Strong Convergence Theorems for Resolvents of Accretive Operators with Possible Unbounded Errors

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Abstract: In this paper, we study the convergence analysis of the sequence generated by an inexact proximal point method with unbounded errors to find zeros of m-accretive operators in Banach spaces. We prove the zero set of the operator is nonempty if and only if the generated sequence is bounded. In this case, we show that the generated sequence converges strongly to a zero of the operator. This process defines a sunny nonexpansive retraction from the Banach space onto the zero set of the operator. We present also some applications and numerical experiments for our results.

Key-Words: Accretive operator, Proximal point method, Resolvent, Strong convergence, Unbounded error.

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1 Introduction

Let X be a real Banach space with norm $\|\cdot\|$. The topological dual of X is denoted by X^*, and the duality pair between $x \in X$ and $w \in X^*$ by $\langle x, w\rangle$. We define the duality mapping $J : X \to \mathcal{P}(X^*)$ by

$$J(x) = \left\{w \in X^* : \langle x, w\rangle = \|x\|^2 = \|w\|^2\right\}$$

for $x \in X$. The operator $A : D(A) \subset X \to \mathcal{P}(X)$ is called accretive, if for each $x, y \in D(A)$, there exists $j(y - x) \in J(y - x)$ such that

$$\langle v - u, j(y - x)\rangle \geq 0, \tag{1}$$

for all $u \in A(x)$ and $v \in A(y)$. An accretive operator A is called maximal accretive if there is no proper accretive extension of A. If I denotes the identity operator on X, we say that the operator A is m-accretive if $R(I + \gamma A) = X$ for all $\gamma > 0$. Every m-accretive operator is maximal accretive, however the converse is not true in general. For every accretive operator A, and for each $\gamma > 0$, we can define a nonexpansive single-valued mapping $J_\gamma^A(\cdot) : R(I + \gamma A) \to D(A)$ by $J_\gamma^A(x) = (I + \gamma A)^{-1}(x)$, which is called the resolvent of A.

The study, [1], proved that if X is a uniformly smooth Banach space, $A : X \to \mathcal{P}(X)$ is an m-accretive operator and $A^{-1}(0) \neq \emptyset$, then for any $x \in X$, the strong $\lim_{\gamma\to\infty} J_\gamma^A(x)$ exists and belongs to $A^{-1}(0)$. The study, [2], investigated an iterative method for finding the common zeros of two accretive operators. Recently, [3], studied the convergence of an inexact proximal point algorithm for finding zeros of maximal monotone operators in Hilbert spaces. If A is a maximal monotone operator in X and $u \in X$ is an arbitrary point, the authors, [4], showed the strong convergence of the sequence generated by an inexact proximal point method to $P_{A^{-1}(0)}(u)$ where P is the generalized projection of u onto $A^{-1}(0)$. For other recent results in this direction see, [5], [6].

Motivated by the above results, we investigate the strong convergence of the sequence generated by an inexact proximal point method with possible unbounded errors to find zeros of m-accretive operators in Banach spaces. We show that the zero set of the operator is nonempty, if and only if the generated sequence is bounded. In this case, the generated sequence converges strongly to $Q_{A^{-1}(0)}(u)$, where A is an m-accretive operator, $u \in X$ is an arbitrary point and $Q_{A^{-1}(0)}$ is the sunny nonexpansive retraction of X onto $A^{-1}(0)$.

2 Preliminaries

A Banach space X is called *strictly convex* if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in X$ with $\|x\| = \|y\| = 1$ and $x \neq y$. X is called a *uniformly convex* if for each $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that for all $x, y \in X$ with $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \varepsilon$, it holds that $\|\frac{x+y}{2}\| < 1 - \delta$. Every uniformly convex Banach space is reflexive and strictly convex. X is called *smooth* if

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2}$$

exists for all $x, y \in B = \{z \in X : \|z\| = 1\}$. If X is smooth, then the duality mapping J is single valued.

A Banach space X is called uniformly smooth if the limit in (2) is attained uniformly for $x, y \in B$. The spaces L^p $(1 < p < \infty)$ and the Sobolev spaces $W^{k,p}$ $(1 < p < \infty)$ are examples of uniformly convex and uniformly smooth Banach spaces.

We denote the strong convergence of a sequence $\{x_k\}$ to $x \in X$ by $x_k \to x$, and weak convergence by $x_k \rightharpoonup x$.

Consider the Banach space ℓ^∞ of all bounded complex-valued sequences. A Banach limit is a continuous linear functional $\psi : \ell^\infty \to \mathbb{C}$ such that for all sequences $x = \{x_n\}$ and $y = \{y_n\}$ in ℓ^∞, and complex numbers α, we have
(i) $\psi(\alpha x + y) = \alpha\psi(x) + \psi(y)$,
(ii) if $x_n \geq 0$ for all $n \in \mathbb{N}$, then $\psi(x) \geq 0$,
(iii) $\psi(x) = \psi(Sx)$, where S is the shift operator defined by $(Sx)_n = x_{n+1}$,
(iv) if $x = \{x_n\}$ is a convergent sequence, then $\psi(x) = \lim_{n\to\infty} x_n$.
In fact, a Banach limit extends the usual notion of a limit for a sequence, is linear, shift-invariant and positive. However, it may not be unique. For more details on Banach limits, we refer the reader to [7].

