Classification of Surfaces of Finite Chen *II*-Type

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Abstract: - In this paper, we delve into the fascinating realm of quadric surfaces, with a specific focus on those of finite type. We first define relations regarding the first and the second Laplace operators corresponding to the second fundamental form II of a surface in the Euclidean space E^3. We focus on quadric surfaces from two sides, on one side, we study quadric surfaces of the first kind whose Gauss map N satisfies a relation of the form $\Delta^{II}n = AN$, where A is a square matrix of order 3 and Δ is the second Laplace operator. On the other side, we study quadric surfaces of the second kind with the same property.

Key-Words: - Surfaces in the Euclidean 3-space, Surfaces of finite Chen-type, Beltrami-Laplace operator, Quadric Surfaces, Gauss map of a surface, Surfaces of coordinate finite type Gauss map.

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1 Introduction

Quadric surfaces of finite type are a class of three-dimensional surfaces in geometry that are defined by second-degree equations in three variables. These surfaces are an essential part of the study of conic sections, and they exhibit a wide range of interesting geometric properties and real-world applications. The study of quadric surfaces of finite type is crucial in various fields, including geometry, physics, engineering, and computer graphics.

Quadric surfaces are used for modeling various objects in computer graphics, providing a mathematical representation of surfaces such as rocks, terrain, and architectural elements.

Quadric surfaces, particularly hyperboloids (Figure 1) and ellipsoids (Figure2), are essential in the analysis of electromagnetic fields. In antenna design, hyperbolic and parabolic surfaces are used to guide and focus electromagnetic waves, improving transmission and reception.

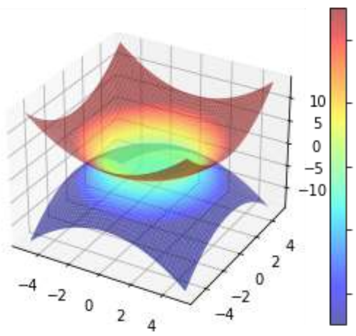

Fig. 1: (Hyperboloid)

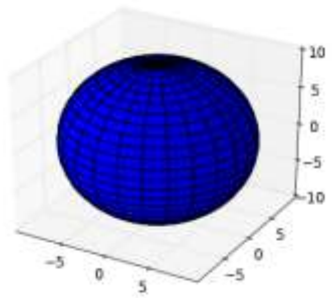

Fig. 2: (Ellipsoid)

The study, [1], brought the concept of surfaces of finite Chen type to become an interesting topic for many differential geometers. As a result, much research has been done in this field by studying certain special classes of surfaces such as quadric surfaces, [2], [3], tubes, [4], [5], translation surfaces, [6], [7], ruled surfaces, [3], [8], [9], surfaces of revolution, [10], [11], [12], spiral surfaces, [13], cycles of Dupin, [14], [15], and helicoidal surfaces, [16], [17].

We consider a surface Q in a Euclidean 3-space E^3 with a system of coordinates v^1, and v^2 to be referred. We denote by (g_{ij}), (b_{ij}), and (e_{ij}) the coefficients of the metrics I, II, and III of Q respectively. Let γ and δ be any two functions defined on Q. The first Laplace operator regarding the fundamental form $J = I, II, III$ of Q is defined by

$$\nabla^J(\gamma,\delta) := c^{ij}\, \gamma_{/i}\, \delta_{/j},$$

where $\gamma_{/i}$ is the partial derivative with respect to the parameter v^i and (c^{ij}) is defined to be the inverse tensor of (g_{ij}), (b_{ij}), and (e_{ij}) for $J = I$, II, and III respectively. The second Laplace operator according to the fundamental form J of Q is defined by:

$$\Delta^J \gamma. = -\frac{1}{\sqrt{c}}(\sqrt{c}c^{ij}\gamma_{/i})_{/j}$$

where $c = \det(c_{ij})$.

