) gives

$$\begin{aligned}
A &= \int_0^T \left\langle u_t^\epsilon, T'(u^\epsilon)\psi(x)\right\rangle \xi(t)dt \\
&= \left\langle \frac{\partial}{\partial t}T(u^\epsilon); \psi\otimes\xi\right\rangle_{\mathcal{D}',\mathcal{D}}. \quad (13)
\end{aligned}$$

The second integral and third integral of (12) gives

$$\begin{aligned}
B &= -\int_0^T\int_\Omega f(u^\epsilon)\psi(x)\xi(t).\nabla T'(u^\epsilon)dxdt \\
&- \int_0^T\int_\Omega f(u^\epsilon)T'(u^\epsilon)\xi.\nabla\psi dxdt \\
&= -\int_0^T\int_\Omega f(u^\epsilon)T''(u^\epsilon).\nabla u^\epsilon\psi\xi dxdt \\
&- \int_0^T\int_\Omega f(u^\epsilon)T'(u^\epsilon)\xi.\nabla\psi dxdt \\
&= -\int_0^T\int_\Omega \nabla.\left(\int_0^{u^\epsilon} f(s)T''(s)ds\right)\xi\psi dxdt \\
&- \int_0^T\int_\Omega f(u^\epsilon)T'(u^\epsilon)\xi.\nabla\psi dxdt \quad (14) \\
&= \int_0^T\int_\Omega \nabla.\left(\int_0^{u^\epsilon} f'(s)T'(s)ds\right)\psi\xi dxdt \\
&= \int_0^T\int_\Omega \mathrm{div}\tilde{f}(T(u^\epsilon))\psi(x)\xi(t)dxdt \quad (15)
\end{aligned}$$

The two last integrals of (12) give

$$\begin{aligned}
E &= \int_0^T\int_\Omega \nabla\phi_\epsilon(u^\epsilon)T'(u^\epsilon)\xi.\nabla\psi dxdt \\
&+ \int_0^T\int_\Omega \nabla\phi_\epsilon(u^\epsilon)\psi.\xi.\nabla T'(u^\epsilon)dxdt \\
&= \epsilon\int_0^T \xi(t)\int_\Omega \nabla T(u^\epsilon).\nabla\psi dxdt \\
&+ \epsilon\int_0^T\int_\Omega T''(u^\epsilon)|\nabla u^\epsilon|^2\psi\xi dxdt \\
&= -\epsilon\langle\Delta T(u^\epsilon), \xi(t)\psi(x)\rangle \\
&+ \epsilon\int_0^T\int_\Omega T''(u^\epsilon)|\nabla u^\epsilon|^2\psi\xi dxdt. \quad (16)
\end{aligned}$$

Now adding (13), (14) and (16), we obtain:

$$\begin{aligned}
&\left\langle \frac{\partial}{\partial t}T(u^\epsilon); \psi\otimes\xi\right\rangle_{\mathcal{D}',\mathcal{D}} \\
&+ \int_0^T\int_\Omega \mathrm{div}\tilde{f}(T(u^\epsilon))\psi(x)\xi(t)dxdt \\
&- \epsilon\int_0^T\int_\Omega \Delta T(u^\epsilon)\xi(t)\psi(x)dxdt \\
&+ \epsilon\int_0^T\int_\Omega T''(u^\epsilon)|\nabla u^\epsilon|^2\psi\xi dxdt = 0. \quad (17)
\end{aligned}$$

We set $v^\epsilon = T(u^\epsilon)$, by density of $\mathcal{D}(0,T)\otimes\mathcal{D}(\Omega)$ in $\mathcal{D}(Q)$ then we have

$$v_t^\epsilon + \mathrm{div}\tilde{f}(v^\epsilon) = \epsilon\Delta v^\epsilon - \mu^\epsilon \text{ in } \mathcal{D}'(Q). \quad (18)$$