Lemma 2.1. [8], [9] *Let X be a Banach space with a uniformly Gâteaux differentiable norm, and $C \subset X$ be a nonempty, closed and convex set. Suppose that $\{x_k\}$ is a bounded sequence in X,* LIM *a Banach limit on ℓ^∞ and $q \in C$, then*

$$\mathrm{LIM}\|x_k - q\|^2 = \min_{y\in C}\Big\{\mathrm{LIM}\|x_k - y\|^2\Big\}$$

if and only if

$$\mathrm{LIM}\langle x - q, J(x_k - q\rangle \leq 0$$

for all $x \in C$.

Lemma 2.2. [10] *Suppose that $A : D(A) \subset X \to \mathcal{P}(X)$ is an accretive operator. Then for $r, \gamma > 0$ and $x \in X$, we have the resolvent identity*

$$J^A_\gamma x = J^A_r\Big(\frac{r}{\gamma}x + (1 - \frac{r}{\gamma})J^A_\gamma x\Big).$$

Assume that $C \subset X$ is nonempty, closed and convex and D is a nonempty subset of C. A mapping $Q : C \to D$ is a retraction whenever $Qx = x$ for all $x \in D$. A retraction $Q : C \to D$ is nonexpansive if Q is nonexpansive. A retraction $Q : C \to D$ is sunny if Q satisfies the property:

$$Q(Qx+t(x-Qx)) = Qx \quad \text{for all } x \in C \text{ and } t > 0, \tag{3}$$

whenever $Qx + t(x - Qx) \in C$. A retraction $Q : C \to D$ is sunny nonexpansive if Q is both sunny and nonexpansive. It is known ([11], [12]) that in a smooth Banach space X, a retraction $Q : C \to D$ is a sunny nonexpansive retraction from C to D if and only if the following inequality holds:

$$\langle x - Qx, J(z - Qx)\rangle \leq 0 \qquad \text{for } x \in C \text{ and } z \in D. \tag{4}$$

In particular, this shows that Q is unique, if it exists.

Lemma 2.3. [12], [13] (Browder fixed point theorem) *Let X be a uniformly convex Banach space. If $C \subset X$ is a nonempty, closed, convex and bounded set, and $T : C \to C$ is a nonexpansive mapping, then T has a fixed point.*

3 Main results

Let X be a uniformly smooth and uniformly convex Banach space, and $A : D(A) \subset X \to \mathcal{P}(X)$ be an m-accretive operator. Let the sequence $\{x_k\}$ be generated by

$$x_{k+1} = J^A_{\gamma_k}(u_k + \alpha_k(x_k + e_k)), \tag{5}$$

where $x_0 \in X$, $\alpha_k \in \mathbb{R}$ and $\gamma_k \in (0, \infty)$ for all k, and $\{u_k\} \subset X$ is an arbitrary sequence such that $u_k \to u$, and $\{e_k\}$ is a sequence of computational errors. We provide a necessary and sufficient condition for the zero set of A to be nonempty, and in this case, show the strong convergence of the sequence generated by (5) to a zero of A.

Lemma 3.1. *Let $A : D(A) \subset X \to \mathcal{P}(X)$ be an m-accretive operator, and the sequence $\{x_k\}$ be generated by* (5)*, where $\{\alpha_k\} \subset \mathbb{R}$ and $\{\gamma_k\} \subset (0,\infty)$ and $\gamma_k \to \infty$ such that $\{\alpha_k\}$ and $\{\alpha_k e_k\}$ are bounded. If $\{x_k\}$ is bounded and $r > 0$, then* $\lim_{k\to\infty} \|J^A_r x_k - x_k\| = 0$.

Proof. Without loss of generality, we assume that $\gamma_k > r$ for all k. Suppose that $z_{k-1} := (u_{k-1} + \alpha_{k-1}(x_{k-1} + e_{k-1}))$. Then we have

$$\begin{aligned}
&\|J^A_r x_k - x_k\| = \|J^A_r x_k - J^A_{\gamma_{k-1}} z_{k-1}\| \\
&= \|J^A_r x_k - J^A_r\Big(\frac{r}{\gamma_{k-1}}z_{k-1} + (1 - \frac{r}{\gamma_{k-1}})J^A_{\gamma_{k-1}} z_{k-1}\Big)\| \\
&\leq \|x_k - \Big(\frac{r}{\gamma_{k-1}}z_{k-1} + (1 - \frac{r}{\gamma_{k-1}})J^A_{\gamma_{k-1}} z_{k-1}\Big)\| \\
&= \|x_k - \Big(\frac{r}{\gamma_{k-1}}z_{k-1} + (1 - \frac{r}{\gamma_{k-1}})x_k\Big)\| \\
&= \frac{r}{\gamma_{k-1}}\|x_k - z_{k-1}\| \\
&\leq \frac{r}{\gamma_{k-1}}\Big(\|x_k - u_{k-1}\| + \mid \alpha_{k-1}(\mid \|x_{k-1}\| + \|e_{k-1}\|)\Big).
\end{aligned}$$

