Λ-fractional Analysis. Basic Theory and Applications

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Abstract: - Fractional Analysis is a mathematical method based on different principles from those governing the well-known mathematical principles of differential and integral calculus. The main difference from ordinary differential analysis lies in its property being a non-local analysis, not a local one. This analysis is essential in studying problems in physics, engineering, biology, biomechanics, and others that fall into the micro and nano areas. However, the main issue in fractional analysis is the mathematical imperfections presented by fractional derivatives. In fact, not all known fractional derivatives meet the differential topology requirements for mathematical derivatives. Hence, Λ-fractional differential geometry is invented and applied in various scientific areas, like physics, mechanics, biology, economy, and other fields. Apart from the basic mathematical theory concerning establishing the Λ-fractional derivative, the corresponding differential geometry, differential equations, variational methods, and fields theory are outlined. Proceeding to the applications, Λ-fractional continuum mechanics, Λ-fractional viscoelasticity, Λ-fractional physics, Λ-fractional beam and plate theory are discussed. It is pointed out that only globally stable states are allowed into the context of Λ-fractional analysis.

Key-Words: - Λ-Fractional Derivative, Λ-Fractional space, Λ-Fractional Differential Geometry, Λ- Fractional Tangent, Λ-Fractional Curvature, Λ-Fractional Focal Curve, Λ-fractional tangent space of surfaces, the first and second fundamental forms of surfaces, the covariant Λ-fractional derivative.

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1 Introduction

The need for global differential analysis in physics and engineering has pushed the blooming of fractional calculus, which is inherently global. In [1], pointing out the importance of the global analysis in micro and nanophysics, refers in the introduction of his book, *"Non-local continuum field theories are concerned with the physics of material bodies whose behavior of a material point is influenced by the state of all points of the body. The non-local theory generalizes the classical field theory in two respects i) the energy balance law is considered valid globally(for the entire body), and ii) the state of the body at a material point is described by the response functionals".*

In fact, Fractional Calculus is a mathematical method of global analysis, demanded in various theories concerning micro and nano theories in physics and engineering and not only. [2], [3] and [4], and not only were highly concerned about fractional integrals and derivatives. [5] have presented the historical evolution of fractional calculus. Fractional calculus is an indispensable tool in describing the non-local character of many phenomena in mechanics, physics, engineering, control theory, and economics, [6]. Systematic description and analysis of the fractional calculus field may be found in the books [7], [8], [9], [10], [11] and various applications are also included.

Local mathematical calculus cannot describe an extensive collection of phenomena in mechanics, physics, and other scientific fields. Since intermolecular attractions are considerable, non-local continuum theories are only accepted. Indeed, singularities are generated by various fields, like the application of concentrated loads to the fields of stresses and strains around the crack tips and dislocations. Further, viscoelastic problems are described through time fractional fields. In fact, only viscoelastic problems have been formulated using fractional time calculus for the last 50 years. The interest concerning fractional calculus was turned from time to space [12], expressing the distribution of non-homogeneous material fields, like microcracks, composite materials, and others, through fractional formulation. Moreover, a lot have been done in the field on stability criteria, [13].

Contrary to the conventional strain, [12] and [14] proposed the fractional strain. That strain exhibits strong non-local character. However, the well-known fractional derivatives are mathematically simple operators, not mathematical derivatives satisfying the prerequisites demanded by Differential Topology,

a. Linearity $D\big(af(x)+bg(x)\big)=aDf(x)+bDg(x)$ (1)

b. Leibniz rule $D\big(f(x)\cdot g(x)\big)=Df(x)\cdot g(x)+f(x)\cdot Dg(x)$ (2)

c. Chain rule $D\big(g(f)\big)(x)=Dg(f(x)\cdot Df(x)$ (3)

Hence, the use of fractional derivatives was not mathematically established. [15] proposed the L-fractional derivative, then the fractional Λ-derivative, which is mathematically correct. The proposed non-local procedure has already been applied to the various problems in mechanics and not only. We propose Λ- fractional deformations and define the corresponding deformation tensors [14], [15], [16], [17], [18].

The fractional strain has been presented in [19], [20], [21], [22], however, they do not comply with fractional differential geometries. Further, the Λ-fractional strain tensors will also be presented. Let us point out that the strain exists only in the Λ-space, having the character of a derivative.

We present the formulation of the Λ-space, where geometry and physics are valid in a conventional way. Further, the results in the Λ-space are not the true results. They should be transferred to the initial space. The problem concerning the extension of a fractional bar under axial load is presented. Its solution exhibits the size effect phenomenon. Similar phenomena have been presented in the strain gradient theories.

Λ-Fractional differential analysis is formulated in the present work. The fractional derivatives are local and conventional in the Λ-fractional space. Hence, differential geometry is generated. The actual results should be located in the initial space and should be found by transferring the results found in the Λ-space into the initial space. The Λ-fractional transformation, governing the Λ-fractional space, is similar to Laplace's transformation. The results are transferred from the Λ-fractional space to the initial one, the true space.

2 Summary of Fractional Calculus

One of the most active and interesting fields in applied mathematics is Fractional Calculus due to its broad applications in micro and nano problems. Many applications in engineering and physics are considered in that context. Due to the high importance of Fractional Calculus in many applications, the field has also been extended theoretically. There exist many definitions of fractional derivatives. Leibniz, looking for the possibility of defining the derivative $\frac{d^n g}{dx^n}$ when $n=\frac{1}{2}$, suggested fractional derivatives. There exists a plethora of those derivatives. Nevertheless, they are all non-local, contrary to the conventional ones that are local. Information about fractional analysis and its applications may be found in the classical books of [8], [9] and [10].

Recalling the n-fold integral of a function *f(x)*

$$ {}_{\mathrm{a}}I_x^n f\left(x\right)= \frac{1}{(n-1)!}\int_{\mathrm{a}}^{x}(x-s)^{n-1}f(s)ds, \quad x>0, n\in N \tag{4}$$

Leibniz defined the *γ*-multiple integral with $0<\gamma<1$ by,

$$ {}_aI_x^\gamma f(x)=\frac{1}{\Gamma(\gamma)}\int_a^x\frac{f(s)}{(x-s)^{1-\gamma}}ds \tag{5}$$

with *Γ(γ)* Euler's Gamma function.

Further, the left Riemann-Liouville (R-L) derivatives are defined by:

$$ {}_a^{RL}D_x^\gamma f(x)=\frac{d}{dx}\left({}_\alpha I_x^{1-\gamma}f(x)\right)= \frac{1}{\Gamma(1-\gamma)}\frac{d}{dx}\int_a^x\frac{f(s)}{(x-s)^\gamma}ds. \tag{6}$$

Moreover, definitions for the right fractional integrals and derivatives have been assigned, [9].

3 The Λ-Fractional Derivative

The authors initially introduced The L-fractional derivative to establish a fractional derivative conforming to the properties of a derivative demanded by the Differential Topology. The objective was for the fractional derivative to correspond to a fractional differential. As has been mentioned, the prerequisites of differential topology for the derivatives are [23]:

a.Linearity:

$$D(af+bg)(x)= aDf(x)+bDg(x) \tag{7}$$

b. Composition (chain rule):

$$D\left(f(g)\right)(x) = Df(g)\cdot D(g)(x) \tag{8}$$

c. Leibniz's (product) rule:

$$D(f\cdot g)(x) = Df(x)\cdot g(x) + f(x)\cdot D(g(x)) \tag{9}$$

The Λ-fractional derivative (Λ-FD) has been introduced as:

$${}_a^{\Lambda}D_x^{\gamma} f(x) = \frac{{}_{a}^{RL}D_x^{\gamma} f(x)}{{}_{a}^{RL}D_x^{\gamma} x} \quad . \tag{10}$$

Hence, according to the definition of the fractional derivative, Eq. (6), the Λ-FD is defined by:

$${}_a^{\Lambda}D_x^{\gamma} f(x) = \frac{\dfrac{d\,{}_aI_x^{1-\gamma} f(x)}{dx}}{\dfrac{d\,{}_aI_x^{1-\gamma} x}{dx}} = \frac{d\,{}_aI_x^{1-\gamma} f(x)}{d\,{}_aI_x^{1-\gamma} x} \tag{11}$$

Assuming $X = {}_aI_x^{1-\gamma} x$ and $F(X) = {}_aI_x^{1-\gamma} f(x)$ the Λ-FD behaves like a conventional derivative, with local behavior in the fractional space *(X, F(X))*. Hence, it is possible to generate Fractional Differential Geometry in the Λ-Fractional space *(X, F(X))*, with the derivative,

$${}_a^{\Lambda}D_x^{\gamma} f(x) = \frac{dF(X)}{dX} \quad . \tag{12}$$

Further the relation,

$${}_{a}^{RL}D_x^{1-\gamma}\left({}_aI_x^{1-\gamma} f(x)\right) = f(x) \tag{13}$$

pulls back to the initial space, the functions generated in the fractional Λ-space. The proposed Λ-Fractional analysis is similar to Laplace's transformation. Further, Eq.(13) is similar to the inverse of Laplace's transformation.

For better understanding, consider in the initial space the function,

$$f(x) = x^2 . \tag{14}$$

Transferring that function in the Λ-fractional space *(X, F(X))* (with *a*=0),

$$X = \frac{x^{2-\gamma}}{\Gamma(3-\gamma)} \tag{15}$$

$$F(X) = {}_0I_x^{1-\gamma} f(x(X)) = \frac{1}{\Gamma(1-\gamma)}\int_0^x \frac{s^2}{(x-s)^{\gamma}} ds = -\frac{2}{\Gamma(4-\gamma)} x^{(3-\gamma)} . \tag{16}$$

Further considering from Eq. (15),

$$x = \left(\Gamma(3-\gamma)X\right)^{\frac{1}{2-\gamma}} . \tag{17}$$

Eq. (16) yields

$$F(X) = \frac{2\left(\left(\Gamma(3-\gamma)X\right)^{\frac{1}{2-\gamma}}\right)^{3-\gamma}}{\Gamma(4-\gamma)} . \tag{18}$$

The curve in the initial plane (*x, f(x)*), shown in Figure 1.

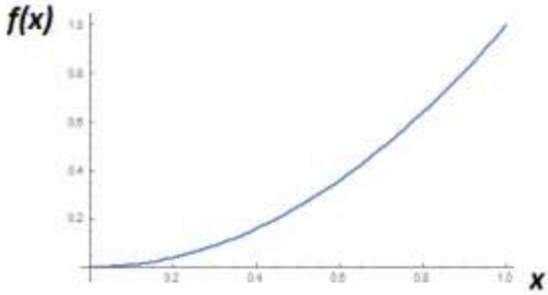

Fig. 1: The original plane (*x*, $f(x)=x^2$)

is transferred to the Λ-fractional plane (space) shown in Figure 2, for *γ*=0.6.