Where $\mu^\epsilon = \epsilon T''(u^\epsilon)|\nabla u^\epsilon|^2$ is an $L^1(Q)$ function thanks (9) we have assumed.
Take $\xi \in \mathcal{C}^\infty(Q)$ and consider $\eta = T'(u^\epsilon)sign_\beta^+(T(u^\epsilon) - a)\xi$ with $a \in [0, u_c]$. Then, there exists α in the same interval such that $a = T(\alpha)$ and for any $\xi \in \mathcal{C}_0^\infty(Q)$, (23) yields

$$\begin{aligned}
&\int_Q T''(u^\epsilon)sign_\beta^+(T(u^\epsilon) - a)\nabla\phi_\epsilon(u^\epsilon)\nabla u^\epsilon\xi dxdt \\
&+ \int_Q sign_\beta'^+(T(u^\epsilon) - a)|T'(u^\epsilon)|^2\nabla\phi_\epsilon(u^\epsilon)\nabla u^\epsilon\xi dxdt \\
&= \int_Q\int_\alpha^{u^\epsilon} T'(\sigma)sign_\beta^+(T(\sigma) - a)d\sigma\xi_t \\
&+ \int_Q\int_\alpha^{u^\epsilon} T'(\sigma)sign_\beta^+(T(\sigma) - a)f'(\sigma)d\sigma\nabla\xi \\
&- \int_Q \nabla\xi\nabla\int_\alpha^{u^\epsilon} T'(\sigma)sign_\beta^+(T(\sigma) - a)[\epsilon + \phi'(\sigma)]d\sigma
\end{aligned}$$

i.e., since $T'\phi' = 0$ and $v^\epsilon = T(u^\epsilon)$,

$$\begin{aligned}&\epsilon\int_Q T''(u^\epsilon)sign_\beta^+(v^\epsilon - a)|\nabla u^\epsilon|^2\xi dxdt\\&+\int_Q sign_\beta'^+(v^\epsilon - a)|T'(u^\epsilon)|^2|\nabla u^\epsilon|^2\xi dxdt\\&=\int_Q\int_\alpha^{u^\epsilon}T'(\sigma)sign_\beta^+(T(\sigma)-a)d\sigma\xi_t\\&+\int_Q\int_\alpha^{u^\epsilon}T'(\sigma)sign_\beta^+(T(\sigma)-a)f'(\sigma)d\sigma\nabla\xi\\&-\epsilon\int_Q\nabla\xi\nabla\int_\alpha^{u^\epsilon}T'(\sigma)sign_\beta^+(T(\sigma)-a)d\sigma dxdt\end{aligned}$$

and, since $T' = 0$ in $\mathbb{R}\backslash]0, u_c[$,

$$\begin{aligned}&\lim_{\beta\to 0}\epsilon\int_Q sign_\beta'^+(v^\epsilon - a)|\nabla v^\epsilon|^2\xi dxdt\\&+\epsilon\int_Q T''(u^\epsilon)sign^+(v^\epsilon - a)|\nabla u^\epsilon|^2\xi dxdt\\&=\int_Q\int_\alpha^{u^\epsilon}T'(\sigma)sign_\beta^+(T(\sigma)-a)d\sigma\xi_t\\&+\int_Q\int_\alpha^{T^{-1}(v^\epsilon)}T'(\sigma)sign_\beta^+(T(\sigma)-a)f'(\sigma)d\sigma\nabla\xi\\&-\epsilon\int_Q\nabla\xi\nabla\int_\alpha^{u^\epsilon}T'(\sigma)sign^+(T(\sigma)-a)d\sigma dxdt.\end{aligned}$$

Since $a = T(\alpha)$, one gets that

$$\begin{aligned}&\varpi_a^\epsilon(\xi)+\int_Q sign^+(v^\epsilon - a)\mu^\epsilon\xi dxdt\\&=\int_Q\int_{T(\alpha)}^{v^\epsilon}sign^+(\sigma-a)d\sigma\xi_t dxdt\\&+\int_Q\int_{T(\alpha)}^{v^\epsilon}sign^+(\sigma-a)f'(T^{-1}(\sigma))d\sigma\nabla\xi dxdt\\&-\epsilon\int_Q\nabla\xi\nabla\int_{T(\alpha)}^{v^\epsilon}sign^+(\sigma-a)d\sigma dxdt\\&=\int_Q sign^+(v^\epsilon - a)\int_{T(\alpha)}^{v^\epsilon}f'(T^{-1}(\sigma))d\sigma\nabla\xi dxdt\\&\int_Q(v^\epsilon - a)^+\xi_t - \epsilon\int_Q\nabla\xi\nabla(v^\epsilon - a)^+dxdt\end{aligned}$$

where

$$\varpi_a^\epsilon(\xi) := \epsilon\lim_{\beta\to 0}\int_Q sign_\beta'^+(v^\epsilon - a)|\nabla v^\epsilon|^2\xi dxdt.$$