Since $\{\alpha_k\}$ and $\{\alpha_k e_k\}$ are bounded, $u_k \to u$ and $\gamma_k \to \infty$ as $k \to \infty$, we get

$$\lim_{k\to\infty} \|J^A_r x_k - x_k\| = 0.$$

□

Theorem 3.2. *Let X be a uniformly convex Banach space, and $A : D(A) \subset X \to \mathcal{P}(X)$ be an m-accretive operator. Suppose that the sequence $\{x_k\}$ is generated by* (5)*, where $\{\alpha_k\} \subset \mathbb{R}$ and $\{\gamma_k\} \subset (0,\infty)$ such that $\gamma_k \to \infty$, $\alpha_k \to 0$ and $\{\alpha_k e_k\}$ is bounded. Also $\{u_k\} \subset X$ is an arbitrary sequence such that $u_k \to u$. Then $A^{-1}(0) \neq \emptyset$ if and only if the sequence $\{x_k\}$ is bounded.*

Proof. Assume $A^{-1}(0) \neq \emptyset$ and let $x^* \in A^{-1}(0)$. Since the resolvent operator $J^A_{\gamma_k}$ is nonexpansive, we have

$$\begin{aligned}\|x^* - x_{k+1}\| &= \|x^* - J_{\gamma_k}(u_k + \alpha_k(x_k + e_k))\| \\ &\leq \|x^* - (u_k + \alpha_k(x_k + e_k))\| \\ &\leq \|x^*\| + \|u_k\| + \mid \alpha_k \mid (\|x_k\| + \|e_k\|)\end{aligned}$$

which implies that

$$\|x_{k+1}\| \leq 2\|x^*\| + \|u_k\| + \mid \alpha_k \mid (\|x_k\| + \|e_k\|) \quad (6)$$

By assumption, we have $u_k \to u$, $\alpha_k \to 0$ and $\{\alpha_k e_k\}$ is bounded. Let

$$L := \sup_k \Big\{2\|x^*\| + \|u_k\| + \mid \alpha_k \mid \|e_k\|\Big\}.$$

Then we have

$$\|x_{k+1}\| \leq \mid \alpha_k \mid \|x_k\| + L \quad (7)$$

Now since $\alpha_k \to 0$, there exists $r \in (0,1)$ and $k_0 \in \mathbb{N}$ such that $\mid \alpha_k \mid < r$ for all $k \geq k_0$. Then the above inequality implies that

$$\|x_{k+1}\| \leq r\|x_k\| + L, \qquad \text{for all } k \geq k_0. \quad (8)$$

Now we have

$$\begin{aligned}\|x_{k+1}\| &\leq r\|x_k\| + L \\ &\leq r\Big[r\|x_{k-1}\| + L\Big] + L \\ &= r^2\|x_{k-1}\| + rL + L \\ &\leq r^2\Big[r\|x_{k-2}\| + L\Big] + rL + L \\ &= r^3\|x_{k-2}\| + r^2L + rL + L \\ &\leq \cdots \\ &\leq r^{k-k_0+1}\|x_{k_0}\| + r^{k-k_0}L + \cdots + rL + L \\ &\leq r^{k-k_0+1}\|x_{k_0}\| + (\frac{1}{1-r})L.\end{aligned}$$

This implies that the sequence $\{x_k\}$ is bounded.

Conversely, let the sequence $\{x_k\}$ be bounded, then there exists a subsequence $\{x_{k_n}\}$ of $\{x_k\}$ such that $x_{k_n} \rightharpoonup p$ where p is a weak cluster point of the sequence $\{x_k\}$. On the other hand, for any $\lambda > 0$, the operator J^A_λ is nonexpansive, and hence it is demiclosed. Now since $x_{k_n} \rightharpoonup p$ and J^A_λ is demiclosed, therefore by Lemma 3.1 we get $p \in \text{Fix}(J^A_\lambda)$, that is $p \in A^{-1}(0)$. □

Theorem 3.3. *Let X be a uniformly smooth and uniformly convex Banach space, and $A : D(A) \subset X \to \mathcal{P}(X)$ be an m-accretive operator. Suppose that the sequence $\{x_k\}$ is generated by* (5)*, where $\{\alpha_k\} \subset \mathbb{R}$ and $\{\gamma_k\} \subset (0,\infty)$ such that $\gamma_k \to \infty$, $\alpha_k \to 0$ and $\alpha_k e_k \to 0$. Also $\{u_k\} \subset X$ is an arbitrary sequence such that $u_k \to u$. If $A^{-1}(0) \neq \emptyset$, then the sequence $\{x_k\}$ converges strongly to $Q_{A^{-1}(0)}(u)$, where $Q_{A^{-1}(0)}$ is the sunny nonexpansive retraction of X onto $A^{-1}(0)$.*