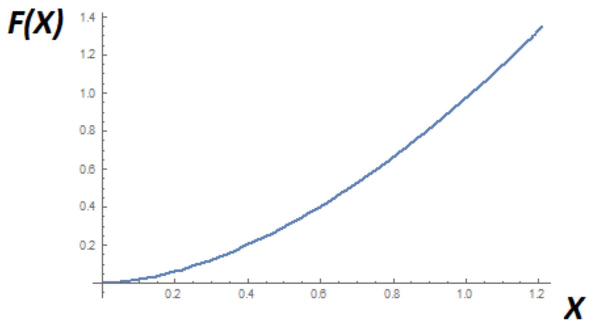

Fig. 2: The curve in the Fractional Space *(X, F(X))* for γ=0.6

Thus, the derivative:

$$D(F(X))=\frac{dF(X)}{dX}=\frac{24(5-\gamma)\Gamma(3-\gamma)(X\Gamma(3-\gamma))^{\frac{3}{2-\gamma}}}{(2-\gamma)\Gamma(6-\gamma)} \quad (19)$$

The Λ-fractional plane for X_0=0.6 and γ=0.6 yields the derivative $D(F(X_0))$=1.1580. Consequently, the equation of the tangent line $Y(X)$ of that curve at a point X_0 is defined by Figure 3

$$Y(X)=F(X_0)+\frac{d}{dX}(F(X_0))(X-X_0). \quad (20)$$

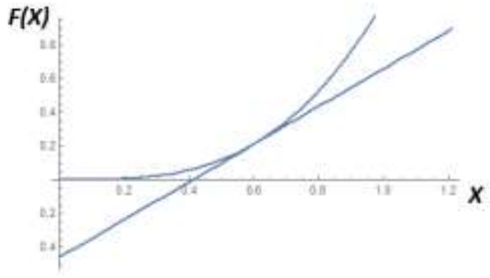

Fig. 3: Λ-Fractional Plane with the tangent space of the curve (x,F(X))

The tangent space is transferred in the original plane $(x, f(x))$ following the procedure explained below:

The point x_0=0.81 in the initial x-axis, corresponding to X_0=0.60 is defined, recalling Eq. (20), Then substituting in the derivative $\frac{dF(X)}{dX}$ in the fractional plane, the X as a function of x, the ${}^{\Lambda}_{0}D^{\gamma}_{x}f(x)$ is defined. Hence, the corresponding function in the real space (x, $f(x)$) may be pulled back by the relation ${}^{RL}_{0}D^{1-\gamma}_{x}\left({}^{\Lambda}_{0}D^{\gamma}_{x}f(x)\right)$. Indeed,

$${}^{RL}_{0}D^{1-\gamma}_{x}\left({}^{\Lambda}_{0}D^{\gamma}_{x}f(x)\right)=\frac{1}{\Gamma(\gamma)}\frac{d}{dx}\int_0^x(x-s)^{\gamma-1}\,{}^{\Lambda}_{0}D^{\gamma}_{x}f(s)\,ds. \quad (21)$$

In the present case for the function $f(x)=x^2$

$${}^{RL}_{0}D^{1-\gamma}_{x}\left({}^{\Lambda}_{0}D^{\gamma}_{x}x^2\right)_{x=0.81}=1.41. \quad (22)$$

Thus, the fractional tangent space $g(x)$ in the original space (x, $f(x)$) is defined by:

$$g(x)=f(x)_{x_0}+{}^{RL}_{0}D^{1-\gamma}_{x}\left({}^{L}_{0}D^{\gamma}_{x}f(x)\right)_{x=x_0}\left(\frac{x^{2-\gamma}}{\Gamma(3-\gamma)}-X_0\right) \quad (23)$$

In the present case at $X_0=0.6$ for $\gamma=0.6$, $x_0=0.81$, the tangent space is expressed by Figure 4.

$$g(x)=(x^2)_{x=0.81}+1.41\left(\frac{1.79\,x^{1.4}}{\Gamma(0.4)}-0.6\right) \quad (24)$$

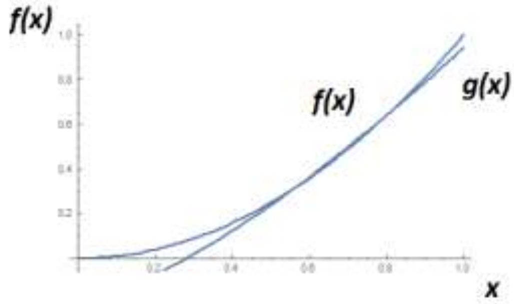

Fig. 4: The fractional tangent space $g(x)$ of the curve f(x) in the original plane for γ=0.6 at x=0.81

4 The fractional Arc-Length

The Λ-Fractional plane $(X, Y(X))$ for a function $y=f(x)$, with its Λ-Fractional Differential of order $0<\gamma<1$, is defined by:

$$dY(X)=\frac{dY(X)}{dX}dX \quad (25)$$

with X and $Y(X)$ defined by $X={}_aI_x^{1-\gamma}x$ and $F(X)={}_aI_x^{1-\gamma}f(x)$. Hence, in the Λ-Fractional Plane, the arc length is expressed by:

$$S(X)=\left(\frac{(dF(X))^2}{(dX)^2}+1\right)^{\frac{1}{2}}dX \quad (26)$$

Likewise, in the original plane, the arc length $s(x)$ is defined by:

$$s(x)={}^{RL}_{0}D^{1-\gamma}_{x}(S(X))=$$
$${}^{RL}_{0}D^{1-\gamma}_{x}\left(S\left(\frac{x^{2-\gamma}}{(2-3\gamma+\gamma^2)\Gamma(1-\gamma)}\right)\right) \quad (27)$$

Nevertheless, for the curves defined parametrically by:

$$x=g(t),\quad y=f(t). \quad (28)$$

The differential of the arc length in the Λ-fractional plane is defined by:

$$dS(T)=\sqrt{\left(\frac{dY(T)}{dT}\right)^2+\left(\frac{dX(T)}{dT}\right)^2}\,dT \quad (29)$$

With the expression of the arc length,

$$S(T)=\int_0^T dS(T). \quad (30)$$

The Integral Equation below yields the arc length *s(t)* in the original plane:

$$s(t) = {}^{RL}_{a}D_x^{1-\gamma}(S(T)) = {}^{RL}_{a}D_x^{1-\gamma}(S(\frac{t^{2-\gamma}}{\left(2-3\gamma+\gamma^2\right)\Gamma\left(1-\gamma\right)})) \quad (31)$$

5 The Fractional Tangent Space of a Space Curve

Assuming that the equation $\mathbf{r}=\mathbf{r}(s)$ represents a space curve equation in the initial space, with s the fractional length of the curve. Transferring the space curve, defined in the initial space, into the Λ-fractional space, the derivative defines the fractional tangent space in the Λ-space:

$$\mathbf{R}_1 = \frac{d^{\gamma}\mathbf{r}}{d^{\gamma}s} = \frac{dI^{1-\gamma}\mathbf{r}}{dI^{1-\gamma}s} = \frac{d\mathbf{R}(S)}{dS} \quad (32)$$

Since,

$$d\left|\mathbf{R}(S)\right| = \left|dS\right|, \quad (33)$$

the length $\left|\mathbf{R}_1\right|$ is unity. Proceeding to the expression of the tangent vector with variables of the original plane:

$$R_1(s) = \mathbf{R}_1(S) = \mathbf{R}_1(\frac{s^{2-\gamma}}{\Gamma\left(3-\gamma\right)}) \quad (34)$$

Further, pulling back that tangent vector into the initial space:

$$r^t(s_0) = {}^{RL}_{a}D_s^{1-\gamma}\mathbf{R}_1(\frac{{s_0}^{2-\gamma}}{\left(2-3\gamma+\gamma^2\right)\Gamma\left(1-\gamma\right)}) \quad (35)$$

Since, at the point $\mathbf{r}_o=\mathbf{r}(s_o)$, the tangent space of the curve $\mathbf{r}^t=\mathbf{r}(s)$ is defined considering the Fractional Space with:

$$\mathbf{R}^t=\mathbf{R}(S_0) + k\mathbf{R}_1(S_0), \quad (36)$$

the tangent space, in the original space, corresponding to the tangent space in the Λ-Fractional space may be defined by:

$$\mathbf{r}^t\left(s\right) = \mathbf{r}(s_0) + ({}^{RL}_{a}D_s^{1-\gamma}\mathbf{R}_1(\frac{{s_0}^{2-\gamma}}{\Gamma\left(3-\gamma\right)}))\left(\frac{s^{2-\gamma}}{\Gamma\left(3-\gamma\right)} - S_0\right) \quad (37)$$

The normal plane to the curve at S_0 orthogonal to the tangent line at $\mathbf{R_0}=\mathbf{R}(S_0)$ is defined by:

$$\left(\mathbf{Y}-\mathbf{R}\left(S_0\right)\right)\cdot\mathbf{T}\left(S_0\right) = \left(\mathbf{Y}-\mathbf{R}\left(S_0\right)\right)\cdot\mathbf{R}_1\left(S_0\right) = 0 \quad (38)$$

The corresponding normal space in the initial space is defined by:

$$\boldsymbol{y}(s) = {}^{RL}_{a}D_s^{\gamma}\,\mathbf{Y}(S(s)). \quad (39)$$

6 Geometry in the Λ-fractional Space

Considering the surface, Figure 5,

$$z=x^2y^2, \quad 0<x<1, \quad 0<y<1 \quad (40)$$

shown in Figure 5.

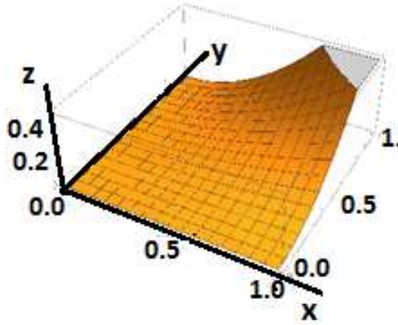

Fig. 5: The surface *z* in the initial space

Assuming the fractional order equal to γ, the Λ-space (*X, Y, Z*) is defined by:

$$X = \frac{x^{2-\gamma}}{\Gamma(3-\gamma)} \quad (41)$$

$$Y = \frac{y^{2-\gamma}}{\Gamma(3-\gamma)} \quad (42)$$

$$Z = {}_bI_y^{1-\gamma}\,{}_{\alpha}I_x^{1-\gamma}z(x,y) = \frac{1}{(\Gamma(1-\gamma))^2}\int_b^y(\int_{\alpha}^x\frac{z(s,t)}{(x-s)^{\gamma}}ds)\frac{dt}{(y-t)^{\gamma}}. \quad (43)$$

With $a=b=0$, Eq.(43) yields,

$$Z = (-\frac{2((X\Gamma[3-\gamma])^{\frac{1}{2-\gamma}})^{3-\gamma}}{\Gamma[4-\gamma]} * Y) \quad (44)$$

In the *Λ*- fractional space, the surface Z is defined for $\gamma=0.6$ by:

$$Z=0.947X^{1.714}Y^{1.714} \quad (45)$$

That surface in the Λ-fractional space is shown in Figure 6.

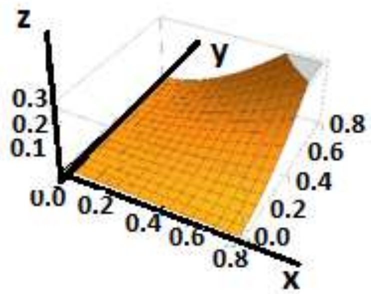

Fig. 6: The surface Z in the Λ- fractional space

Proceeding to the definition of the tangent space of that surface, at the point *X=Y=0.6*,

$$Z = (0.947X^{1.714}Y^{1.714})_{(X=Y=0.6)} + \frac{dZ(X=Y=0.6)}{dX}(X-0.6) + \frac{dZ(X=Y=0.6)}{dY}(Y-0.6), \quad (46)$$

simplified by,

$$Z=0.164+0.469(X-0.6)+0.469(Y-0.6) \quad (47)$$

The tangent space of the surface in the Λ-fractional space is shown in Figure 7.

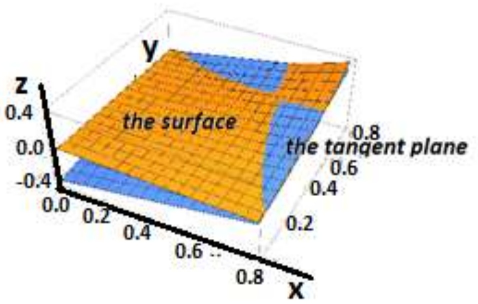

Fig. 7: The tangent plane of the surface in the Λ-fractional space

The corresponding surface in the initial space to the tangent plane in the Λ-fractional space is defined by:

$$z= x^2y^2{}_{(x=y=0.81)} + \left({}^{RL}_{0}D^{1-\gamma}_{y=0.81}\,{}^{RL}_{0}D^{1-\gamma}_{x=0.81}\left(\frac{dZ}{dX}\right)\right)(X(x)-0.6)) + \left({}^{RL}_{0}D^{1-\gamma}_{y=0.81}\,{}^{RL}_{0}D^{1-\gamma}_{x=0.81}\left(\frac{dZ}{dY}\right)\right)(Y(y)-0.6)) \quad (48)$$

The corresponding tangent surface in the initial space is shown in Figure 8.