Considering the position vector $\mathbf{z} = \mathbf{z}(v^1,v^2)$, of Q in E^3, authors in [18], showed the relation:

$$\Delta^{III}\mathbf{z} = -\nabla^{II}\left(\frac{2H}{K}, \mathbf{z}\right) - \frac{2H}{K}\boldsymbol{N}.$$

where K and H are the Gauss and the mean curvature of Q respectively, and $\boldsymbol{N}$ is its Gauss map. Moreover, they proved that a surface satisfying the condition:

$$\Delta^{III}\mathbf{z} = \lambda\mathbf{z}, \quad \lambda\in\mathbb{R},$$

i.e., a surface M: $\mathbf{z} = \mathbf{z}(v^1,v^2)$ for which all coordinate functions are eigenfunctions of Δ^{III} with the same eigenvalue λ, is a part of a sphere ($\lambda = 2$) or a minimal surface ($\lambda = 0$).

2 Fundamentals

Consider the parametric representation:

$$\boldsymbol{r}(x,y) = \{r_1(x,y), r_2(x,y), r_3(x,y)\},$$
$$(x,y) \in \mathrm{B} \subset \mathbb{R}^2$$

of a surface Q.

The fundamental form I of Q is

$$I = Edx^2 + 2Fdxdy + Gdy^2.$$

For some function $\varphi(x, y)$ on $\mathrm{D} \subset \mathbb{R}^2$, Δ^I is found to be [19]:

$$\Delta^I\varphi = -\frac{1}{\sqrt{EG-F^2}}\left[\left(\frac{G\varphi_x - F\varphi_y}{\sqrt{EG-F^2}}\right)_x - \left(\frac{F\varphi_x - E\varphi_y}{\sqrt{EG-F^2}}\right)_y\right]$$

The second metric is:

$$II = Ldx^2 + 2Mdxdy + Ndy^2.$$

Also, we have, [19]

$$\Delta^{II}\varphi = -\frac{1}{\sqrt{LN-M^2}}\left[\left(\frac{N\varphi_x - M\kappa_y}{\sqrt{LN-M^2}}\right)_x - \left(\frac{M\varphi_x - L\varphi_y}{\sqrt{LN-M^2}}\right)_y\right]. \quad (1)$$

For the vector-valued function $\boldsymbol{r} = \{r_1, r_2, r_3\}$, its well known that:

$$\Delta^J\boldsymbol{r} = \{\Delta^J r_1, \Delta^J r_2, \Delta^J r_3\}, J = I, II..$$

Definition 1. A surface Q is said to be of coordinate finite type regarding the metric II, or briefly of coordinate finite II-type if the vector-valued function $\boldsymbol{r}$ of Q satisfies the relation:

$$\Delta^{II}\boldsymbol{r} = A\boldsymbol{r},$$

where A is a square matrix of order 3. Besides, if we consider the unit normal vector field $\boldsymbol{N}$ of the surface Q, then we also have:

Definition 2. A surface Q is said to be of coordinate finite II-type Gauss map if the unit normal vector field $\boldsymbol{N}$ of Q satisfies:

$$\Delta^{II}\boldsymbol{N} = A\boldsymbol{N}. \quad (2)$$

In this article, we pay attention to quadrics whose unit normal vector field $\boldsymbol{N}$ satisfies a relation of the form (2).

Interesting research also, one can follow the idea in [20] by defining the first and second Laplace operator using the definition of the fractional vector operators. Application within this subject can be found in [21], [22].

3 Quadric Surfaces

For the quadric surface Q in $\mathbb{R}^3$ we have the following three cases:

Case I. Q is ruled surface. In geometry, a surface Q is ruled if through every point of Q there is a straight line that lies on S. Examples include the plane, lateral surfaces of a cylinder or cone, a conical surface with elliptical directrix, the right conoid, the helicoidal, and the tangent devolaple of a smooth curve in space.

A ruled surface can be described as a set of points swept by a moving straight line. For example, a cone is formed by keeping one point of a line fixed whilst moving another point along a circle. A surface is doubly ruled if through every one of its points, there are two distinct lines that lie on the surface (Figure 3).

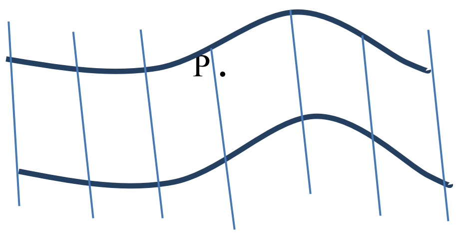

Fig. 3: (Ruled surface)

This case of ruled surfaces was studied in [8], and it was proved:

Theorem 1. There are no ruled surfaces in the Euclidean 3-space that satisfy the relation (2).