Proof. If $A^{-1}(0) \neq \emptyset$, then the sequence $\{x_k\}$ is bounded by Theorem 3.2. Now assume that

$$y_t := tu + (1-t)J^A_r y_t,$$

where $r > 0$ and $t \in (0,1)$. Note that y_t is well defined, because the map $y \longmapsto tu + (1-t)J^A_r(y)$ from X to X is a contraction and hence by the Banach fixed point theorem, it has a unique fixed point. Let $p \in A^{-1}(0)$. Then we have

$$\begin{aligned}\|y_t - p\| &= \|tu + (1-t)J^A_r y_t - p\| \\ &\leq t\|u-p\| + (1-t)\|J^A_r y_t - p\| \\ &\leq t\|u-p\| + (1-t)\|y_t - p\|.\end{aligned}$$

Therefore we get

$$\|y_t - p\| \leq \|u - p\|,$$

which implies that $\{y_t\}$ is bounded, and subsequently $\{J^A_r y_t\}$ is bounded too. Note that

$$\|y_t - J^A_r y_t\| = t\|u - J^A_r y_t\| \to 0 \quad (9)$$

as $t \to 0$. Now we show that $\{y_t\}$ is strongly convergent to an element in $A^{-1}(0)$ as $t \to 0$. Let $t_m \to 0$ and define $\psi : X \to [0,+\infty)$ by

$$\psi(x) = \text{LIM}\|y_{t_m} - x\|^2, \qquad x \in X,$$

where LIM is a Banach limit on ℓ^∞. Since X is reflexive, ψ is convex and continuous, and $\psi(x) \to +\infty$ as $\|x\| \to +\infty$, ψ has a minimum in X. Let

$$K = \Big\{y \in X : \psi(y) = \min_{x \in X}\Big\{\text{LIM}\|y_{t_m} - x\|^2\Big\}\Big\}.$$

It is easy to see that K is nonempty, closed, convex and bounded. Also, K is invariant under J^A_r. In fact, since $\|y_t - J^A_r y_t\| \to 0$ by (9), for any $y \in K$, we have

$$\begin{aligned}\psi(J^A_r y) &= \text{LIM}\|y_{t_m} - J^A_r y\|^2 \\ &= \text{LIM}\|J^A_r y_{t_m} - J^A_r y\|^2 \\ &\leq \text{LIM}\|y_{t_m} - y\|^2 \\ &= \psi(y).\end{aligned}$$

Therefore J_r^A has a fixed point q in K by Lemma 2.3. Now Lemma 2.1 implies that

$$\text{LIM}\langle x-q, J(y_{t_m}-q)\rangle \leq 0, \quad \text{for all } x \in X. \tag{10}$$

On the other hand, we have

$$\begin{aligned}&\|y_t - q\|^2 = \langle y_t - q, J(y_t - q)\rangle \\ &= t\langle u - q, J(y_t - q)\rangle + (1-t)\langle J_r^A y_t - q, J(y_t - q)\rangle \\ &\leq t\langle u - q, J(y_t - q)\rangle + (1-t)\|y_t - q\|^2.\end{aligned}$$

This implies that

$$\|y_t - q\|^2 \leq \langle u - q, J(y_t - q)\rangle. \tag{11}$$

Letting $t = t_m$ and using (11), we get

$$\text{LIM}\|y_{t_m} - q\|^2 \leq \text{LIM}\langle u - q, J(y_{t_m} - q)\rangle. \tag{12}$$

Taking $x = u$ in (10) and using (12), we have $\text{LIM}\|y_{t_m} - q\|^2 \leq 0$ which implies that $\text{LIM}\|y_{t_m} - q\|^2 = 0$. Therefore there is a subsequence $\{y_{t_{m_i}}\}$ of $\{y_{t_m}\}$ such that $y_{t_{m_i}} \to q$.

In the sequel, we prove that $y_t \to q$ as $t \to 0$. Let $\{y_{t_j}\}$ be another subsequence of $\{y_t\}$ such that $y_{t_j} \to z$. Replacing $t = t_j$ in (11) and taking the limit, we get

$$\|z - q\|^2 \leq \langle u - q, J(z - q)\rangle \tag{13}$$

By using (9), it is clear that z is a fixed point of J_r^A. Therefore a similar inequality as in (11) holds for q replaced by z. Replacing $t = t_{m_i}$ in that inequality and taking the limit, we get

$$\|q - z\|^2 \leq \langle u - z, J(q - z)\rangle \tag{14}$$

Adding both sides of (13) and (14), we get

$$2\|q - z\|^2 \leq \langle q - z, J(q - z)\rangle = \|q - z\|^2, \tag{15}$$

which implies that $q = z$ and hence $y_t \to q$ as $t \to 0$.

In the sequel, since $y_t = tu + (1-t)J_r^A y_t$, we have

$$y_t - u = -\frac{1-t}{t}(I - J_r^A)y_t \tag{16}$$

Now, for any $p \in \text{Fix}(J_r^A) = A^{-1}(0)$, we have

$$\begin{aligned}&\langle y_t - u, J(y_t - p)\rangle = -\frac{1-t}{t}\langle (I - J_r^A)y_t, J(y_t - p)\rangle \\ &= -\frac{1-t}{t}\langle (I - J_r^A)y_t - (I - J_r^A)p, J(y_t - p)\rangle \\ &\leq 0.\end{aligned}$$

Letting $t \to 0$, we get

$$\langle q - u, J(q - p)\rangle \leq 0, \quad \text{for all } p \in A^{-1}(0). \tag{17}$$

Now since $A^{-1}(0)$ is a nonempty, closed and convex subset of X and u can be chosen arbitrarily in X, then by (4), (17) implies that q is the sunny nonexpansive retraction of u onto $A^{-1}(0)$. From now on, we also denote q by $Q_{A^{-1}(0)}(u)$ where $Q_{A^{-1}(0)}(u)$ is the sunny nonexpansive retraction of u onto $A^{-1}(0)$.