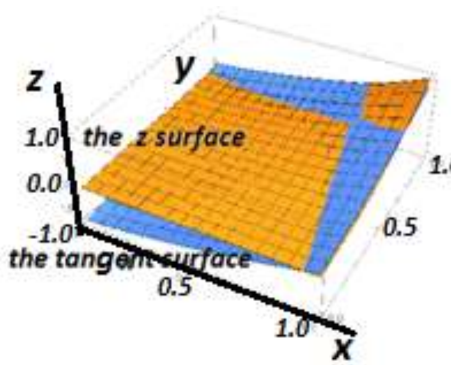

Fig. 8: The surface with its tangent surface at the point (*x=y=0.8106*) at the initial space

7 The Fractional Field Theorems

The well-known field theorems are expressed by:

a. **Green's theorem**. Let $Q_x(x,y)$, $Q_y(x,y)$, be smooth real functions in a domain Ω, with its boundary a smooth closed curve $\partial\Omega$. Then,

$$\int_{\partial\Omega}\left(Q_x dx + Q_y dy\right) = \iint_{\Omega} dxdy\left(\frac{dQ_x}{dy} - \frac{dQ_y}{dx}\right) \quad (49)$$

Corollary: When $Q_x(x,y)$, $Q_y(x,y)$, are derived by a potential function Φ(x,y) with $Q_x = \frac{d\Phi}{dx}, Q_y = \frac{d\Phi}{dy}$, the RHS of Eq.(49) becomes zero. That means that the curvilinear integral along a closed smooth boundary is zero.

b. **Stoke's theorem**:
For a smooth vector field $\boldsymbol{F}$ defined on a simple surface Ω with the boundary $\partial\Omega$, Stoke's theorem is expressed by, $\int_{\partial\Omega}(\boldsymbol{F}, d\boldsymbol{L}) = \iint_{\Omega} (\boldsymbol{\nabla}\times\boldsymbol{F}, d\boldsymbol{S})$ (50)

where **(·,·)** denotes the scalar product.

c. **The Gauss' (divergence) theorem:**
For a space region Ω with a smooth surface boundary $\partial\Omega$, the volume integral of the divergence of a vector field $\boldsymbol{F}$ over Ω is equal to the surface integral of $\boldsymbol{F}$ over the boundary $\partial\Omega$:

$$\int_{\partial\Omega}(\boldsymbol{F}, d\boldsymbol{S}) = \iiint_{\Omega} \boldsymbol{\nabla}\cdot\boldsymbol{F}\, d\Omega \quad (51)$$

It is pointed out that those field theorems are valid in the Λ-fractional space. However, the results may be transferred into the initial space.

8 Fractional Variational Calculus and Fractional Multiple Integrals

For a double integral in the fractional Λ-space:

$$I = \iint_{\Omega} L(X,Y,W,W_X,W_Y)dXdY \quad (52)$$

The extremizing function is expressed by:

$$\frac{\partial L}{\partial W} - \frac{\partial}{\partial X}\left(\frac{\partial L}{\partial W_X}\right) - \frac{\partial}{\partial Y}\left(\frac{\partial L}{\partial W_Y}\right) = 0, \quad (53)$$

along with the condition,

$$\frac{\partial L}{\partial W_X}\frac{dY}{dX} - \frac{\partial L}{\partial W_Y} = 0, \quad (54)$$

on boundary *C*.

9 Λ-fractional Differential Equations. Existence and Uniqueness Theorems

The problem of the existence and uniqueness of theorems concerning Λ-fractional ordinary differential equations has been discussed:

$$\frac{dY}{dX} = F(X, Y) \tag{55}$$

satisfying the Lipschitz condition, in the Λ-fractional space:

$$\left|F\left(X, \tilde{Y}\right) - F(X, Y)\right| < K, \tag{56}$$

Satisfying the initial condition, $Y(X_0)=Y_0$. Those properties are transferred into the initial space through the equation:

$$y(x) = \frac{1}{\Gamma(\gamma)}\frac{d}{dx}\int_0^x \frac{Y(\frac{s^{2-\gamma}}{\Gamma(3-\gamma)})}{(x-s)^{1-\gamma}} \mathrm{ds} \ . \tag{57}$$

Applying Sonia Kowalewski's theorem, concerning the existence of a solution of a Λ-fractional partial differential equation, [24], p.49, in the Λ-space the following existence conditions are derived:

With $G(Y)$ and all its partial Λ-derivatives continuous for $|Y - Y_0| < \Delta$, if X_0 is a given number and $Z_0=G(Y,0)$, $Q_0=G'(Y_0)$, and if $F(X,Y,Z,G)$ in the region S defined by:

$$|X - X_0| < \Delta, |Y - Y_0| < \Delta \, , |Z - Z_0| < \Delta \tag{58}$$

then there exists a unique function $\Phi(X,Y)$ such that,

a) $\Phi(X,Y)$ and all its partial derivatives are continuous in the region R defined by:

$$|X - X_0| < \Delta_1, \quad |Y - Y_0| < \Delta_2 \tag{59}$$

b) For all (X,Y) in R, $Z=\Phi(X,Y)$ is a solution of the equation:

$$\frac{\partial Z}{\partial X} = F(X, Y, Z, \frac{\partial Z}{\partial Y}) \tag{60}$$

c) For all values of Y in the interval:

$$|Y - Y_0| < \Delta_1 \ , \Phi(X_0,Y)=G(Y) \tag{61}$$

The solution may be transferred into the initial space through the equation:

$$z(x, y) = \frac{1}{\Gamma(\gamma)^2}\frac{\partial^2}{\partial x \partial y}\int_0^x \int_0^y \frac{\Phi(\frac{s^{2-\gamma}}{\Gamma(3-\gamma)}, \frac{t^{2-\gamma}}{\Gamma(3-\gamma)})}{(x-s)^{1-\gamma}(y-t)^{1-\gamma}} ds \, dt. \tag{62}$$

10 Linear Oscillations with Fractional Dissipation

An elastic spring with k elastic modulus acts along the axis x upon a body of mass m. In addition, friction of coefficient μ is acting upon the mass. Then, the Λ- space is defined with the T (time) axis corresponding to the t-axis by:

$$T = \frac{1}{\Gamma(1-\gamma)}\int_0^t \frac{s}{(t-s)^\gamma} ds = \frac{t^{2-\gamma}}{\Gamma(3-\gamma)}. \tag{63}$$

Then, the Lagrangian is defined by:

$$L = T - V = \frac{1}{2}m(\frac{d(X(T))}{dT})^2 - \frac{1}{2}kX(T)^2 \tag{64}$$

Therefore, the Λ-space equation of motion is defined by:

$$\frac{d^2(X(T))}{dT^2} + \mu\frac{dX(T)}{dT} + \omega^2 X(T) = 0 \tag{65}$$

with $\omega^2 = \frac{k}{m}$ and the initial conditions, $X(0)=X_0$,

$$\frac{dX(0)}{dT} = X_1 \tag{66}$$

Solving the differential equation for $T>0$ gives:

$$X(T) = X_0 \cos\left(T\sqrt{\omega^2 + \frac{\mu^2}{4}}\right) e^{-\frac{\mu T}{2}} + (X_1 + \frac{\mu}{2}X_0)\frac{\sin\left(T\sqrt{\omega^2+\frac{\mu^2}{4}}\right)}{\sqrt{\omega^2+\frac{\mu^2}{4}}} e^{-\frac{\mu T}{2}}. \tag{67}$$

That solution is valid in the Λ-fractional space. For the parameters:

$$X_0 = X_1 = \omega = \mu = 1, \tag{68}$$

the solution is shown in Figure 9.

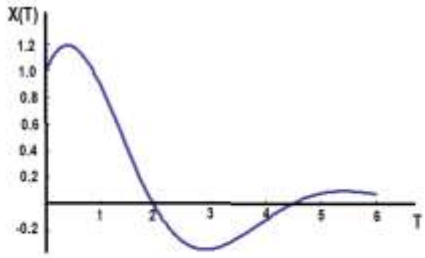

Fig. 9: The solution $X(T)$ in the Λ-fractional space

The solution in the initial space $(t, x(t))$, is defined with the help of the expression of $T(t)$ with,

$$T(t) = \frac{t^{2-\gamma}}{\Gamma(3-\gamma)} \ . \tag{69}$$

In fact, it should be substituted into the solution $X(T)$ in the Λ-space, so

$$X\big(T(t)\big) = X(\frac{t^{2-\gamma}}{\Gamma(3-\gamma)}). \tag{70}$$

The corresponding solution, $x(t)$ in the initial space, is expressed by the equation:

$$x(t) = \frac{1}{\Gamma(\gamma)}\frac{d}{dt}\int_0^t \frac{X(\frac{s^{2-\gamma}}{\Gamma(3-\gamma)})}{(t-s)^{1-\gamma}} ds. \tag{71}$$

The solution *x(t)* of the fractional equation for fractional order $\gamma_1=0.5$ is shown in Figure 10 and for γ_1=0.8 in Figure 11.

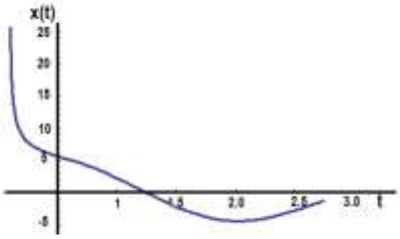

Fig. 10: The initial space solution *x(t)* for γ=0.5

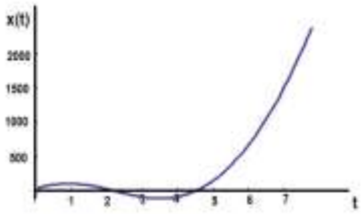

Fig. 11: The initial space solution *x(t)* for γ=0.8

11 The Wave Equation

Let us now discuss the wave equation:

$$\frac{\partial^2 Y(X,T)}{\partial X^2} = \frac{1}{C^2}\frac{\partial^2 Y(X,T)}{\partial T^2} \quad (72)$$

in the Λ- space with the initial conditions:

$$Y(X,0)=\eta(X) \quad , \quad \frac{\partial Y(X,0)}{\partial T} = v(X) \quad (73)$$

The solution is presented in [24], page 220. In fact, where:

$$\eta(X)=\frac{\varepsilon X}{\alpha} \qquad 0 \le X < \alpha \quad (74)$$

$$\eta(X)= \varepsilon\frac{(3\alpha-2X)}{\alpha} \qquad \alpha < X \le 2\alpha \quad (75)$$

$$\eta(X)= \varepsilon\frac{(X-3\alpha)}{\alpha} \qquad 2\alpha < X < 3\alpha \quad (76)$$

and $\upsilon(X)=0$, the solution in the Λ-space is presented by:

$$Y(X,T) = \frac{9\varepsilon}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{2n\pi}{3}\sin\frac{2n\pi X}{3\alpha}\cos\frac{2n\pi cT}{3\alpha}. \quad (77)$$

Assuming specific values ε=1, n=2, α=1, and c=1, the solution *Y(X,T)* in the Λ-space is shown in Figure 12.

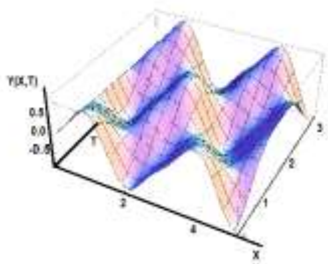

Fig. 12: Solution of *Y(X,T)* in the Λ-space

The solution obtained in the Λ-fractional space is transferred to the initial space, considering the fractional orders of time, γ_1, of the x coordinate γ_2, simulating the non-homogeneous distribution of the media in that space direction.

Examples are porous or composite materials, and not only.

Considering homogeneous media, i.e. media with γ_2=1, the solution *Y(X,T)* is transferred into the initial space, recalling that:

$$X = x \qquad T(t) = \frac{t^{2-\gamma_1}}{\Gamma(3-\gamma_1)} \quad (78)$$

Introducing *(X,T)* into the function of *Y(X,T)*,

$$Y(x,t)=Y(X,T(t))= Y(x, \frac{t^{2-\gamma_1}}{\Gamma(3-\gamma_1)}) \quad (79)$$

Then, the solution of the equation *y(x,t)* in the initial space is defined through the transformation,

$$y(x,t)= \frac{1}{\Gamma(\gamma_1)}\frac{d}{dt}\int_0^t \frac{Y(x,\tau)}{(t-\tau)^{1-\gamma_1}}d\tau \quad (80)$$

Figure 13 shows the solution of the equation in the initial space for $\gamma_1=0.5$ and Figure 14 for $\gamma_1=0.8$.

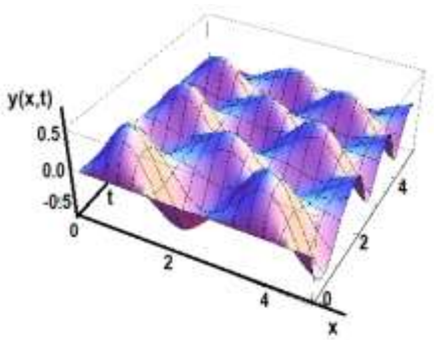

Fig. 13: The wave solution *y(x,t)* for γ_1=0.5

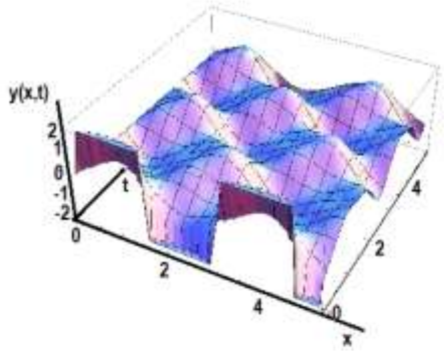

Fig. 14: The solution in the initial space *y(x,t)* for γ_1=0.8

In fact, non-homogeneous media, such as porous or composite materials, and not only are simulated.