Denoting by $\tilde{f}(t) = \int_0^{T^{-1}[\min(u_c,t^+)]}f'(\sigma)T'(\sigma)d\sigma$, for any $a \in [0, u_c]$ and any non-negative $\xi \in \mathcal{C}_0^\infty(Q)$, we obtain

$$\begin{aligned}&\int_Q sign^+(v^\epsilon - a)(\tilde{f}(v^\epsilon)-\tilde{f}(a)).\nabla\xi dxdt\\&-\int_Q\epsilon\nabla(v^\epsilon - a)^+.\nabla\xi dxdt+\int_Q(v^\epsilon - a)^+\xi_t dxdt\\&=\int_0^T\int_\Omega sign^+(v^\epsilon - a)\mu^\epsilon\xi dxdt+\varpi_a^\epsilon(\xi).\end{aligned}\tag{19}$$

If $a > u_c$, then both sides of the above equation are null.
If $a < 0$, then, since $\xi \in C_0^\infty(Q)$ and we have $sign_\beta'^+(v^\epsilon - a) = 0$ if $\beta < -a$,

$$\begin{aligned}&\int_Q(v^\epsilon - a)^+\xi_t + sign^+(v^\epsilon - a)(\tilde{f}(v^\epsilon)-\tilde{f}(a)).\nabla\xi\\&-\int_Q\epsilon\nabla(v^\epsilon - a)^+.\nabla\xi dxdt\\&=\int_Q v^\epsilon\xi_t+\tilde{f}(v^\epsilon).\nabla\xi-\epsilon\nabla v^\epsilon.\nabla\xi dxdt\\&=\int_0^T\int_\Omega\mu^\epsilon\xi dxdt\\&=\int_0^T\int_\Omega sign^+(v^\epsilon - a)\mu^\epsilon\xi dxdt+\varpi_a^\epsilon(\xi).\end{aligned}$$

Note that ϖ_a^ϵ exists, irrespective of the approximation of $sign_\beta^+$, and is non negative due to the fact that the limit of the other terms exists, and of course, it is a non-negative measure.
Now, using the convergence of u^ϵ to u and the continuity of the function T, we obtain firstly that

$$\lim_{\epsilon\to 0^+}J_\epsilon = J$$

where

$$\begin{aligned}J_\epsilon =&\int_Q(v^\epsilon - a)^+\xi_t+sign^+(v^\epsilon - a)\Big(\tilde{f}(v^\epsilon)-\tilde{f}(a)\Big).\nabla\xi\\&-\int_Q\epsilon\nabla(v^\epsilon - a)^+.\nabla\xi dxdt\\J =&\int_Q(v-a)^+\xi_t+sign^+(v-a)(\tilde{f}(v)-\tilde{f}(a)).\nabla\xi.\end{aligned}$$

Since $sign^+(v^\epsilon - a)\mu^\epsilon$ is bounded in $L^1(Q)$, up to a subsequence denoted in the same way, it converges weakly in the sense of the bounded measures to a bounded measure denoted by μ_a.

Thus, ϖ_a^ϵ converges in the sense of the distribution to a non negative distribution ϖ_a, *i.e.* a non negative Radon measure. Then,

$$\int_Q (v-a)^+\xi_t + sign^+(v-a)\left(\tilde{f}(v) - \tilde{f}(a)\right).\nabla\xi dxdt = \langle \mu_a, \xi\rangle + \langle \varpi_a;\xi\rangle \tag{20}$$

and, for $b > a$,

$$\int_Q (v-b)^+\xi_t + sign^+(v-b)\left(\tilde{f}(v) - \tilde{f}(b)\right).\nabla\xi dxdt = \langle \mu_b, \xi\rangle + \langle \varpi_b;\xi\rangle \tag{21}$$

with $v = T(u)$ solution of (2). Here the quantities μ_a and μ_b are two measures finite up to the boundary and ϖ_a, ϖ_b are non negative measure. Then by Lemma 3.4, we have that ϖ_a, ϖ_b are finite up to boundary. Since (20) and (21), we deduce

$$T_{[a,b]}(v)_t + div\tilde{f}(T_{[a,b]}(v)) = -\gamma_{a,b} \text{ in } D'(Q),$$