In the sequel, we prove that the sequence $\{x_k\}$ converges strongly to $q = Q_{A^{-1}(0)}(u)$. Let $M := \sup\{\|x_k - y_t\| : t \in (0,1),\ k \geq 0\}$. Then we have

$$\begin{aligned}&\|y_t - x_k\|^2 = t\langle u - x_k, J(y_t - x_k)\rangle \\ &+ (1-t)\langle J_r^A y_t - x_k, J(y_t - x_k)\rangle \\ &= t\langle u - y_t + y_t - x_k, J(y_t - x_k)\rangle \\ &+ (1-t)\langle J_r^A y_t - J_r^A x_k + J_r^A x_k - x_k, J(y_t - x_k)\rangle \\ &= t\langle u - y_t, J(y_t - x_k)\rangle + t\|y_t - x_k\|^2 \\ &+ (1-t)\langle J_r^A y_t - J_r^A x_k, J(y_t - x_k)\rangle \\ &+ (1-t)\langle J_r^A x_k - x_k, J(y_t - x_k)\rangle \\ &\leq t\langle u - y_t, J(y_t - x_k)\rangle + t\|y_t - x_k\|^2 \\ &+ (1-t)\|y_t - x_k\|^2 + M\|J_r^A x_k - x_k\|.\end{aligned}$$

This shows that

$$\langle u - y_t, J(x_k - y_t)\rangle \leq \frac{M}{t}\|J_r^A x_k - x_k\|, \quad \forall\, t \in (0,1). \tag{18}$$

Since $\|J_r^A x_k - x_k\| \to 0$ by Lemma 3.1, we get

$$\limsup_{k\to\infty}\langle u - y_t, J(x_k - y_t)\rangle \leq 0, \quad \forall\, t \in (0,1). \tag{19}$$

We note that

$$\begin{aligned}&|\ \langle u - q, J(x_k - q)\rangle - \langle u - y_t, J(x_k - y_t)\rangle\ | \\ &= |\ \langle u - q, J(x_k - q) - J(x_k - y_t)\rangle \\ &+ \langle q - y_t, J(x_k - y_t)\rangle\ | \\ &\leq |\ \langle u - q, J(x_k - q) - J(x_k - y_t)\rangle\ | + M\|q - y_t\|.\end{aligned}$$

Now since $y_t \to q$ as $t \to 0$ and J is uniformly norm to weak* continuous on bounded subsets of X, we have

$$\lim_{t\to 0}\ |\ \langle u - q, J(x_k - q)\rangle - \langle u - y_t, J(x_k - y_t)\rangle\ | = 0 \tag{20}$$

uniformly for $k \geq 0$. Therefore (19) and (20) imply that

$$\limsup_{k\to\infty}\langle u - q, J(x_k - q)\rangle \leq 0. \tag{21}$$

On the other hand, by (5) and the definition of the resolvent operator, we have

$$v_k := \frac{u_k + \alpha_k(x_k + e_k) - x_{k+1}}{\gamma_k} \in A(x_{k+1}) \tag{22}$$

Since $q \in A^{-1}(0)$ and A is an accretive operator, we have

$$\langle 0 - v_k, J(q - x_{k+1})\rangle \geq 0. \tag{23}$$

Let $z_k := u_k + \alpha_k(x_k + e_k)$, then we get

$$\langle z_k - x_{k+1}, J(q - x_{k+1})\rangle \leq 0, \tag{24}$$

which implies that

$$\begin{aligned}\|q - x_{k+1}\|^2 &= \langle q - x_{k+1}, J(q - x_{k+1})\rangle \\ &\leq \langle z_k - q, J(x_{k+1} - q)\rangle. \end{aligned} \tag{25}$$

On the other hand, since $\alpha_k \to 0$, $\alpha_k e_k \to 0$ and $u_k \to u$ as $k \to \infty$, we have $z_k \to u$ as $k \to \infty$. Therefore (21) shows that

$$\begin{aligned}&\limsup_{k\to\infty}\langle z_k - q, J(x_{k+1} - q)\rangle \\ &= \limsup_{k\to\infty}\langle u - q, J(x_{k+1} - q)\rangle \leq 0. \end{aligned} \tag{26}$$

Now, (25) and (26) imply that the sequence $\{x_k\}$ converges strongly to $q = Q_{A^{-1}(0)}(u)$.

□

Remark 3.4. According to Theorem 3.3, the limit is unique as long as u, which is the limit of an arbitrary sequence $\{u_k\}$, is fixed.