It is recalled that:

$$X(x) = \frac{x^{2-\gamma_2}}{\Gamma(3-\gamma_2)}, \quad T(t) = \frac{t^{2-\gamma_1}}{\Gamma(3-\gamma_1)} \qquad (81)$$

Then, introducing these values into the function of *Y(X,T):*

$$Y(x,t)=Y(X(x),T(t))=Y\left(\frac{x^{2-\gamma_2}}{\Gamma(3-\gamma_2)}, \frac{t^{2-\gamma_1}}{\Gamma(3-\gamma_1)}\right) \qquad (82)$$

Then, the solution *y(x,t)* is given by:

$$y(x,t)=\frac{1}{\Gamma(\gamma)^2}\frac{d}{dx}\frac{d}{dt}\int_0^x\int_0^t \frac{Y(s,\tau)}{(x-s)^{1-\gamma_2}(t-\tau)^{1-\gamma_1}}d\tau ds \qquad (83)$$

Figure 15 shows the solution of the equation in the initial space for fractional time order γ_1=0.5 and space fractional order γ_2=0.5. Nevertheless, the solution *y(x,t)* for time fractional order $\gamma_1=0.5$ and space fractional order $\gamma_2=0.8$ is shown in Figure 16. Figure 17 corresponds to the solution *y(x,t)* in the initial space forγ_1=0.8 and γ_2=0.5. Figure 18 corresponds to the solution y(x,t) in the initial space for γ_1=0.8 and γ_2=0.8

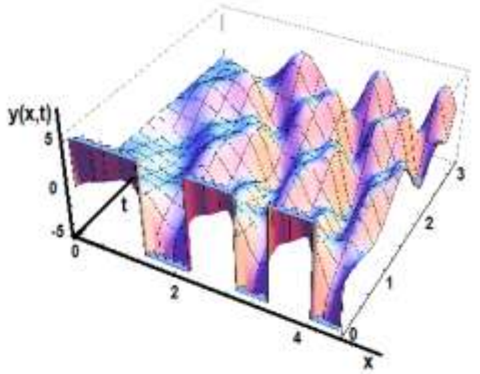

Fig. 15: The solution y(x,t) in the initial space for γ_1=0.5 and γ_2=0.5

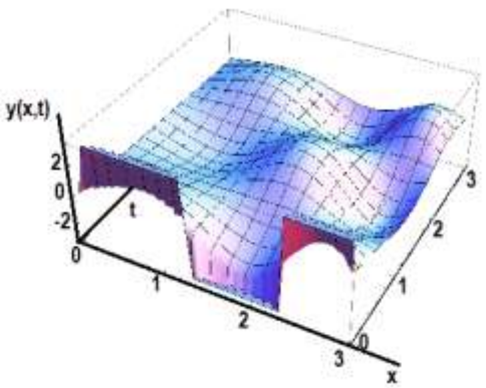

Fig. 16: The solution *y(x,t)* in the initial space for γ_1=0.5 and γ_2=0.8

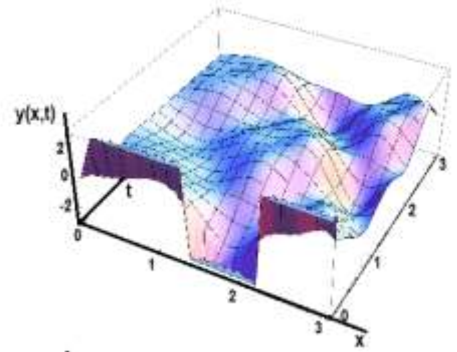

Fig. 17: The solution *y(x,t)* in the initial space for γ_1=0.8 and γ_2=0.5

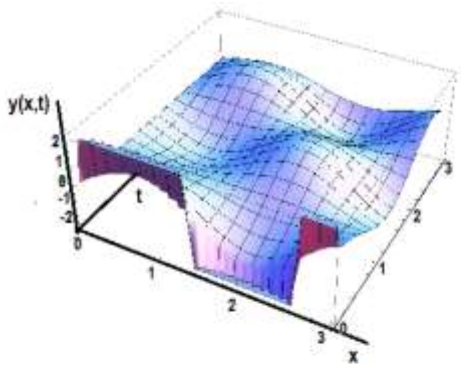

Fig. 18: The solution y(x,t) in the initial space for γ_1=0.8 and γ_2=0.8

Figure 15, Figure 16, Figure 17 amd Figure 18 clearly show the influence of the time(viscoelastic) and space (porosity) order effects.

12 The Λ-fractional Diffusion Equation

The one-dimensional Λ-fractional diffusion equation is:

$$\frac{\partial^2 Y}{\partial X^2} = \frac{1}{k}\frac{\partial Y}{\partial T}\ , \qquad (84)$$

with the b.cs,

$$Y(X,0) = \sum_{n=0}^{\infty} c_n \cos(nX+\varepsilon_n), \qquad (85)$$

and $\quad Y \to 0 \ \ as \ \ T \to \infty.$

For the solution to the diffusion equation, see [24],

$$Y(X,T) = \sum_{n=0}^{\infty} c_n \cos(nX+\varepsilon_n) e^{-n^2 kT} \qquad (86)$$

First-time fractional space is considered with fractional order γ_1 only. In that case:

$$T = \frac{t^{2-\gamma_1}}{\Gamma(3-\gamma_1)}, \quad X=x. \qquad (87)$$

Hence,

$$Y\big(x,T(t)\big) = Y\left(x,\frac{t^{2-\gamma_1}}{\Gamma(3-\gamma_1)}\right) = \sum_{n=0}^{\infty} c_n \cos(nX + \varepsilon_n) e^{-n^2 k\frac{t^{2-\gamma_1}}{\Gamma(3-\gamma_1)}} \qquad (88)$$

The solution is transferred to the initial space with:

$$z(x,t)=\frac{1}{\Gamma(\gamma_1)}\frac{d}{dt}\int_0^t\frac{\sum_{n=0}^{\infty}c_n cos(nX+\varepsilon_n)e^{-n^2k\frac{\tau^{2-\gamma_1}}{\Gamma(3-\gamma_1)}}}{(t-\tau)^{1-\gamma_1}}d\tau \quad (89)$$

A simplified solution with the initial condition:

$$Y(X,0)=\cos(X), \quad (90)$$

is considered with k=1. Hence, the solution in the Λ-fractional space is defined by:

$$Y(X,T)=cosXe^{-T}. \quad (91)$$

Figure 19 pictures the solution of the diffusion equation in the Λ-fractional space.

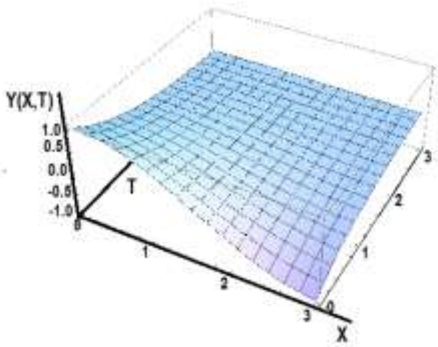

Fig. 19: The solution of the diffusion equation in the Λ-space

Hence,

$$Y\left(x,T(t)\right)=Y\left(x,\frac{t^{2-\gamma_1}}{\Gamma(3-\gamma_1)}\right)=cosx\ e^{-\left(\frac{t^{2-\gamma_1}}{\Gamma(3-\gamma_1)}\right)} \quad (92)$$

Therefore, the solution in the initial space is defined by:

$$y(x,t)=\frac{1}{\Gamma(\gamma_1)}\frac{d}{dt}\int_0^t\frac{cos(x)e^{-n^2k\frac{\tau^{2-\gamma_1}}{\Gamma(3-\gamma_1)}}}{(t-s)^{1-\gamma_1}}d\tau \quad (93)$$

Figure 20 represents the solution of the diffusion equation in the initial space for fractional order γ_1=0.5. However, Figure 21 shows the solution for γ_1=0.8.

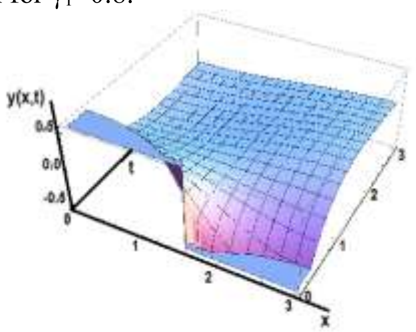

Fig. 20: The solution of the diffusion equation in the initial space for fractional order γ_1=0.5

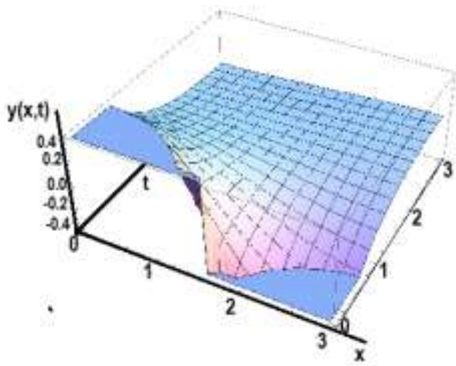

Fig. 21: The solution of the diffusion equation in the initial space for fractional order γ_1=0.8

Proceeding to the interaction of time γ_1 and space γ_2 fractional orders:

$$T=\frac{t^{2-\gamma_1}}{\Gamma(3-\gamma_1)},\quad X=\frac{x^{2-\gamma_2}}{\Gamma(3-\gamma_2)} \quad (94)$$

Hence,

$$Y\left(X(x),T(t)\right)=Y\left(\frac{x^{2-\gamma_2}}{\Gamma(3-\gamma_2)},\frac{t^{2-\gamma_1}}{\Gamma(3-\gamma_1)}\right)=\sum_{n=0}^{\infty}c_n\,cos\left(n\frac{x^{2-\gamma_2}}{\Gamma(3-\gamma_2)}+\varepsilon_n\right)e^{-n^2k\frac{t^{2-\gamma_1}}{\Gamma(3-\gamma_1)}} \quad (95)$$

Transferring the solution to the initial space:

$$y(x,t)=\frac{1}{\Gamma(\gamma_1)\Gamma(\gamma_2)}\frac{d}{dx}\frac{d}{dt}\int_0^x\int_0^t\frac{\sum_{n=0}^{\infty}c_n cos\left(n\frac{s^{2-\gamma_2}}{\Gamma(3-\gamma_2)}+\varepsilon_n\right)e^{-n^2k\frac{\tau^{2-\gamma_1}}{\Gamma(3-\gamma_1)}}}{(x-s)^{1-\gamma_2}(t-\tau)^{1-\gamma_1}}d\tau ds \quad (96)$$

Simplifying the algebra the solution:

$$Y(X,0)=\cos(X), \quad (97)$$

is taken into consideration with k=1. The solution in the Λ-fractional space, is defined by:

$$Y(X,T)=cosXe^{-T} \quad (98)$$

Figure 22 shows the solution of the diffusion equation in the Λ-fractional space.

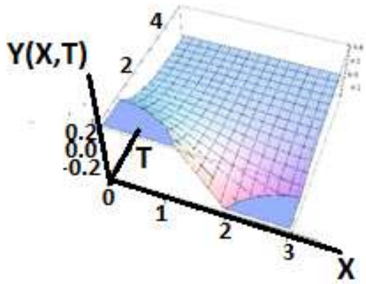

Fig. 22: The solution of the diffusion equation in the Λ-space

Hence,

$$Y\big(X(x),T(t)\big) = Y\left(\frac{x^{2-\gamma_2}}{\Gamma(3-\gamma_2)}, \frac{t^{2-\gamma_1}}{\Gamma(3-\gamma_1)}\right) = cos\frac{x^{2-\gamma_2}}{\Gamma(3-\gamma_2)}\ e^{-\left(\frac{t^{2-\gamma_1}}{\Gamma(3-\gamma_1)}\right)}. \tag{99}$$

Hence, in the initial space:

$$y(x,t) = \frac{1}{\Gamma(\gamma_2)\Gamma(\gamma_1)}\frac{d}{dx}\frac{d}{dt}\int_0^x\int_0^t \frac{cos\left(\frac{s^{2-\gamma_2}}{\Gamma(3-\gamma_2)}\right)e^{-n^2k\frac{\tau^{2-\gamma_1}}{\Gamma(3-\gamma_1)}}}{(x-s)^{1-\gamma_2}(t-\tau)^{1-\gamma_1}}d\tau\, ds\ . \tag{100}$$

The solution of the diffusion equation in the initial space for time fractional order γ_1=0.5 and space fractional order γ_2=0.5 is shown in Figure 23. Nevertheless, in Figure 24 **we can see t**he solution for time fractional order γ_1=0.5 and space fractional order γ_2=0.8. Figure 25 pictures the solution of the equation for time order γ_1=0.8 and space order γ_2=0.5. Figure 26 indicates the solution of the equation for time order γ_1=0.8 and space order γ_2=0.8.