Case II. Q is of the form:

$$Z^2 = c + nX^2 + mY^2, \quad n, m, c \in \mathbb{R}, \quad nm \neq 0,\ c > 0, \tag{3}$$

Case III. Q is of the form

$$Z = \frac{n}{2}X^2 + \frac{m}{2}Y^2, \quad n, m \in \mathbb{R},\ n, m > 0. \tag{4}$$

For case II mentioned above, we prove that a quadric of the form (3) satisfies the (2), exactly when $n = m = -1$, that is, Q is a part of a sphere. Next, we prove that for a quadric of the form (4) condition (2) cannot be satisfied.

3.1 Quadrics of the Form (3)

Putting $X = x$ and $Y = y$, then $Z = \pm\sqrt{c + nx^2 + my^2}$. Thus a parameterization of this form is locally represented by:

$$\boldsymbol{r}(x, y) = \left\{x, y, \sqrt{c + nx^2 + my^2}\right\}. \tag{5}$$

We have:

$$\boldsymbol{r}_x = \left\{1, 0, \frac{nx}{\sqrt{\omega}}\right\}, \boldsymbol{r}_y = \left\{0, 1, \frac{my}{\sqrt{\omega}}\right\},$$

where

$$\omega = c + nx^2 + my^2.$$

The coefficients of the metric I are:

$$E = \langle \boldsymbol{r}_x, \boldsymbol{r}_x \rangle = 1 + \frac{(nx)^2}{\omega},$$
$$F = \langle \boldsymbol{r}_x, \boldsymbol{r}_y \rangle = \frac{nmxy}{\omega},$$
$$G = \langle \boldsymbol{r}_y, \boldsymbol{r}_y \rangle = 1 + \frac{(my)^2}{\omega}.$$

So, we obtain:

$$I = \left(1 + \frac{(nx)^2}{\omega}\right)dx^2 + 2\frac{nmxy}{\omega}dx\,dy + \left(1 + \frac{(my)^2}{\omega}\right)dy^2.$$

The normal vector $\boldsymbol{N}$ is:

$$\boldsymbol{N} = \frac{\boldsymbol{r}_x \times \boldsymbol{r}_y}{\sqrt{EG - F^2}} = \left\{\frac{-nx}{\sqrt{W}}, \frac{-my}{\sqrt{W}}, \frac{\sqrt{\omega}}{\sqrt{W}}\right\} \tag{6}$$

where

$$W = c + n(n+1)x^2 + m(m+1)y^2.$$

We have:

$$\boldsymbol{N}_x = \left\{\frac{n^2x^2(n+1) - nW}{W\sqrt{W}}, \frac{nm(n+1)xy}{W\sqrt{W}}, \frac{nWx - n\omega(n+1)x}{W\sqrt{W}\sqrt{\omega}}\right\}$$

$$\boldsymbol{N}_y = \left\{\frac{nm(m+1)xy}{W\sqrt{W}}, \frac{m^2y^2(m+1) - mW}{W\sqrt{W}}, \frac{mWy - m\omega(m+1)y}{W\sqrt{W}\sqrt{\omega}}\right\}$$

then

$$L = -\langle \boldsymbol{N}_x, \boldsymbol{r}_x \rangle = \frac{n(c + my^2)}{\omega\sqrt{W}},$$
$$M = -\frac{1}{2}\left(\langle \boldsymbol{N}_x, \boldsymbol{r}_y \rangle + \langle \boldsymbol{N}_y, \boldsymbol{r}_x \rangle\right) = -\frac{nmxy}{\omega\sqrt{W}},$$
$$N = -\langle \boldsymbol{N}_y, \boldsymbol{r}_y \rangle = \frac{m(c + nx^2)}{\omega\sqrt{W}}.$$

And

$$\sqrt{|LN - M^2|} = \frac{\sqrt{nmc}}{\sqrt{\omega}\sqrt{W}}.$$

The Second fundamental form of the surface is given by:

$$II = \frac{n(c + my^2)}{\omega\sqrt{W}}dx^2 - \frac{2mnxy}{\omega\sqrt{W}}dxdy + \frac{m(c + nx^2)}{\omega\sqrt{W}}dy^2.$$