Corollary 3.5. *Let X be a uniformly smooth and uniformly convex Banach space, and $A : D(A) \subset X \to \mathcal{P}(X)$ be an m-accretive operator. Suppose that the sequence $\{x_k\}$ is generated by* (5)*, where $\{\alpha_k\} \subset \mathbb{R}$ and $\{\gamma_k\} \subset (0,\infty)$ such that $\gamma_k \to \infty$, $\alpha_k \to 0$ and $\alpha_k e_k \to 0$. Also $\{u_k\} \subset X$ is an arbitrary sequence such that $u_k \to u$. Then the sequence $\{x_k\}$ is bounded if and only if the sequence $\{x_k\}$ is strongly convergent to $Q_{A^{-1}(0)}(u)$, where $Q_{A^{-1}(0)}$ is the sunny nonexpansive retraction of X onto $A^{-1}(0)$.*

Proof. This is a direct consequence of Theorems 3.2 and 3.3. □

The following corollary extends a result of Reich [1, Theorem 1].

Corollary 3.6. *Let X be a uniformly smooth and uniformly convex Banach space, and $A : D(A) \subset X \to \mathcal{P}(X)$ be an m-accretive operator. Suppose that $x \in X$ and $\{\gamma_k\} \subset (0,\infty)$ such that $\gamma_k \to \infty$ as $k \to \infty$. Then the following statements are equivalent:*
(i) $A^{-1}(0) \neq \emptyset$,
(ii) *the sequence $\{J^A_{\gamma_k}(x)\}$ is bounded,*
(iii) *$\{J^A_{\gamma_k}(x)\}$ converges strongly to $Q_{A^{-1}(0)}(x)$ as $k \to \infty$, where $Q_{A^{-1}(0)}$ is the sunny nonexpansive retraction of X onto $A^{-1}(0)$.*

Proof. The proof follows directly from Theorems 3.2 and 3.3 by taking $\alpha_k \equiv 0$ and $u_k \equiv x$. □

4 Applications

In this section, we apply our main results to approximate the solution of a variational inequality and the fixed point of a nonexpansive operator.

Let C be a nonempty, closed and convex subset of a Banach space X and let $A : C \to X$ be an operator. A variational inequality for A and C, abbreviated as $\mathrm{VI}(A,C)$ consists of finding $x^* \in C$ such that

$$\langle Ax^*, J(y - x^*)\rangle \geq 0, \quad \forall y \in C. \tag{27}$$

We denote the set of all solutions to (27) by $S(A,C)$. The associated dual variational inequality, denoted by $\mathrm{DVI}(A,C)$, is expressed as finding $x^* \in C$ such that $\langle Ay, J(x^* - y)\rangle \leq 0$ for all $y \in C$. It is clear that $\mathrm{VI}(A,C)$ has a solution if and only if the zero set of the operator A is nonempty.

Let $T : C \to C$ be a map, and $A(x) = x - Tx$ for $x \in C$. Then an example of the above variational inequality problem is to find $x^* \in C$ such that

$$\langle x^* - Tx^*, J(y - x^*)\rangle \geq 0, \quad \forall y \in C. \tag{28}$$

If we set $y = Tx^*$, then (28) implies that $Tx^* = x^*$. Therefore $\mathrm{VI}(A,C)$ is equivalent to the fixed point problem for the mapping $T : C \to C$.

Note that if the mapping $T : C \to C$ is nonexpansive, then we have

$$\begin{aligned}\langle Tx - Ty, J(x - y)\rangle &\leq \|Tx - Ty\|\|x - y\| \\ &\leq \|x - y\|^2 \\ &= \langle x - y, J(x - y)\rangle. \end{aligned} \tag{29}$$

This implies that

$$\langle (x - Tx) - (y - Ty), J(x - y)\rangle \geq 0. \tag{30}$$

Therefore the operator $A : C \to X$ defined by $A(x) = x - Tx$ is accretive. Hence the following statements are equivalent:
(i) $A^{-1}(0) \neq \emptyset$.
(ii) $S(A,C) \neq \emptyset$.
(iii) $\mathrm{Fix}(T) \neq \emptyset$.

Now we apply our main results to find a zero of the accretive operator A which will be a solution to the variational inequality problem $\mathrm{VI}(A,C)$, or a fixed point of the nonexpansive mapping T.

Let X be a uniformly smooth and uniformly convex Banach space, and $T : C \to C$ be a nonexpansive mapping. Let the sequence $\{x_k\}$ be generated by

$$\begin{aligned}x_{k+1} &= J^{I-T}_{\gamma_k}(u_k + \alpha_k(x_k + e_k)) \\ &= (I + \gamma_k(I - T))^{-1}(u_k + \alpha_k(x_k + e_k)), \end{aligned} \tag{31}$$

where $x_0 \in X$, $\alpha_k \in \mathbb{R}$ and $\gamma_k \in (0,\infty)$ for all k, and $\{u_k\} \subset X$ is an arbitrary sequence such that $u_k \to u$, and $\{e_k\}$ is a sequence of computational errors. We first show that the sequence $\{x_k\}$ is well defined. Then we present the associated results. For simplicity, denote $v_k = u_k + \alpha_k(x_k + e_k)$. Note that the map $y \longmapsto \frac{1}{1+\gamma_k}v_k + \frac{\gamma_k}{1+\gamma_k}Ty$ from X to X is a contraction and hence by the Banach fixed point theorem, it has a unique fixed point x_{k+1}. Then we have $x_{k+1} = \frac{1}{1+\gamma_k}v_k + \frac{\gamma_k}{1+\gamma_k}Tx_{k+1}$ which implies that $x_{k+1} = (I + \gamma_k(I-T))^{-1}(v_k)$.