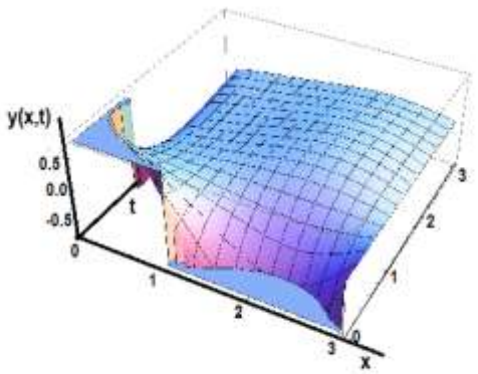

Fig. 23: The solution of the equation for time order γ_1=0.5 and space order γ_2=0.5

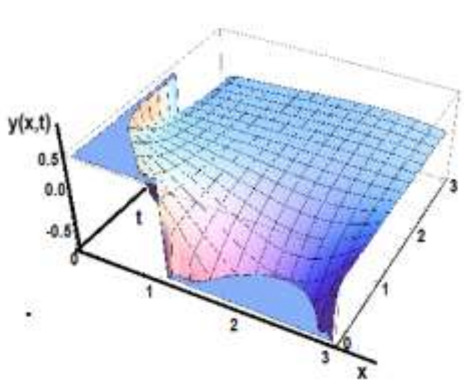

Fig. 24: The solution of the equation for time order γ_1=0.5 and space order γ_2=0.8

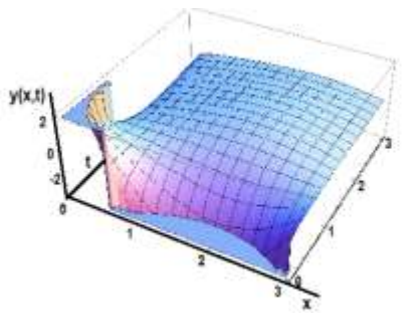

Fig. 25: The solution of the equation for time order γ_1=0.8 and space order γ_2=0.5

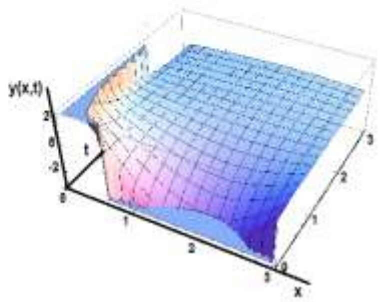

Fig. 26: The solution of the equation for time order γ_1=0.8 and space order γ_2=0.8

13 Branching of the Λ-fractional Differential Equations

Branching problems are considered into the context of Λ-fractional analysis.

The beam deflection due to branching is represented by *w(x)*, (Figure 27), corresponding to the buckling of a simply supported beam, where the axial load p is applied upon the support at *x=l* and with axis 0<*x*<*l*.

Fig. 27: Branching curve in the initial plane (*x*,*w(x)*)

The geometry of the branching problem in the Λ-fractional space is defined by:

$$X = \frac{1}{\Gamma(1-\gamma)}\frac{d}{dx}\int_0^x \frac{s}{(x-s)^{\gamma}}ds\ , \tag{101}$$

and the beam deflection *W(X)* ,(Figure 28)

$$W(X) = \frac{1}{\Gamma(1-\gamma)}\int_0^x \frac{w(s)}{(x-s)^{\gamma}}ds. \tag{102}$$

Fig. 28: Branching curve in the Λ-space (X,W(X))

The length *l* and the axial force p . correspond to L and P in the Λ-fractional space, where:

$$L=\frac{l^{1-\gamma}}{\Gamma(2-\gamma)} \quad \text{and} \quad P=\frac{p^{1-\gamma}}{\Gamma(2-\gamma)} \tag{103}$$

Further, the free energy of the beam is given by:

$$V=\frac{EI}{2}\int_0^L K^2 dS-P\delta L \tag{104}$$

where the various characteristics of the beam in the Λ-space are defined as follows: S represents the arc length of the inextensible elastic curve of the beam in the Λ-space, EI stands for the stiffness of the beam, and δL denotes the displacement of the load P. Hence, defining the free energy of the beam, Eq.(104):

$$V=\frac{EI}{2}\int_0^L\left(\frac{W''(S)}{\sqrt{1+W'(S)^2}}\right)^2 dS-$$
$$-P\int_0^L\left(1-\frac{1}{\sqrt{1+W'(S)^2}}\right)dS \tag{105}$$

Considering small deflections, $|W(S)|\ll 1$, the free energy function is expressed by:

$$V=\int_0^L\frac{EI}{2}\left(W''(S)^2\left(1-\frac{1}{2}W'(S)^2\right)^2-\frac{P}{2}W'(S)^2\right)dS=\int_0^L\Omega(S,P)dS. \tag{106}$$

The variational equation, derived through the potential function V, yields the equilibrium equation:

$$\frac{d^2}{dS^2}\frac{\partial\Omega(S,P)}{\partial W''(S)}-\frac{d}{dS}\frac{\partial\Omega(S,P)}{\partial W'(S)}=0. \tag{107}$$

Consequently, the equilibrium equation is:

$$W^{(4)}(S)+\frac{P}{EI}W''(S)=\frac{9}{2}W'(S)^2W''(S)+W''(S)^3+4W'(S)W''(S)W'''(S), \tag{108}$$

with the b.cs,

$$W(0)=W''(0)=W(L)=W''(L)=0. \tag{109}$$

Following the principles of branching theory, [13], [25], the homogeneous linear problem:

$$W^{(4)}(S)+\frac{P}{EI}W''(S)=0, \tag{110}$$

with the above b.cs, is expressed by:

$$W(S)=\xi\sin(\lambda S), \tag{111}$$

$$\text{with } \lambda^2=\frac{P}{EI}=\left(\frac{\pi}{L}\right)^2 . \tag{112}$$

Further, increasing the loading by:

$$\frac{P}{EI}=\lambda^2(1+k^2), \tag{113}$$

the branching equation becomes:

$$W^{(4)}(S)+\lambda^2W''(S)=R(S,k)=-\lambda^2k^2W''(S)-\frac{9}{2}W'(S)^2W''(S)+W''(S)^3+4W'(S)W''(S)W'''(S) \tag{114}$$

According to Fredholm's alternative theorem, the branching equation has a solution if:

$$\int_0^L R(S,K)dS=0, \tag{115}$$

Hence, the deflection of the branching curve is defined as a function of the incremental axial force by:

$$\xi=\frac{1.26}{\pi}kL \ . \tag{116}$$

Consequently, the branching elastic curve in the Λ-space is defined by:

$$W(X)=\frac{1.26}{\pi}kL\ \sin(\lambda X). \tag{117}$$

Expressing the elastic line in the Λ-space with variables of the initial space:

$$W(X(x))=\xi\sin\left(\frac{\pi}{L}\frac{x^{2-\gamma}}{\Gamma(3-\gamma)}\right) \ . \tag{118}$$

Finally, the branching equation in the initial space is defined by:

$$w(x)=\frac{1}{\Gamma(\gamma)}\frac{d}{dx}\int_0^x\frac{\xi\sin\left(\frac{\pi}{L}\frac{s^{2-\gamma}}{\Gamma(3-\gamma)}\right)}{(x-s)^{1-\gamma}}ds. \tag{119}$$

14 The Λ-fractional Euler-Lagrange Equation

We transfer the variational problem, [26] from the initial space to the Λ-fractional space [27], [28]. The various functions are defined through:

$$X=\frac{1}{\Gamma(1-\gamma)}\int_0^x\frac{x}{(x-s)^\gamma}ds=\frac{x^{2-\gamma}}{\Gamma(3-\gamma)} \tag{120}$$

$$x=(X\Gamma(3-\gamma))^{1/(2-\gamma)}. \tag{121}$$

The extermination of the functional in the Λ-fractional space is expressed by:

$$V=\int_0^L F\big(X,Y(X),Y'(X)\big)dX \tag{122}$$

with some corresponding boundary conditions.

The extremal equation is defined by:

$$\frac{d}{dX}\frac{\partial F(X,Y(X),Y'(X))}{\partial Y'(X)}-\frac{\partial W(X,Y(X),Y'(X))}{\partial Y(X)}=0 \tag{123}$$

with the boundary conditions. Since Λ-fractional analysis is a global procedure, globally stable variational problems will be considered. In other words, continuous solutions of the extremal Eq.(123) are adopted with non-continuous derivatives; hence, the Weierstrass-Erdman corner conditions, [13], [27], should additionally be satisfied with:

$F_{Y'}[X = c - 0] - F_{Y'}[X = c + 0] = 0$ (124)

$(F - Y'F_{Y'})[X = c - 0] - (F - Y'F_{Y'})[X = c + 0] = 0.$ (125)

With the help of the Eqs. (120), we should transfer the extremal function *Y(X)* into the initial space.

Indeed,

$$Y(X) = Y\left(\frac{x^{2-\gamma}}{\Gamma(3-\gamma)}\right). \quad (126)$$

Further, using Eq. (126) the problem is transferred into the initial space as:

$$y(x) = \frac{1}{\Gamma(\gamma)}\frac{d}{dx}\int_0^x \frac{Y\left(\frac{s^{2-\gamma}}{\Gamma(3-\gamma)}\right)}{(x-s)^{1-\gamma}}ds. \quad (127)$$

15 The Λ-fractional Refraction of Light

Two mediums I and II are sparated by the line y=0. The light ray transverses a path from point **a**=(x_1,y_1) of medium *I* with velocity v_1 to point **b**=(x_2,y_2) of medium *II* with velocity v_2, considering the shortest time interval.

The points **A**=(X_1,Y_1) and **B**=(X_2,Y_2) in the Λ-fractional space correspond to the points **a** and **b** in the initial space, Figure 29.

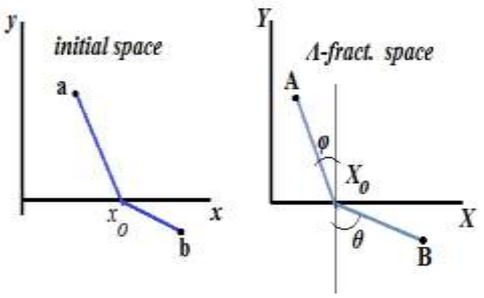

Fig. 29: The Λ-fractional light refraction problem

We define the Λ-fractional space by(γ is the fractional order of various material non-homogeneous distributions):

$$X = \frac{x^{2-\gamma}}{\Gamma(3-\gamma)}, \qquad Y = \frac{y^{2-\gamma}}{\Gamma(3-\gamma)}, \quad (128)$$

where (x,y) are the corresponding points of the initial space.

We formulate the Λ-fractional light refraction problem in the Λ-fractional space. In order to solve it, the following integral must be minimized:

$$J = \int_{X_1}^{X_o}\frac{\sqrt{1+Y'^2}}{v_1}dX + \int_{X_o}^{X_2}\frac{\sqrt{1+Y'^2}}{v_2}dX \quad (129)$$

Both the Euler-Lagrange equation and the corner conditions must be considered to minimize globally the integral.

The minimum time for the light traveling from point ***A*** to point ***B*** is defined for the zigzag straight line AX_0B, which can be given from the relation:

$$\frac{\sin\varphi}{\sin\theta} = \frac{v_1}{v_2} = \text{constant}. \quad (130)$$

From that relation, we can define the point X_0 where the light beam meets the axis *X*. The point x_0 in the initial space, corresponding to the X_0, is defined with the formula:

$$x_o = (X_o\Gamma(3-\gamma))^{1/(2-\gamma)}. \quad (131)$$

We transfer straight lines in the Λ-fractional space as straight lines in the initial space; the path should pass through the points **a, x₀,** and **b** in the initial space, and the zigzag curve axon defines the fractional refraction problem.