where $\gamma_{a,b} = \gamma_a + \gamma_b$. Then v is quasi-solution, therefore as $\tilde{f}$ is non degenerate (else $f' \circ T^{-1} = cst$ on an interval (a, b), then $f' = cst$ on an interval $(T^{-1}(a), T^{-1}(b))$ and $\tilde{f}$ is non degenerate), then by Panov argument the strong trace exist for $v = T(u)$. Moreover, $\phi(u) = \lim_{\epsilon\to 0}\phi(u^\epsilon)$ in $L^2(0,T;H^1(\Omega))$ and $\phi(u^\epsilon) \to \phi(u))$ strongly in $L^2(Q)$ then thanks to Lemma 2.5, there exists a strong trace $\gamma\phi(u)$ of $\phi(u)$ on Σ in the L^2 sense and then in the L^1 sense.

Let us note $\Psi(x) = T(x) + \phi(x) - x^-$. The function Ψ is a continuous bijection from $[0, ||u||_\infty]$ onto $\Psi([0, ||u||_\infty])$ and $\Psi(u) = v + \phi(u)$. v and $\phi(u)$ have a strong trace, γv and $\gamma\phi(u)$, then by to Lemma 2.3 $\Psi(u)$ possess a strong trace $\gamma\Psi = \gamma v + \gamma\phi(u)$. Afterwards, Ψ^{-1} is continuous on $\Psi[0, ||u||_\infty]$, therefore by Lemma 2.4 u possesses a strong trace

$$\gamma u = \Psi^{-1}(\gamma\Psi(u)) = \Psi^{-1}(\gamma T(u) + \phi(u)_{|_\Sigma}) = \Psi^{-1}(\gamma T(u) + \gamma\phi(u)).$$

5 Application for Boundary Value Problems

Here, we apply the result of Theorem 4.1 of existence of strong trace of the solution for three kind of boundary condition (zero-flux, Robin, Dirichlet).

Corollary 5.1 *(Case of zero flux boundary problem) Consider* (E) *with the zero-flux boundary problem under assumption* $(f(.), \phi(.))$ *non-degenerate. Then If* $\ell \geq 1$*, there exists an entropy solution* u *that has strong trace* γu *on the boundary. In particular, if* $\ell = 1$*, the unique entropy solution has strong boundary trace.*

Proof. This is a consequence of Theorem 4.1.

Remark 5.2 *In the same case, we can extend this corollary to the Robin boundary problem. For* $\ell \geq 1$ *there exists an entropy solution* u *of that has strong trace* γu *on the boundary. In particular, if* $\ell = 1$*, the unique entropy solution has strong boundary trace.*

From now, we consider the Dirichlet boundary problem in a bounded domain $\Omega \subset \mathbb{R}^\ell$ with Lipschitz boundary

$$\begin{cases} u_t + \text{div}(f(u) - \nabla\phi(u)) = 0 & \text{in } Q \\ u(0,x) = u_0 & \text{in } \Omega \\ u(t,x) = u^D(t,x) & \text{on } \Sigma. \end{cases} \tag{DP}$$

Here u_0 and u^D are bounded measurable functions. We approximate $\phi(u)$ by $\phi_\epsilon(u^\epsilon) = \phi(u^\epsilon) + \epsilon Id(u^\epsilon)$ for each $\epsilon > 0$. We obtain the following regularized problem:

$$\begin{cases} u_t^\epsilon + \text{div} f(u^\epsilon) - \Delta\phi_\epsilon(u^\epsilon) = 0 & \text{in } Q \\ u^\epsilon(0,x) = u_0^\epsilon(x) & \text{in } \Omega, \\ u^\epsilon(t,x) = u^D(t,x) & \text{on } \Sigma, \end{cases} \tag{DP$_\epsilon$}$$

where $(u_0^\epsilon)_\epsilon$ is a sequence of smooth functions that converges to u_0 a.e and respects the minimum/maximum values of u_0.

Theorem 5.3 *There exists a function solution* $u^\epsilon \in L^2(0,T;H^1(\Omega)) \cap L^\infty(Q) \cap C([0,T], L^2(\Omega))$ *for the problem* (DP$_\epsilon$) *such that* $u^\epsilon(0,.) = u_0^\epsilon$, $u^\epsilon_{|_\Sigma} = u^D$, $u_t^\epsilon \in L^2(0,T;H^{-1}(\Omega))$ *and for all* $\xi(t,x) \in L^2(0,T;H_0^1(\Omega))$*:*

$$\int_0^T \langle u_t^\epsilon, \xi\rangle_{H^{-1}(\Omega);H_0^1(\Omega)} dt - \int_0^T \int_\Omega (f(u^\epsilon) - \nabla\phi_\epsilon(u^\epsilon)).\nabla\xi dxdt = 0 \tag{22}$$

Proof. The existence (and uniqueness) of such solution to (DP$_\epsilon$) follows from standard arguments, [30], [31].