Therefore from (1), the Laplace operator Δ^{II} of Q is given as follows:

$$\Delta^{II} = -\frac{\sqrt{W}}{c}\left[2x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + 2xy\frac{\partial^2}{\partial x\partial y} + \frac{(c + nx^2)}{n}\frac{\partial^2}{\partial x^2} + \frac{(c + my^2)}{m}\frac{\partial^2}{\partial y^2}\right] \tag{7}$$

We denote by (N_1, N_2, N_3) the components of the vector $\boldsymbol{N}$. For the partial derivatives of N_1, we have:

$$\frac{\partial N_1}{\partial x} = \frac{n^2(n+1)x^2 - nW}{W\sqrt{W}}$$
$$\frac{\partial N_1}{\partial y} = \frac{mn(m+1)xy}{W\sqrt{W}}$$
$$\frac{\partial^2 N_1}{\partial x^2} = \frac{3n^2(n+1)x}{W\sqrt{W}} - \frac{3n^3(n+1)^2x^3}{W^2\sqrt{W}}$$
$$\frac{\partial^2 N_1}{\partial y^2} = \frac{mn(m+1)x}{W\sqrt{W}} - \frac{3nm^2(m+1)^2xy^2}{W^2\sqrt{W}}.$$

Similar relations can be drawn for the partial derivatives of N_2. Applying (7) for the functions N_1 and N_2, we find:

$$\Delta^{II}N_1 = \frac{2nx}{c} - \frac{2n^2(n+1)x^3}{cW} - \frac{4mn(m+1)xy^2}{cW} + \frac{6mn^2(m+1)(n+1)x^3y^2}{cW^2} - \frac{3n(n+1)(c + nx^2)x}{cW} + \frac{3n^2(n+1)^2(c + nx^2)x^3}{cW^2} + \frac{3mn(m+1)^2(c + my^2)xy^2}{cW^2} - \frac{n(m+1)(c + my^2)x}{cW}$$

After a lot of computations, we write the above equation as follows:

$$\Delta^{II} N_1 = -\frac{1}{W^2}\left[n^2(n+1)x^3 + nc(3n+1)x\right] - \frac{1}{cW^2} f(x,y) \tag{8}$$

where

$$f(x,y) = [-4mn^2(n+1)(m+1)x^3y^2 + 2nm^2(m+1)^2xy^4 + 3nm(n+1)(m+1)(c+nx^2)xy^2 - 2mn(m+1)^2(c+my^2)xy^2 + n^2(n+1)(m+1)(c+my^2)x^3 + cn(m+1)(c+my^2)x].$$

$$\Delta^{II} N_2 = \frac{2my}{c} - \frac{2m^2(m+1)y^3}{cW} - \frac{4mn(n+1)x^2y}{cW} + \frac{6m^2n(m+1)(n+1)x^2y^3}{cW^2} - \frac{3m(m+1)(c+my^2)y}{cW} + \frac{3m^2(m+1)^2(c+my^2)y^3}{cW^2} + \frac{3mn(n+1)^2(c+nx^2)x^2y}{cW^2} - \frac{m(n+1)(c+nx^2)y}{cW}$$

$$= -\frac{1}{W^2}\left[m^2(m+1)y^3 + mc(3m+1)y\right] - \frac{1}{cW^2} g(x,y) \tag{9}$$

where

$$g(x,y) = [-4nm^2(n+1)(m+1)x^2y^4 + 2mn^2(n+1)^2yx^4 + 3nm(n+1)(m+1)(c+my^2)yx^2 - 2mn(n+1)^2(c+nx^2)yx^2 + m^2(n+1)(m+1)(c+nx^2)y^3 + cm(n+1)(c+nx^2)y].$$