16 Conventional Deformation vs Λ - Fractional Deformation Geometry

Let us consider a material body **b** in its undeformed initial placement with its boundary **∂b.** The deformed configuration is **δ** with the boundary **∂δ.** If **a** material point **x** in **b** moves to the position **ψ** in **δ, (**Figure 30), the local deformation is defined by [13], [16], [17], [29]:

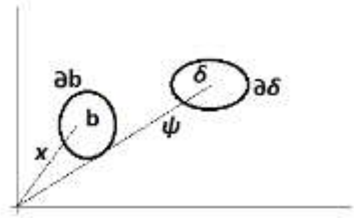

Fig. 30: The initial and deformed placements at the initial space

$$\mathbf{F}(\boldsymbol{\psi}(\mathbf{x})) = \frac{\partial(\boldsymbol{\psi})}{\partial\mathbf{x}} \quad (132)$$

The right Cauchy-Green deformation tensor is:

$$\mathbf{C}=\mathbf{F}^T\mathbf{F} \quad (133)$$

whereas the left Cauchy-Green deformation tensor is given by:

$$\mathbf{B}=\mathbf{F}\,\mathbf{F}^T \quad (134)$$

where $(\,)^T$ denotes the transpose matrix.

We then define the Euler-Lagrange strain tensor as,

$$\mathbf{E}=1/2\ (\mathbf{C-I}) \tag{135}$$

and the Euler-Almansi strain tensor:

$$\mathbf{e} = 1/2\left(\mathbf{I} - \mathbf{B}^{-1}\right) \tag{136}$$

Besides, the linear strain tensor is defined by:

$$\mathbf{E}^{\mathbf{l}} = \frac{1}{2}\left(\mathbf{F} + \mathbf{F}^{\mathbf{T}}\right) - \mathbf{I} = \frac{1}{2}\left(\mathbf{H} + \mathbf{H}^{\mathbf{T}}\right) \tag{137}$$

where **H =F-I** is the displacement gradient tensor. The configurations **B** and **Δ** in the Λ-space correspond to the configurations of the reference placement **b** and the deformed one **δ** in the initial space. In fact the transformation:

$$\mathbf{X} = \frac{x_i^{2-\gamma}\mathbf{e}_i}{\Gamma(3-\gamma)},\ i=1,..,3 \tag{138}$$

is considered.
(The tensor contraction and the vectors $\mathbf{e}_i$ are the unit ones). Likewise, the transformation **Ψ** of the current placement **ψ(x)** in the fractional Λ-space (Figure 31) is defined by:

$$\bar{\mathbf{\Psi}}(\mathbf{x}) = \frac{\mathbf{E}_i}{\Gamma(1-\gamma)^3}\iiint_{\alpha_3\alpha_2\alpha_1}^{x_3 x_2 x_3}\frac{\psi_i(s,t,u)\ ds\ dt\ du}{\left((x_1-s)(x_2-t)(x_3-u)\right)^{\gamma}} \tag{139}$$

where $\mathbf{E}_i$, i=1,2,3, are the unit vectors in the conjugate Λ-space.

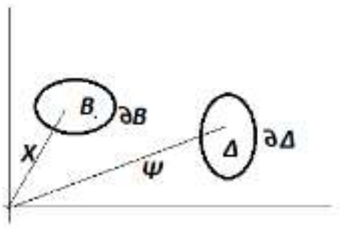

Fig. 31: The initial and the deformed placements at the Λ-space

Eq.(138) defines the transformation of the current placement in the Λ-space. With the help of Eq.(138), the current placement is defined:

$$\mathbf{\Psi}=\mathbf{\Psi}(\mathbf{X}) \tag{140}$$

with the **X** vector defining the initial placement in the conjugate Λ-fractional space.

Therefore, the Λ- fractional deformation tensor is defined by:

$${}^{\Lambda}\mathbf{F}(\mathbf{\Psi}(\mathbf{X})) = \frac{\partial(\mathbf{\Psi})}{\partial\mathbf{X}} \tag{141}$$

The right Cauchy-Green deformation tensor in the Λ- fractional is defined by:

$${}^{\Lambda}\mathbf{C}={}^{\Lambda}\mathbf{F}^{\mathbf{T}}\,{}^{\Lambda}\mathbf{F} \tag{142}$$

Further, expressing the left Λ- fractional Cauchy-Green deformation tensor:

$${}^{\Lambda}\mathbf{B}={}^{\Lambda}\mathbf{F}\ {}^{\Lambda}\mathbf{F}^{\mathbf{T}} \tag{143}$$

with $(\ )^{\mathrm{T}}$ denoting the transpose matrix.

Then, the Λ- fractional Euler-Lagrange strain tensor is defined by:

$${}^{\Lambda}\mathbf{E}=1/2\ ({}^{\Lambda}\mathbf{C-I}) \tag{144}$$

where **I** is the identity matrix. Yet the Λ- fractional Euler-Almansi strain tensor:

$${}^{\Lambda}\mathbf{e} = 1/2\left(\mathbf{I} - {}^{\Lambda}\mathbf{B}^{-1}\right) \tag{145}$$

Further, the Λ-fractional linear strain tensor is defined by:

$${}^{\Lambda}\mathbf{E}^{\mathbf{l}} = \frac{1}{2}\left({}^{\Lambda}\mathbf{F} + {}^{\Lambda}\mathbf{F}^{\mathbf{T}}\right) - \mathbf{I} = \frac{1}{2}\left({}^{\Lambda}\mathbf{H} + {}^{\Lambda}\mathbf{H}^{\mathbf{T}}\right) \tag{146}$$

with ${}^{\Lambda}\mathbf{H}$ the Λ-fractional displacement gradient in the Λ-fractional space.

At this point, we must note that strains may not be transferred in the initial space since strains are, in fact, derivatives. Nevertheless, if strain is needed in the initial space, those strains will be transferred as functions.

We transfer the various deformation tensors in the Λ-fractional space back to the original space through the transformation:

$$\mathbf{q}(\mathbf{x}) = {}_{a}^{RL}D_{\mathbf{x}(\mathbf{X})}^{1-\gamma}(\mathbf{Q}(\mathbf{X})) \tag{147}$$

In the case of both sides transformation **Ψ(X)** of the displacement function **ψ,** Eq.(139) becomes:

$$\bar{\mathbf{\Psi}}(\boldsymbol{x})= \frac{\mathbf{E}_i}{(2\Gamma(1-\gamma))^3}\frac{d}{dx_3}\left(\frac{d}{dx_2}\int_{a_2}^{x_2}\left(\frac{d}{dx_1}\left(\int_{a_1}^{x_1}\frac{\psi_i(s,t,u)\ ds}{\left((x_1-s)(x_2-t)(x_3-u)\right)^{\gamma}} - + \int_{x_1}^{b_1}\frac{\psi_i(s,t,u)\ ds}{\left((x_1-s)(x_2-t)(x_3-u)\right)^{\gamma}}\right)dt\right)du\right) \tag{148}$$

In order to make the problem more accessible, we will consider only the left fractional derivative in the first chapters.

17 The Elasticity Problem

The strain energy density function is defined, [16], [17], [30] as:

$$W = W(\mathbf{C}), \tag{149}$$

where **C** is the right Cauchy-Green deformation tensor, Eq.(142); moreover, for isotropic materials holds that:

$$W = W(I_1, I_2, I_3) \tag{150}$$

where I_1, I_2, I_3 are the principal invariants of the tensor **C** or **B.** Then the stress tensor **T** is defined by [30]:

$$\mathbf{T} = \chi_0\mathbf{I} + \chi_1\mathbf{B} + \chi_{-1}\mathbf{B}^{-1} \quad (151)$$

where **I** is the unit 3x3 matrix, and if

$$W_i = \frac{\partial W}{\partial I_i}, \quad (152)$$

$$\chi_0 = \frac{2}{I_3^{1/2}}(I_2W_2 + I_3W_3) \quad (153)$$

$$\chi_1 = \frac{2}{I_3^{1/2}}W_1 \quad (154)$$

$$\chi_{-1} = -2I_3^{1/2}W_2 \quad (155)$$

Further, for incompressible isotropic materials with I_3=1, the stress tensor is expressed by:

$$\mathbf{T} = -p\mathbf{I} + 2W_1\mathbf{B} - 2W_2\mathbf{B}^{-1} \quad (156)$$

where p is the pressure due to the constraint of incompressibility.

Let us point out that for neo-Hookean materials:

$$W = C_1(I_1 - 3) \quad (157)$$

with

$$\mathbf{T} = -p\mathbf{I} + 2C_1\mathbf{B}_1 \quad (158)$$

Besides, for the Mooney-Rivlin incompressible materials with:

$$W = C_1(I_1 - 3) + C_2(I_2 - 3) \quad (159)$$

the stress tensor is expressed by:

$$\mathbf{T} = -p\mathbf{I} + 2C_1\mathbf{B} - 2C_2\mathbf{B}^{-1} \;. \quad (160)$$

Likewise, for linear elasticity with infinitesimal deformations, we get:

$$T_{KL} = 2\mu E_{KL} + \lambda E_{MM}\delta_{KL} \quad (161)$$

where,

$$E_{KL} = \frac{1}{2}\left(u_{K,L} + u_{L,K}\right) \quad (162)$$

is the linear deformation tensor.

The coefficients (λ,μ) are the well-known *Lame* coefficients. Furthermore, we may transfer the various results only as functions to the original space. Moreover, the equation of the balance of linear momentum is defined by:

$$div[\mathbf{T}\;] + \rho\mathbf{b} = 0 \quad (163)$$

where **b** is the body loading and ρ is the material density. The symmetry of Cauchy stress tensor yields,

$$\mathbf{T} = \mathbf{T}^T \quad (164)$$

All the relations concerning elasticity theory are evidently valid in the Λ-Fractional space, not the initial space. The results from the Λ-space are transferred into the initial space, and those are the final results.

In the case of the Λ-space, all the functions will be considered with the Λ-upper left index.

18 Application

A material point (x,y) moves from the initial to the deformed position through the formula: $(\chi,\psi)=\xi(x+x^2y,\ y+y^2x)$, with $|\xi| \ll 1$, in the initial plane. Therefore, this point, in its undeformed and deformed status, can be described in the corresponding positions (X,Y) in the **Λ**-fractional space, where, see Eq.(135):

$$X = \frac{x^{2-\gamma}}{\Gamma(3-\gamma)} \quad (165)$$

$$Y = \frac{y^{2-\gamma}}{\Gamma(3-\gamma)} \quad (166)$$

Meanwhile, in order for the algebra not to become lengthy, the application presented will have fractional order γ=0.6.

Furthermore, the initial plane's current displacement (u,v) corresponds to the displacement (*U*, *V*) in the Λ-fractional space. According to Eq.(135),we have:

$$U = 0.973\xi X^{1.71}Y \quad (167)$$

$$V = 0.973\xi XY^{1.71} \quad (168)$$

The Λ-fractional (non-local) displacement gradient ${}^{\Lambda}\mathbf{H}$ is defined by:

$${}^{\Lambda}\mathbf{H} = \frac{\partial(U,V)}{\partial(X,Y)} \quad (169)$$

The Λ- fractional linear strain tensor, Eq.(135), is defined by using the computerized Mathematica pack, [31]. In that case:

$${}^{\Lambda}\mathbf{E}^{\mathbf{l}}(X,Y) = \xi\begin{bmatrix} 1.668X^{0.714}Y & 0.487(X^{0.714}+Y^{0.714}) \\ 0.487(X^{0.714}+Y^{0.714}) & 1.668XY^{0.714} \end{bmatrix} \quad (170)$$

Similarly, we may define all the necessary Λ-fractional deformation and strain tensors using similar procedures with the help of the Mathematica computerized algebra. In order to pull back to the initial space all the various deformation tensors, the following method should be followed:

Through the Eqs. (165, 166, 147), the variables *X* and *Y* are expressed in *x* and *y*. Then the Eq.(147) yields:

$$\mathbf{q}(x,y) = {}^{RL}_{a}D_{y}^{1-\gamma}\,{}^{RL}_{a}D_{x}^{1-\gamma}(\mathbf{Q}(X(x),Y(y))) \quad (171)$$

Expressing the fractional Riemann-Liouville derivatives in Eq.(171), it appears:

$$\mathbf{q}(x,y) = \frac{1}{(\Gamma(\gamma))^2}\frac{\partial^2}{\partial x\partial y}\int_0^y \frac{1}{(y-t)^{1-\gamma}}\left(\int_0^x \frac{\mathbf{Q}(s,t)}{(x-s)^{1-\gamma}}ds\right)dt \qquad (172)$$

The corresponding matrices in the original space (*x*,*y*) are extracted by applying Eq.(172) to the Λ-fractional deformation and Λ-strain matrices. Indeed, we may compute the fractional linear strain $\mathbf{e}^{\mathbf{l}}$ in the initial space through the relation:

$$\mathbf{e}^{\mathbf{l}}(x,y) = {}^{RL}_{a}D_{y}^{1-\gamma}\,{}^{RL}_{a}D_{x}^{1-\gamma}\left({}^{\Lambda}\mathbf{E}^{\mathbf{l}}(X(x),Y(y))\right) \qquad (173)$$

Performing the computation with the help of Mathematica computerized algebra, it is found:

$$\mathbf{e}^{\mathbf{l}}(x,y) = \xi\begin{bmatrix} 1.599x^{0.6}y & 0.335\left(\frac{x^2}{y^{0.4}}+\frac{y^2}{x^{0.4}}\right) \\ 0.335\left(\frac{x^2}{y^{0.4}}+\frac{y^2}{x^{0.4}}\right) & 1.599xy^{0.6} \end{bmatrix} \qquad (174)$$

Proceeding to the definition of the stresses in the Λ-space:

$$\frac{\Sigma_{11}}{(2\mu+\lambda)\xi} = E_{11} + \frac{\mu}{2\mu+\lambda}E_{22} \qquad (175)$$

$$\frac{\Sigma_{22}}{(2\mu+\lambda)\xi} = E_{22} + \frac{\mu}{2\mu+\lambda}E_{11} \qquad (176)$$

$$\frac{\Sigma_{12}}{(2\mu+\lambda)\xi} = \frac{2\mu}{2\mu+\lambda}E_{12} \quad . \qquad (177)$$

We have computed the stresses $\frac{\sigma_{ij}}{(2\mu+\lambda)\xi}$ in the original space for $\frac{\mu}{2\mu+\lambda} = 0.30$. (Figure 32, Figure 33 and Figure 34).