Remark 5.4 *For any non-negative* $\varphi \in C_0^\infty([0,T[\times\Omega)$*, any Lipschitz-continuous function* η *and any constant* α*, one has that*

$$\int_Q \eta'(u^\epsilon)\nabla\phi_\epsilon(u^\epsilon)\nabla u^\epsilon\varphi dxdt - \int_\Omega\int_\alpha^{u_0^\epsilon}\eta(\sigma)d\sigma\varphi(0,.)dx = \int_Q\int_\alpha^{u^\epsilon}\eta(\sigma)d\sigma\varphi_t + \int_Q\int_\alpha^{u^\epsilon}\eta(\sigma)f'(\sigma)d\sigma\nabla\varphi - \int_Q \nabla\varphi\nabla\int_\alpha^{u^\epsilon}\eta(\sigma)[\epsilon + \phi'(\sigma)]d\sigma dxdt. \tag{23}$$

In particular, for any parameter k, if $\alpha = k$ and η is an approximation of $sign(. - k)$, one gets that

$$\int_Q (u^\epsilon - k)^\pm \varphi_t + sign^\pm(u^\epsilon - k)[f(u^\epsilon) - f(k)]\nabla\varphi - \int_Q \nabla\varphi\nabla(\phi(u^\epsilon) - \phi(k))^\pm dxdt - \epsilon\int_Q \nabla|u^\epsilon - k|\nabla\varphi dxdt + \int_\Omega |u_0^\epsilon - k|\varphi(0,.)dx \geq 0. \quad (24)$$

In particular, u^ϵ is a "local entropy solution" of (DP_ϵ) . *Furthermore, set $M :=$* $\max\{||u^D||_{L^\infty(\Sigma)}, ||u_0||_{L^\infty(\Omega)}\}$. *If $k \geq M$ (respectively $k \leq -M$) we can extend the inequalities* (24) *up to the boundaries $\Sigma \cup (\{0\} \times \Omega)$.*

Lemma 5.5 *Assume that $u_0 \in L^\infty(\Omega)$, $u^D \in L^2(0,T; H^{1/2}(\partial\Omega)) \cap L^\infty(\Sigma)$ and $u_t^D \in L^1(0,T; L^2(\Omega))$. Then the estimates* (9) *are satisfied.*

Remark 5.6 *Notice that in the case where $\Omega = (a,b)$ is a bounded interval of $\mathbb{R}$ and u^D is constant in t, the assumptions of Lemma 5.5 hold.*

Proof. First, take $k = M$ (respectively $k = -M$) in the up-to-the-boundary inequality (24) (with $sign^+$ respectively $sign^-$), we get $||u^\epsilon||_{L^\infty(Q)} \leq M$. In the sequel, we take f, ϕ restricted to $[-M, M]$. We take $(u^\epsilon - u^D)\mathbf{1}_{[0,t]}$ as test function in (22)

$$\int_0^t \left\langle (u^\epsilon - u^D)_t; u^\epsilon - u^D \right\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \int_0^t\int_\Omega u_t^D(u^\epsilon - u^D)dxds - \int_0^t\int_\Omega f(u^\epsilon).\nabla(u^\epsilon - u^D)dxds + \int_0^t\int_\Omega \nabla\phi_\epsilon(u^\epsilon).\nabla(u^\epsilon - u^D)dxds = 0. \quad (25)$$