Let A = [a_{ij}]. Applying relation (2) for the position vector (6) we find:

$$\Delta^{II} N_1 = \Delta^{II}\left(-\frac{nx}{\sqrt{W}}\right) = a_{11}N_1 + a_{12}N_2 + a_{13}N_3, \tag{10}$$

$$\Delta^{II} N_2 = \Delta^{II}\left(-\frac{my}{\sqrt{W}}\right) = a_{21}N_1 + a_{22}N_2 + a_{23}N_3, \tag{11}$$

$$\Delta^{II} N_3 = \Delta^{II}\left(\frac{\sqrt{\omega}}{\sqrt{W}}\right) = a_{31}N_1 + a_{32}N_2 + a_{33}N_3. \tag{12}$$

From (8) and (10) we have:

$$-\frac{1}{W^2}\left[n^2(n+1)x^3 + nc(3n+1)x\right] - \frac{1}{cW^2} f(x,y) = -a_{11}\frac{nx}{\sqrt{W}} - a_{12}\frac{my}{\sqrt{W}} + a_{13}\frac{\sqrt{\omega}}{\sqrt{W}}.$$

Putting $y = 0$ in the above equation, then we get:

$$n^2(n+1)x^3 + nc(3n+1)x + n^2(n+1)(m+1)x^3 + cn(m+1)x = (-a_{11}nx + a_{13}\sqrt{c+nx^2})W\sqrt{W}. \tag{13}$$

Making some computations on (13) we obtain a polynomial of the variable x which must hold for all the values of x from which we conclude that all the coefficients of the terms of the variable x of the polynomial must be zeros, and since $n \neq 0$ so we will have $n + 1 = 0$ and $a_{13} = 0$.

From (9) and (11) we have:

$$-\frac{1}{W^2}\left[m^2(m+1)y^3 + mc(3m+1)y\right] - \frac{1}{cW^2} g(x,y) = -a_{21}\frac{nx}{\sqrt{W}} - a_{22}\frac{my}{\sqrt{W}} + a_{23}\frac{\sqrt{\omega}}{\sqrt{W}}.$$

Putting $x = 0$ in the above equation, then we get

$$m^2(m+1)y^3 + mc(3m+1)y + m^2(n+1)(m+1)y^3 + cm(n+1)y = (-a_{22}my + a_{23}\sqrt{c+my^2})W\sqrt{W}. \tag{14}$$

Similarly, by making some computations on (14) we obtain a polynomial of the variable y which must hold for all the values of y from which we conclude that all the coefficients of the terms of the variable y of the polynomial must be zeros, and since $m \neq 0$ so we will have $m + 1 = 0$ and $a_{23} = 0$.

Putting $n = -1$, $m = -1$, and $a_{13} = 0$, then from (13) we get $a_{11} = -\frac{2}{\sqrt{c}}$. In the same way, if we put $n = -1$, $m = -1$, and $a_{23} = 0$, then from (14) we also get $a_{22} = -\frac{2}{\sqrt{c}}$.

Besides, relation (7) becomes:

$$\Delta^{II} = -\frac{1}{\sqrt{c}}\left[2x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + 2xy\frac{\partial^2}{\partial x\partial y} + (x^2-c)\frac{\partial^2}{\partial x^2} + (y^2-c)\frac{\partial^2}{\partial y^2}\right] \tag{15}$$

The component N_3 of $\boldsymbol{N}$ becomes:

$$N_3 = \frac{\sqrt{c-x^2-y^2}}{\sqrt{c}}. \tag{16}$$

From (15) and (16) we find:

$$\Delta^{II} N_3 = -\frac{2\sqrt{c-x^2-y^2}}{c}$$

On account of (12), we get:

$$-\frac{2\sqrt{c-x^2-y^2}}{c}=a_{31}\frac{x}{\sqrt{c}}+a_{32}\frac{y}{\sqrt{c}}+a_{33}\frac{\sqrt{c-x^2-y^2}}{\sqrt{c}}.$$

Or

$$-(\frac{2}{\sqrt{c}}+a_{33})\sqrt{c-x^2-y^2}=a_{31}x+a_{32}y.$$

Since the last equation holds for all the values of the variables x and y, it is easily verified that we must have:

$$a_{31}=a_{32}=0, \text{ and } a_{33}=-\frac{2}{\sqrt{c}}.$$

So, we proved the following:

Theorem 2. Spheres are the only quadric surfaces of this kind $Z^2=c+nX^2+mY^2$, that satisfies the relation $\Delta^{II}\boldsymbol{N}=A\boldsymbol{N}$.