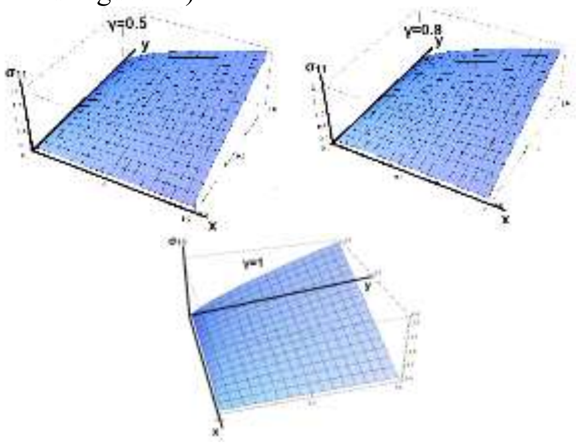

Fig. 32: The stress σ_{11} in the initial space for γ=0.5, 0.8, 1.0

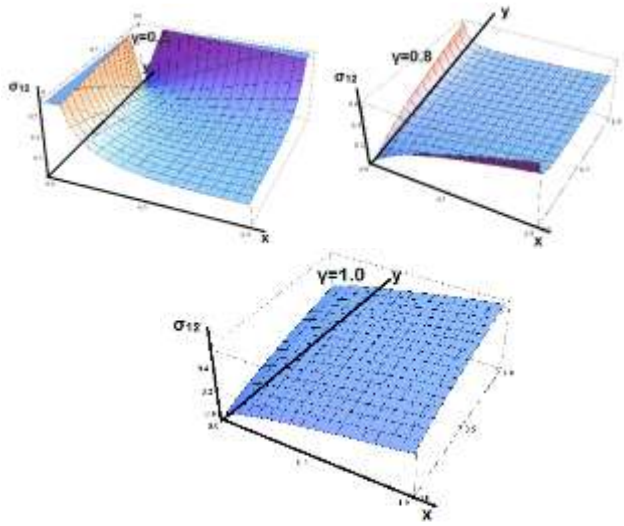

Fig. 33: The stress σ_{12} in the initial space for γ=0.5,0.8,1.0

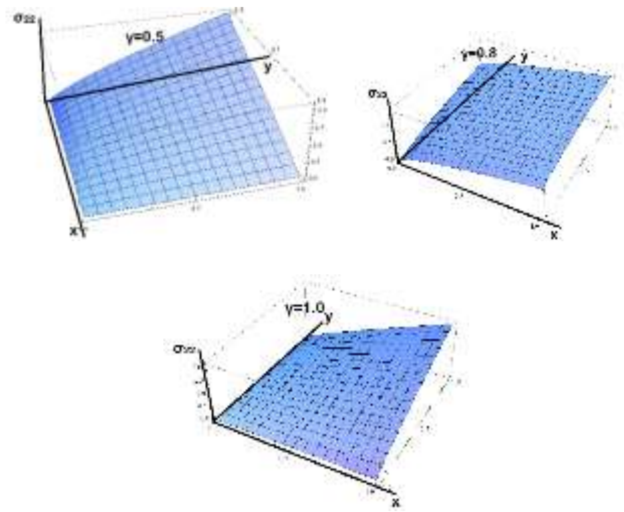

Fig. 34: The stress σ_{22} in the initial space for γ=0.5,0.8,1.0

The high importance of the contribution of the fractional-order γ is evident.

19 The Bar Extension

This section describes the problem of the bar extension using the Λ-strain, as implemented in deformation problems. Let's assume a bar, which is fixed at one end, is fractionally deformed due to a force p at its free end (Figure 35).

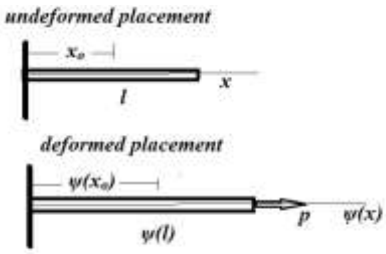

Fig. 35: The bar in the initial space

This initial space force p corresponds to a force P in the fractional Λ-space, which is defined by the formula:

$$\boldsymbol{P} = \frac{1}{\Gamma(1-\gamma)}\int_0^l \frac{\boldsymbol{p}}{(l-s)^\gamma} ds = \frac{\boldsymbol{p}l^{1-\gamma}}{\Gamma(2-\gamma)} \qquad (178)$$

Transferring into the Λ-space, the length l of the bar:

$$L = \frac{l^{2-\gamma}}{\Gamma(3-\gamma)}\ , \qquad (179)$$

(Figure 36).

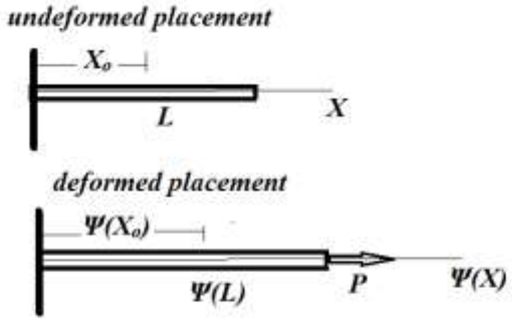

Fig. 36: The fractional Λ-space

Likewise, the constant cross-section area is transferred to the Λ-space through the formula:

$$A \ = \frac{1}{\Gamma(1-\gamma)}\int_0^x \frac{a}{(l-s)^\gamma} ds = \frac{a x^{1-\gamma}}{\Gamma(2-\gamma)} \qquad (180)$$

Hence, the linear Hook's law in the Λ-space, is defined by:

$$\frac{dY}{dX} = \frac{P}{EA} = \frac{p l^{1-\gamma}}{E a x^{1-\gamma}} \qquad (181)$$

where E is Young's modulus of Elasticity in the Λ-space. Furthermore:

$$x = (\Gamma(3-\gamma)X)^{\frac{1}{2-\gamma}} \qquad (182)$$

Hence, Eq.(181) yields:

$$Y(X) = \int_0^X \frac{p l^{1-\gamma}\ dX}{Ea(\Gamma(3-\gamma)X)^{\frac{1-\gamma}{2-\gamma}}} = \frac{(2-\gamma)p l^{1-\gamma} X^{\frac{1}{2-\gamma}}}{Ea} \qquad (183)$$

where $Y(X)$ signifies the displacement in the fractional Λ-space.

Likewise, the displacement Y may be defined as a function of the initial placement x through Eqs.(165),(166). Indeed,

$$Y(x) = \frac{p x^{2-\gamma}\left(l\left(x^{2-\gamma}\right)^{\frac{1}{-2+\gamma}}\right)^{1-\gamma}}{Ea\Gamma(1-\gamma)} \qquad (184)$$

Moreover, the displacement y(x) in the initial plane (x,y) is transferred through,

$$y(x) = \frac{1}{\Gamma(\gamma)}\frac{d}{dx}\int_0^x \frac{Y(s)}{(x-s)^{1-\gamma}} ds. \qquad (185)$$

The displacement in the initial space x is defined by:

$$y(x) = \frac{(-2+\gamma)l^{1-\gamma} p x^\gamma}{E\alpha\Gamma(1+\gamma)\Gamma(3-\gamma)^{\frac{1}{2-\gamma}}} \qquad (186)$$

Figure 37 shows the displacement functions for a bar of length l=2 and γ=0.6, 0.8, 1.

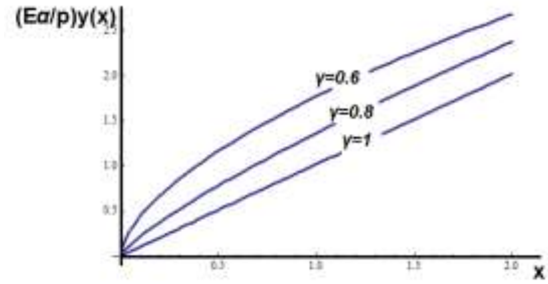

Fig. 37: The bar displacement $y(x)$ for various values of the fractional-order γ

We can conclude that the displacements increase for smaller values of the fractional orders γ.

Since the conventional mechanics' rules are valid in the Λ-space, the axial force is equal to $\boldsymbol{P}$, see Eq.(178). Hence, the axial stress $\Sigma(X)$, is defined through the formula:

$$\Sigma(X) = \frac{P}{A} = \frac{P\Gamma(2-\gamma)}{a(\Gamma(3-\gamma)X)^{\frac{1-\gamma}{2-\gamma}}} \qquad (187)$$

Furthermore, the stress $\Sigma(X)$ in the Λ-space may be expressed in the x variable of the initial space by:

$$\Sigma(X(x)) = \frac{p l^{1-\gamma}}{\alpha x^{1-\gamma}} \qquad (188)$$

Transferring the stress from the Λ-space to the initial space:

$$\sigma(x) = {}_0^{RL}D_x^{1-\gamma}\Sigma(X(x)) = \frac{1}{\Gamma(\gamma)}\frac{d}{dx}\int_0^x \frac{p l^{1-\gamma}}{\alpha s^{1-\gamma}(x-s)^{1-\gamma}} ds = \frac{p l^{1-\gamma}\Gamma(\gamma)x^{-2+2\gamma}}{\alpha\Gamma(2\gamma-1)} \qquad (189)$$

Let's point out that the size effect phenomenon is present in this analysis, precisely as the one that appeared in gradient elasticity theories, [32].

Diagrams of the stresses in the initial space for a bar of initial length *l*=2 is shown in Figure 38.

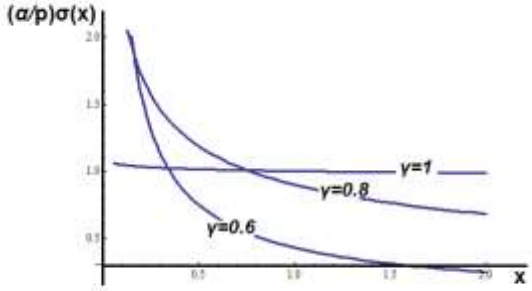

Fig. 38: Stresses in the initial space for various fractional orders *γ*

20 The Fractal Bar Extension

Λ- derivative and the use of the Λ-fractional space will be applied in fractals. Consider bars with variable cross-section area *a(x)* defined through the fractal function with Hausdorff dimension d_H=1.5,

$$a(x) = 1 + \sum_{n=1}^{\infty} n^{-0.5n} sin(n^n x) \quad (190)$$

and approximated by Figure 39

$$a(x) \approx 1 + \sum_{n=1}^{5} n^{-0.5n} sin(n^n x) \quad (191)$$

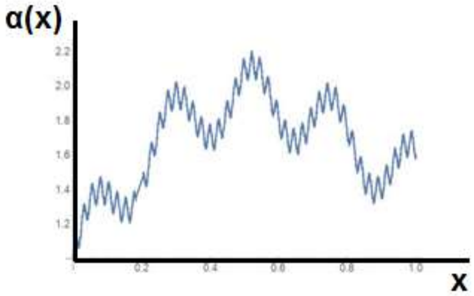

Fig. 39: The unit length fractal cross-sectional area *a(x)*

Then, the cross-sectional area in the Λ-space is expressed by:

$$A(x) = \frac{1}{\Gamma(1-\gamma)} \int_0^x \frac{a(s)}{(x-s)^\gamma} ds \,. \quad (192)$$

Moreover,

$$X = \frac{x^{2-\gamma}}{\Gamma(3-\gamma)} \,. \quad (193)$$

Solving for *x* from Eq. (193),

$$x = (\Gamma(3-\gamma)X)^{\frac{1}{2-\gamma}} \,. \quad (194)$$

Hausdorff dimension and fractal order are simply connected; [33], [34], [35], [36], [37].