Corollary 4.1. *Let X be a uniformly smooth and uniformly convex Banach space, and $T : X \to X$ be a nonexpansive mapping. Suppose that the sequence $\{x_k\}$ is generated by* (31)*, where $\{\alpha_k\} \subset \mathbb{R}$ and $\{\gamma_k\} \subset (0,\infty)$ such that $\gamma_k \to \infty$, $\alpha_k \to 0$ and $\alpha_k e_k \to 0$. Also $\{u_k\} \subset X$ is an arbitrary sequence such that $u_k \to u$. Then the following statements are equivalent:*
(i) Fix$(T) \neq \emptyset$,
(ii) *the sequence $\{x_k\}$ is bounded,*
(iii) *$\{x_k\}$ is strongly convergent to $Q_{\text{Fix}(T)}(u)$, where $Q_{\text{Fix}(T)}$ is the sunny nonexpansive retraction of X onto* Fix(T).

Proof. Since $T : X \to X$ is a nonexpansive mapping, the operator $I - T$ is accretive. Now we show that $I-T$ is an m-accretive operator. In fact, we show that $R(I + \lambda(I-T)) = X$ for all $\lambda > 0$. Take $y \in X$ and note that the map $z \longmapsto \frac{1}{1+\lambda}y + \frac{\lambda}{1+\lambda}Tz$ from X to X is a contraction and hence by the Banach fixed point theorem, it has a unique fixed point x. Therefore we have $x = \frac{1}{1+\lambda}y + \frac{\lambda}{1+\lambda}Tx$ which implies that $(I+\lambda(I-T))x = y$. Now the proof follows from Theorems 3.2 and 3.3. ☐

The following corollary is a result of Corollary 4.1.

Corollary 4.2. *Let X be a uniformly smooth and uniformly convex Banach space, and $T : X \to X$ be a nonexpansive mapping. Suppose that $x \in X$ and $\{\gamma_k\} \subset (0,\infty)$ such that $\gamma_k \to \infty$ as $k \to \infty$. Then the following statements are equivalent:*
(i) Fix$(T) \neq \emptyset$,
(ii) *the sequence $\{J^{I-T}_{\gamma_k}(x)\}$ is bounded,*
(iii) *$\{J^{I-T}_{\gamma_k}(x)\}$ converges strongly to $Q_{\text{Fix}(T)}(x)$ as $k \to \infty$.*

Proof. The proof follows directly from Corollary 4.1 by taking $\alpha_k \equiv 0$ and $u_k \equiv x$. ☐

Let H be a real Hilbert space, $C \subset H$ be nonempty, closed and convex, and let $f : H \times H \to \mathbb{R}$ be a bifunction. An equilibrium problem for f and C consists of finding $x^* \in C$ such that

$$f(x^*, y) \geq 0, \quad \forall y \in C. \tag{32}$$

We denote the set of all solutions for (32) by $S(f,C)$. It is well known that equilibrium problems unify many problems in nonlinear analysis and optimization like convex optimization problems, fixed point problems variational inequalities , Nash equilibrium problems, etc. We introduce some standard conditions on the bifunction f that are generally used for the study of the convergence analysis:
B1 : $f(x,x) = 0$ for all $x \in H$.
B2 : $f(\cdot,x) : H \to \mathbb{R}$ is upper semicontinuous for all $x \in H$.
B3 : $f(x,\cdot) : H \to \mathbb{R}$ is convex and lower semicontinuous for all $x \in X$.
B4 : $f(x,y) + f(y,x) \leq 0$ for all $x,y \in H$, i.e. f is monotone.

It was shown in Theorem 4 of [14], that when f satisfies B1–B4, then $U^f : H \to \mathcal{P}(H)$ defined by

$$U^f(x) = \partial f(x,\cdot)(x) = \big\{u^* \in H : \langle y - x, u^*\rangle \leq f(x,y) - f(x,x), \forall\, y \in H\big\} \tag{33}$$

is an m-accretive (maximal monotone) operator. Moreover note that any solution of VI(U^f, C) belongs to $S(f,C)$ when f satisfies B1–B4.

Suppose that $f : H \times H \to \mathbb{R}$ satisfies B1-B4, and let the sequence $\{x_k\}$ be generated by

$$x_{k+1} = J^{U^f}_{\gamma_k}(u_k + \alpha_k(x_k + e_k)), \tag{34}$$

where $x_0 \in H$, $\alpha_k \in \mathbb{R}$ and $\gamma_k \in (0,\infty)$ for all k, and $\{u_k\} \subset H$ is an arbitrary sequence such that $u_k \to u$, and $\{e_k\}$ is a sequence of computational errors.