3.2 Quadrics of the Form (4)

Putting $X=x$ and $Y=y$, then $Z=\frac{nx^2}{2}+\frac{my^2}{2}$. And so a parametric representation of this kind is locally given by:

$$\boldsymbol{r}(x,y)=\{x,y,\frac{nx^2}{2}+\frac{my^2}{2}\}. \tag{17}$$

We have:

$$\boldsymbol{r}_x=\{1,0,nx\},\quad \boldsymbol{r}_y=\{0,1,my\}.$$

The coefficients of the metric I are:

$$E=1+(nx)^2,\quad F=nmxy,\quad G=1+(my)^2.$$

So, we obtain:

$$\mathrm{I}=[1+(nx)^2]dx^2+2nmxydxdy+[1+(my)^2]dy^2.$$

The Gauss map $\boldsymbol{N}$ is:

$$\boldsymbol{N}=\left\{-\frac{nx}{\sqrt{g}},-\frac{my}{\sqrt{g}},\frac{1}{\sqrt{g}}\right\}, \tag{18}$$

where $g=1+(nx)^2+(my)^2$. We have:

$$\boldsymbol{N}_x=\left\{\frac{-ng+n^2x^2}{g\sqrt{g}},\frac{mnxy}{g\sqrt{g}},-\frac{nx}{g\sqrt{g}}\right\},$$

$$\boldsymbol{N}_y=\left\{\frac{mnxy}{g\sqrt{g}},\frac{-mg+m^2y^2}{g\sqrt{g}},-\frac{my}{g\sqrt{g}}\right\},$$

The components of the second fundamental form are defined as follows:

$$L=\frac{n}{\sqrt{g}},\quad M=0,\quad N=\frac{m}{\sqrt{g}}$$

The Second fundamental form of the surface is given by:

$$II=\frac{n}{\sqrt{g}}dx^2+\frac{m}{\sqrt{g}}dy^2.$$

Therefore from (1), the Laplace operator $\Delta^{II}$of Q is given as follows:

$$\Delta^{II}=-\frac{\sqrt{g}}{\sqrt{mn}}\left[\frac{\sqrt{m}}{\sqrt{n}}\frac{\partial^2}{\partial x^2}+\frac{\sqrt{n}}{\sqrt{m}}\frac{\partial^2}{\partial y^2}\right]. \tag{19}$$

Let (N_1, N_2, N_3) be the components of the vector $\boldsymbol{N}$. For the partial derivatives of N_1, we have:

$$\frac{\partial N_1}{\partial x}=\frac{-ng+n^3x^2}{g\sqrt{g}},\quad \frac{\partial N_1}{\partial y}=\frac{m^2nxy}{g\sqrt{g}}$$

$$\frac{\partial^2 N_1}{\partial x^2}=\frac{3n^3x}{g\sqrt{g}}-\frac{3n^5x^3}{g^2\sqrt{g}}$$

$$\frac{\partial^2 N_1}{\partial y^2}=\frac{m^2nx}{g\sqrt{g}}-\frac{3nm^4xy^2}{g^2\sqrt{g}}$$

Similar relations can be drawn for the partial derivatives of N_2. Applying (19) for the function N_1 and N_2, we find:

$$\Delta^{II}N_1=\frac{3nx(n^3x^2+m^3y^2)}{g^2}-\frac{n(m+3n)x}{g}$$

$$=\frac{nx}{g^2}(2m^3y^2-n^2mx^2-3nm^2y^2-3n-m), \tag{20}$$

$$\Delta^{II}N_2=\frac{3my(n^3x^2+m^3y^2)}{g^2}-\frac{m(n+3m)y}{g}$$

$$=\frac{my}{g^2}(2n^3x^2-m^2ny^2-3mn^2x^2-3m-n). \tag{21}$$

For the partial derivatives of N_3, we have:

$$\frac{\partial N_3}{\partial x}=\frac{-n^2x}{g\sqrt{g}},\quad \frac{\partial N_3}{\partial y}=\frac{-m^2y}{g\sqrt{g}}$$