Denote the four terms in the left-hand side of (25) by A, B, E and G respectively, we calculate:

$$A = \frac{1}{2}||u^\epsilon(t,.) - u^D(t,.)||^2_{L^2(\Omega)} - \frac{1}{2}||u^\epsilon(0,.) - u^D(0,.)||^2_{L^2(\Omega)} = \frac{1}{2}||u^\epsilon(t,.) - u^D(t,.)||^2_{L^2(\Omega)} + C. \quad (26)$$

$$|-B| \leq C(u^D)||u^\epsilon - u^D||_{L^\infty(0,t;L^2(\Omega))} \leq \frac{1}{4}||u^\epsilon - u^D||^2_{L^\infty(0,t;L^2(\Omega))} + (C(u^D))^2 \leq \frac{1}{4}||u^\epsilon - u^D||^2_{L^\infty(0,t;L^2(\Omega))} + C. \quad (27)$$

$$|E| = \left|-\int_0^t\int_\Omega f(u^\epsilon).\nabla u^\epsilon + \int_0^t\int_\Omega f(u^\epsilon).\nabla u^D\right| \leq \left|\int_0^t\int_\Omega f(u^\epsilon).\nabla u^\epsilon\right| + \left|\int_0^t\int_\Omega f(u^\epsilon).\nabla u^D\right| \leq \left|\int_0^t\int_\Omega \mathrm{div}\left(\int_0^{u^\epsilon} f(r)dr\right)dxds\right| + C(u^D)||f||_\infty \leq \left|\int_0^t\int_{\partial\Omega}\left(\int_0^{u^\epsilon} f(r)dr\right).\eta d\mathcal{H}^{\ell-1}ds\right| + C \leq C. \quad (28)$$

$$G = \epsilon\int_0^t\int_\Omega \nabla u^\epsilon.\nabla(u^\epsilon - u^D)dxdt + \int_0^t\int_\Omega \nabla\phi(u^\epsilon).\nabla u^\epsilon dxdt - \int_0^t\int_\Omega \nabla\phi(u^\epsilon).\nabla u^D dxdt \geq \epsilon\int_0^t |\nabla u^\epsilon|^2 dxdt - \frac{\epsilon}{2}\int_0^t\int_\Omega |\nabla u^\epsilon|^2 dxdt - \frac{\epsilon}{2}\int_0^t\int_\Omega |\nabla u^D|^2 + \frac{1}{||\phi'||_\infty}\int_0^t\int_\Omega |\nabla\phi(u^\epsilon)|^2 - \frac{\alpha}{2}\int_0^t\int_\Omega |\nabla\phi(u^\epsilon)|^2 - \frac{1}{2\alpha}\int_0^t\int_\Omega |\nabla u^D|^2. \quad (29)$$

For all $t \leq T$, from (26)-(29) we have

$$\frac{1}{2}||u^\epsilon(t,.) - u^D(t,.)||^2_{L^2(\Omega)} + \epsilon\int_0^t\int_\Omega |\nabla u^\epsilon|^2 dxdt + C\int_0^t\int_\Omega |\nabla\phi(u^\epsilon)|^2 dxdt \leq C + \frac{1}{4}||u^\epsilon - u^D||^2_{L^\infty(0,t;L^2(\Omega))}. \quad (30)$$

Taking the sup over t in $[0,T]$, we obtain

$$\frac{1}{4}||u^\epsilon - u^D||^2_{L^\infty(0,T;\Omega)} + \epsilon\int_0^T\int_\Omega |\nabla u^\epsilon|^2 dxdt + C\int_0^T\int_\Omega |\nabla\phi(u^\epsilon)|^2 dxdt \leq C + \frac{1}{4}||u^\epsilon - u^D||^2_{L^\infty(0,T;L^2(\Omega))}.$$

Therefore

$$\epsilon\int_0^T |\nabla u^\epsilon|^2 dxdt + C\int_0^T\int_\Omega |\nabla\phi(u^\epsilon)|^2 dxdt \le C. \tag{31}$$

Corollary 5.7 *(Case of Dirichlet boundary problem) Assume that* (f,ϕ) *is non degenerate and* $u_0 \in L^\infty(\Omega)$*,* $u^D \in L^2(0,T;H^{1/2}(\partial\Omega)) \cap L^\infty(\Sigma)$*,* $u_t^D \in L^1(0,T;L^2(\Omega))$*. Then the strong trace* γu *of solution* u *for the Dirichlet boundary problem exists. In particular, for problem* (DP$_\epsilon$) *considered, there exists a subsequence* u^ϵ *that converges to a limit* u *a.e.. Moreover, the limit* u *is a local entropy solution and admits a strong boundary trace.*

We refer to [32], for the up-to-the-boundary entropy formulation and uniqueness of entropy solution u to (DP).

Remark 5.8 *Analogous arguments apply for the stationary problem associated to* (DP)*. In particular, the result holds for stationary problem associated to* (DP).

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