Corollary 4.3. *Suppose that $f : H \times H \to \mathbb{R}$ satisfies B1-B4. Let the sequence $\{x_k\}$ be generated by* (34)*, where $\{\alpha_k\} \subset \mathbb{R}$ and $\{\gamma_k\} \subset (0,\infty)$ such that $\gamma_k \to \infty$, $\alpha_k \to 0$ and $\alpha_k e_k \to 0$. Also let $\{u_k\} \subset H$ be an arbitrary sequence such that $u_k \to u$. Then the following statements are equivalent:*
(i) $S(f,C) \neq \emptyset$,
(ii) *the sequence $\{x_k\}$ is bounded,*
(iii) *$\{x_k\}$ is strongly convergent to $P_{S(f,C)}(u)$, where $P_{S(f,C)}$ is the projection of H onto $S(f,C)$.*

Proof. Since f satisfies B1–B4, then $U^f : H \to \mathcal{P}(H)$ is m-accretive (maximal monotone) by Theorem 4 of [14]. Now the proof follows from Theorems 3.2 and 3.3. ☐

Remark 4.4. Let $\phi : H \to \mathbb{R}$ be a proper, convex and lower semicontinuous function. Define $f(x,y) = \phi(y) - \phi(x)$ for all $x,y \in H$. Since ϕ is convex and lower semicontinuous, the bifunction $f : H \times H \to \mathbb{R}$ satisfies B1-B4. Then Corollary 4.3 may be applied to find a minimum point of ϕ.

We end this paper by providing a numerical example where our main result can be applied.

Example 4.5. Let $C = \mathbb{R}^3$ and define the bifunction $f : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ by $f(z, y) = \phi(y) - \phi(z)$ where $\phi(z) = zAz^t + Bz^t$ and

$$A = \begin{bmatrix} 3 & 2 & 0 \\ -2 & 4 & -1 \\ 0 & 1 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 4 & -1 & 3 \end{bmatrix}$$

where z^t denotes the transpose of the vector $z = \begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix}$. Note that $f(z, \cdot) : \mathbb{R}^3 \to \mathbb{R}$ is convex for all $z \in \mathbb{R}^3$, because A is positive definite. Clearly f satisfies B1-B4. Also, $S(f, C) \neq \varnothing$. Indeed, the unique solution of the equilibrium problem is $x^* = \begin{bmatrix} \frac{-2}{3} & \frac{1}{8} & \frac{-3}{10} \end{bmatrix}$, because $\phi(z) = 3z_1^2 + 4z_2^2 + 5z_3^2 + 4z_1 - z_2 + 3z_3$.

In order to illustrate an application of Corollary 4.3, we take $\alpha_k = \frac{1}{k+1}$, $\gamma_k = k^3 + 1$, $x_1 = \begin{bmatrix} -2 & 1 & 3 \end{bmatrix}$, $u_k = \begin{bmatrix} 4 + \frac{2}{k+1} & -1 + \frac{3}{k+1} & 2 + \frac{2}{k+1} \end{bmatrix}$, $e_k = \begin{bmatrix} 2\sqrt{k} + 5 & -\sqrt{k} + 4 & 3\sqrt{k} - 2 \end{bmatrix}$. It is easy to see that $u_k \to u = \begin{bmatrix} 4 & -1 & 2 \end{bmatrix}$ and $P_{S(f,K)}(u) = \begin{bmatrix} \frac{-2}{3} & \frac{1}{8} & \frac{-3}{10} \end{bmatrix}$. Note that the error sequence $\{e_k\}$ is unbounded and the conditions of Corollary 4.3 are satisfied. If $\{x_k\}$ is the sequence generated by

$$x_{k+1} = J_{\gamma_k}^{Uf}(u_k + \alpha_k(x_k + e_k)),$$

then Corollary 4.3 ensures that $\{x_k\}$ converges to x^*. We performed the numerical experiment for this example and the numerical results are displayed in Table 1. The table shows that the sequence $\{x_k\}$ is convergent to $\begin{bmatrix} \frac{-2}{3} & \frac{1}{8} & \frac{-3}{10} \end{bmatrix}$, which is the solution of EP(f, C). This problem was solved by the Optimization Toolbox in Matlab R2020a on a Laptop Intel(R) Core(TM) i7- 8665U CPU @ 1.90GHz RAM 8.00 GB.

Table 1. The numerical experiment for Example 4.5

k	x_{k+1}	$\|x_{k+1} - x_k\|$	$\|x_{k+1} - x^*\|$
1	(1.5714, 1.0556, 0.7273)	4.2336	2.6325
2	(0.5309, 0.4169, -0.0418)	1.4429	1.2594
3	(-0.1826, 0.2167, -0.1907)	0.7559	0.5047
10	(-0.6258, 0.1274, -0.2837)	0.0093	0.0440
100	(-0.6658, 0.1249, -0.2994)	1.5803×10^{-5}	0.0010
200	(-0.6664, 0.1249, -0.2998)	2.7028×10^{-6}	3.4551×10^{-4}
500	(-0.6666, 0.1250, -0.2999)	2.7841×10^{-7}	8.5606×10^{-5}

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