$$\frac{\partial^2 N_3}{\partial x^2}=-\frac{n^2(m^2y^2+1)}{g^2\sqrt{g}}$$

$$\frac{\partial^2 N_3}{\partial y^2}=-\frac{m^2(n^2x^2+1)}{g^2\sqrt{g}}$$

Applying (19) for the function N_3, we find:

$$\Delta^{II}N_3=\frac{1}{g^2}[n(m^2y^2+1)+m(n^2x^2+1)]. \tag{22}$$

Let $A=[a_{ij}]$. Applying relation (2) for the position vector (18) we find:

$$\Delta^{II}N_1=\Delta^{II}\left(-\frac{nx}{\sqrt{g}}\right)=a_{11}N_1+a_{12}N_2+a_{13}N_3, \tag{23}$$

$$\Delta^{II}N_2=\Delta^{II}\left(-\frac{my}{\sqrt{g}}\right)=a_{21}N_1+a_{22}N_2+a_{23}N_3, \tag{24}$$

$$\Delta^{II}N_3=\Delta^{II}\left(\frac{1}{\sqrt{g}}\right)=a_{31}N_1+a_{32}N_2+a_{33}N_3. \tag{25}$$

From (20) and (23) we get:

$$\frac{nx}{g^2}(2m^3y^2 - n^2mx^2 - 3nm^2y^2 - 3n - m)$$
$$= -a_{11}\frac{nx}{\sqrt{g}} - a_{12}\frac{my}{\sqrt{g}} + a_{13}\frac{1}{\sqrt{g}}.$$

Let $x = 0$, then

$$a_{12}\frac{-my}{\sqrt{1+m^2y^2}} + a_{13}\frac{1}{\sqrt{1+m^2y^2}} = 0\cdot$$

It clearly can be seen that $a_{12} = a_{13} = 0$, from which we conclude that $a_{11} = 0$.

In the same way, from (21) and (24) we get:

$$\frac{my}{g^2}(2n^3x^2 - m^2ny^2 - 3mn^2x^2 - 3m - n)$$
$$= -a_{21}\frac{nx}{\sqrt{g}} - a_{22}\frac{my}{\sqrt{g}} + a_{23}\frac{1}{\sqrt{g}}.$$

Let $y = 0$, then

$$a_{21}\frac{-nx}{\sqrt{1+n^2x^2}} + a_{23}\frac{1}{\sqrt{1+n^2x^2}} = 0\cdot$$

We find that $a_{21} = a_{23} = 0$, from which we also conclude that $a_{22} = 0$.

So, (20) and (21) become respectively:

$$\frac{nx}{g^2}(2m^3y^2 - n^2mx^2 - 3nm^2y^2 - 3n - m) = 0,$$
$$\frac{my}{g^2}(2n^3x^2 - m^2ny^2 - 3mn^2x^2 - 3m - n) = 0.$$

The last two equations hold for all the values of the variables x and y only when $n = m = 0$, which is impossible since n and m are positive. So, we proved:

Theorem 3. There are no quadric surfaces of the second kind $Z = \frac{nx^2}{2} + \frac{my^2}{2}$ whose Gauss map satisfies the relation $\Delta^{II}\boldsymbol{N} = A\boldsymbol{N}$.

4 Conclusion

This research article was divided into three sections, where after the introduction, the needed definitions and relations regarding this interesting field of study were given. Then a formula for the Laplace operator corresponding to the first fundamental form I was proved once for the position vector and another for the Gauss map of a surface Q by using Cartan's method of the moving frame. Finally, we classify the quadric surfaces Q satisfying the relation ΔG = MG, for a real square matrix M of order 3. An interesting study can be drawn, if this type of study can be applied to other classes of surfaces that have not been investigated yet such as spiral surfaces, or tubular surfaces.

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References:

[1] B.-Y. Chen, Some open problems and conjectures on submanifolds of finite type, *Soochow J. Math*, Vol. 17, 1991.

[2] H. Al-Zoubi, T. Hamadneh, M. Abu Hammad, M. Al- Sabbagh, and Mehmet Ozdemir, Ruled and Quadric Surfaces Satisfying $\Delta^{II}N = \Lambda N$, *Symmetry*, Vol. 15 No. 2, 2023, 1-15. https://doi.org/10.3390/sym15020300

[3] B.-Y. Chen, F. Dillen, Quadrics, of finite type, *J. Geom*. Vol. 38, 1990, 16-22.

[4] J. S. Ro, D. W. Yoon. Tubes of Weingarten types in Euclidean 3-space, *J. Cungcheong Math. Soc.* Vol 22 (2009), 359-366.

[5] B.-Y. Chen, Surfaces of finite type in Euclidean 3-space, *Bull. Soc. Math. Belg*., Vol 39 (1987), 243-254.

[6] B. Senousi, H. Al-Zoubi, Translation surfaces of finite type in Sol_3, *Commentationes Mathematicae Universitatis Carolinae,* Vol. 22, No. 2, 2020, pp 237–256. DOI: 10.14712/1213-7243.2020.018

[7] M. Bekkar and B. Senoussi, Translation surfaces in the 3-dimensional space satisfying $\Delta^{III}r_i=m_ir_i$, *J. Geom*., Vol 103 (2012), 367-374.

[8] H. Al-Zoubi, H. Alzaareer, A. Zraiqat, T. Hamadneh, and W. Al-Mashaleh, On Ruled Surfaces of Coordinate Finite Type, *WSEAS Transactions on Mathematics*, Vol. 21, 2022, 765-769. DOI: 10.37394/23206.2022.21.87

[9] B.-Y. Chen, F. Dillen, L. Verstraelen, L. Vrancken, Ruled surfaces of finite type, Bull. Austral. Math. Soc. 42, 447-553 (1990).

[10] F. Dillen, J. Pass, L. Verstraelen, On the Gauss map of surfaces of revolution, *Bull. Inst. Math. Acad. Sinica* Vol. 18, 1990, 239-246.

[11] O. Garay, On a certain class of finite type surfaces of revolution, *Kodai Math. J.* Vol. 11, 1988, 25-31.

[12] Y. H. Kim, C. W. Lee, and D. W. Yoon, On the Gauss map of surfaces of revolution without parabolic points, *Bull. Korean Math. Soc*. Vol 46 (2009), 1141–1149.

[13] Ch. Baikoussis, L. Verstraelen, The Chen-type of the spiral surfaces, *Results. Math*. Vol. 28, 1995, 214-223.

[14] F. Denever, R. Deszcz, L. Verstraelen, The Chen type of the noncompact cyclides of Dupin, *Glasg. Math. J*. 36, 1994, 71-75.

[15] F. Denever, R. Deszcz, L. Verstraelen, The compact cyclides of Dupin and a conjecture by B.-Y Chen, *J. Geom*. Vol. 46, 1993, 33-38.

[16] Ch. Baikoussis, L. Verstraelen, On the Gauss map of helicoidal surfaces, *Rend. Semi. Mat. Messina Ser II* Vol. 16, 1993, 31-42.

[17] B. Senoussi and M. Bekkar, Helicoidal surfaces with $\Delta^J r = Ar$ in 3-dimensional Euclidean space, *Stud. Univ. Babes-Bolyai Math*. 60 2015, pp 437–448.

[18] S. Stamatakis, H. Al-Zoubi, On surfaces of finite Chen type. *Result. Math.* Vol. 43, 2003, 181-190.

[19] B.-Y. Chen, *Total mean curvature and submanifolds of finite type*, world Scientific, Singapore, 1984.

[20] M. Mhailan, M. Abu Hammad, M. Al Horani, R. Khalil, On fractional vector analysis, *J. Math. Comput. Sci*. \Vol 10 (2020), 2320-2326. https://doi.org/10.28919/jmcs/4863

[21] I. Batiha, J. Oudetallah, A. Ouannas, A. Al-Nana, I. Jebril. Tuningthe fractional-order PID-controller for blood Glucose level of diabetic patients, *Int. J. Adv. Soft Comp. App*. 13 No. 2 1-10 (2021).

[22] I. Batiha, S. Njadat R. Batyha. A. Zraiqat, A. Dabbabneh, Sh. Momani. Design fractional-order PID controller for single-joint robot arm model. *Int. J. Adv. Soft Comp. App*. 14 No.2 96-114 (2022).

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Conflict of Interest

The authors have no conflicts of interest to declare.