The cross-sectional area function *A(X)* in the Λ-space, for the initial cross-sectional area *a(x)* for *γ*=0.6, is shown in Figure 40.

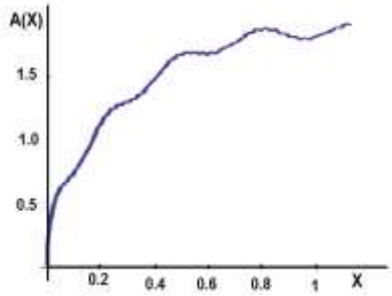

Fig. 40: The cross-sectional area *A(X)* versus the axial coordinate *X* in the Λ-space

Transferring all the characteristics of the bar in the Λ-space, the length *L* and the force P corresponding to *l* and *p* are defined by:

$$L = \frac{1}{\Gamma(1-\gamma)} \int_0^l \frac{1}{(l-s)^\gamma} ds = \frac{l^{1-\gamma}}{\Gamma(2-\gamma)} \,. \quad (195)$$

$$P = \frac{1}{\Gamma(1-\gamma)} \int_0^l \frac{p}{(l-s)^\gamma} ds = \frac{pl^{1-\gamma}}{\Gamma(2-\gamma)} \quad (196)$$

in the Λ-space. The stress in the Λ-space is defined by Figure 41:

$$\Sigma(X) = \frac{P}{A(X)} = \frac{pl^{1-\gamma}}{\Gamma(2-\gamma)A(X)} \quad (197)$$

Size effects, as seen in [31], are present.

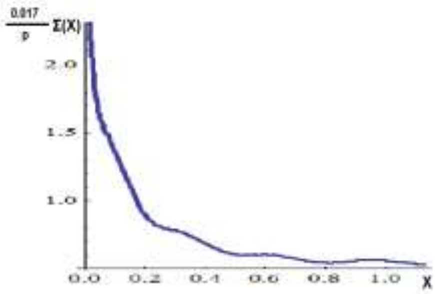

Fig.41: The fractional stress *Σ(X)* diagram in the Λ-space for *γ*=0.6

Moreover, the fractional strain in the Λ-space is defined by (considering the elastic modulus *E* constant in the initial space):

$$E(X) = \frac{\Sigma(X)}{E^\Lambda} = \frac{P}{E^\Lambda A(X)} = \frac{pl^{1-\gamma}}{Ex^{1-\gamma}A(X)} \quad (198)$$

Furthermore, considering Eq. (198), the fractional strain *E(X)* is defined by:

$$E(X) = \frac{pl^{1-\gamma}}{E(\Gamma(2-\gamma)X)^{\frac{1-\gamma}{2-\gamma}}A(X)} \quad (199)$$

A diagram of the fractional strain in the Λ-space is presented in Figure 42.

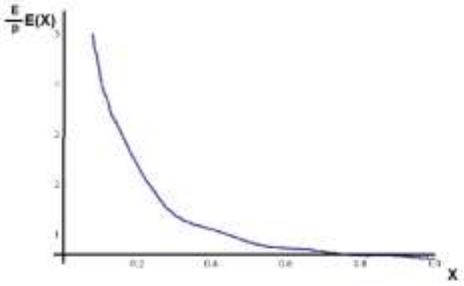

Fig. 42: The strain *E(X)* in the Λ-space with *γ*=0.6

Since the strain in the Λ-space:

$$\frac{dY(X)}{dX} = E(X) \quad (200)$$

the displacement *Y(X)* in the Λ-space is defined by,

$$Y(X) = \int_0^X E(X)dX \quad (201)$$

For the initial unit length fractal bar, the displacement function in the Λ-space for the order *γ*=0.6 is shown in Figure 43.

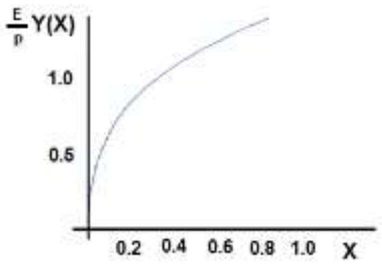

Fig.43: The fractional displacement field for the bar of unit initial length with γ=0.6 in the Λ-fractional space

The axial stress is defined by:

$$\sigma(x) = \frac{1}{\Gamma(\gamma)}\frac{d}{dx}\int_0^x \frac{\Sigma(s)}{(x-s)^{1-\gamma}}ds \quad (202)$$

The stress function in the initial space is shown in Figure 44.

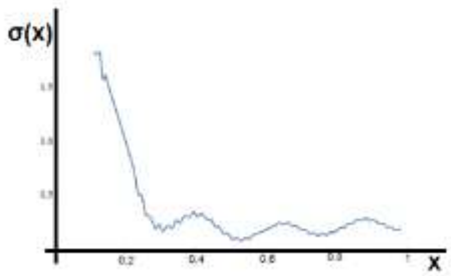

Fig. 44: The fractional stress field for the unit bar with γ=0.6 in the initial space

Furthermore, the displacement in the initial space may be found through the formula:

$$y(x) = \frac{1}{\Gamma(\gamma)}\frac{d}{dx}\int_0^x \frac{Y(s)}{(x-s)^{1-\gamma}}ds. \quad (203)$$

Figure 45 indicates the displacement function in the initial space.

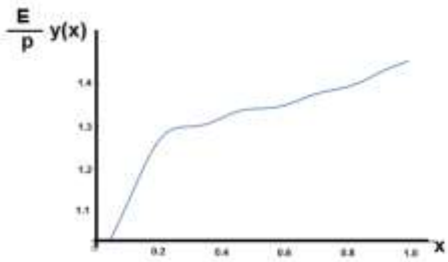

Fig.45: The fractional displacement field for the bar of unit initial length with γ=0.6 in the initial space

21 Plane Linear Λ-fractional Elasticity with both Sides Fractional Derivatives

In the current section, we present the simple problem of homogeneous deformation in the plane infinitesimal elasticity due to Λ-fractional deformations with fractional derivatives of both sides. The purpose of the present paragraph is to present the analysis, explain its various steps for a fractional problem, and consider derivatives of both sides.

We use the biharmonic equation to solve the plane elastic problems in linear elasticity:

$$\frac{\partial^4\varphi}{\partial^4 x} + 2\frac{\partial^4\varphi}{\partial^2 x\partial^2 y} + \frac{\partial^4\varphi}{\partial^4 y} = 0 \quad (204)$$

The plane stresses are defined through the stress function *φ* and the relations:

$$\sigma_{xx} = \frac{\partial^2\varphi}{\partial y^2} \quad (205)$$

$$\sigma_{yy} = \frac{\partial^2\varphi}{\partial x^2} \quad (206)$$

$$\sigma_{xy} = -\frac{\partial^2\varphi}{\partial x\partial y} \quad (207)$$

Hence, the homogeneous elasticity linear Λ-fractional problem in the Λ-fractional space is completely defined by the biharmonic function:

$$\Phi = A_o X^2 + A_1 XY + A_2 Y^2 \quad (208)$$

Recalling Eqs. (205-207) we define the homogeneous Λ-stresses by:

$$\Sigma_{xx} = 2A_2, \ \Sigma_{xy} = -A_1, \quad \Sigma_{yy} = 2A_0 \quad (209)$$

For any constant Ξ in the fractional Λ-space (X, Y), the corresponding function ξ in the initial space (x,y) equals to:

$$\xi=\frac{\Xi}{(2\Gamma(\gamma))^2}\frac{d}{dx}\left(\int_\alpha^x\frac{1}{(x-s)^{1-\gamma}}\left(\frac{d}{dy}\left(\int_\eta^y\frac{dt}{(y-t)^{1-\gamma}}-\int_y^\theta\frac{dt}{(t-y)^{1-\gamma}}\right)\right)ds-\int_x^\beta\frac{1}{(s-x)^{1-\gamma}}\left(\frac{d}{dy}\left(\int_\eta^y\frac{dt}{(y-t)^{1-\gamma}}-\int_y^\theta\frac{dt}{(t-y)^{1-\gamma}}\right)\right)ds\right) \quad (210)$$

Performing the algebra, Eq.(210) is simplified and with Ξ=1, Eq. (210) yields,

$$\zeta(x,y)=\frac{((\beta-x)^{-1+\gamma}+(-\alpha+x)^{-1+\gamma})((\theta-y)^{-1+\gamma}+(-\eta+y)^{-1+\gamma})}{4\Gamma(\gamma)^2} \quad (211)$$

Therefore,

$$\sigma_{xx}=\Sigma_{xx}\zeta(x,y)=2A_2\,\zeta(x,y), \quad (212)$$
$$\sigma_{yy}=\Sigma_{yy}\zeta(x,y)=2A_0\zeta(x,y), \quad (213)$$
$$\sigma_{xy}=\Sigma_{xy}\zeta(x,y)=-A_1\zeta(x,y). \quad (214)$$

We have computed the function ζ(x, y) with the help of Mathematica computerized pack and for the values, (α=η=0, β=3, θ=2, γ=0.6). The function *ζ(x,y)* has a picture of Figure 46.

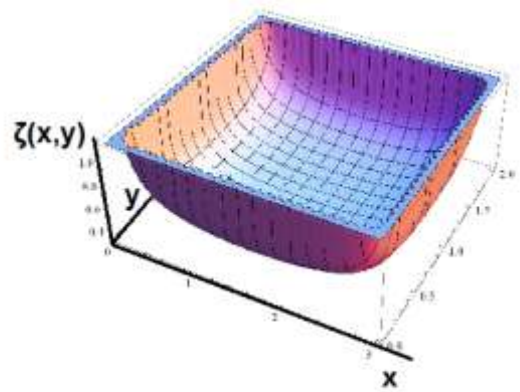

Fig. 46: The function ζ(x,y) in the initial space for the specific values (α=η=0, β=3, θ=2, γ=0.6)

Hence, if we applied the traction to a body in the initial plane, it should be governed by Eqs. (212-214). The stresses are not constant in the initial space but multiples of the function ζ(x,y).

If we want to define the displacement field, the strain in the Λ-space is defined for the plane strain problem by:

$$E_{xx}=\frac{1+\nu}{E}\left[(1-\nu)\Sigma_{xx}-\nu\Sigma_{yy}\right]=\frac{2(1+\nu)}{E}[(1-\nu)A_2-\nu A_0]=K \quad (215)$$

$$E_{yy}=\frac{1+\nu}{E}\left[(1-\nu)\Sigma_{yy}-\nu\Sigma_{xx}\right]=\frac{2(1+\nu)}{E}[(1-\nu)A_0-\nu A_2]=M \quad (216)$$

with ν denoting Poisson's ratio and E Young's modulus. It is assumed that for zero x-displacement along x=0 and y-displacement along y=0, the displacement functions in the Λ-space are defined by:

$$U=KX \quad (217)$$
$$\text{and } V=MY\quad. \quad (218)$$

Since,

$$X=\frac{x^{2-\gamma}}{\Gamma(3-\gamma)} \quad \text{and } Y=\frac{y^{2-\gamma}}{\Gamma(3-\gamma)} \quad (219)$$
$$x=(\Gamma(3-\gamma)X)^{\frac{1}{2-\gamma}},\ y=(\Gamma(3-\gamma)Y)^{\frac{1}{2-\gamma}} \quad (220)$$

We define the displacements (u,v) in the initial plane (x,y) that correspond to the displacements (*U*, *V*), Eqs. (217,218) in the Λ-Fractional space by:

$$u=\frac{K}{(2\Gamma(\gamma))^2}\frac{d}{dx}\left(\int_\alpha^x\frac{s^{2-\gamma}}{\Gamma(3-\gamma)}\ \frac{1}{(x-s)^{1-\gamma}}\left(\frac{d}{dy}\left(\int_\eta^y\frac{dt}{(y-t)^{1-\gamma}}-\int_y^\theta\frac{dt}{(t-y)^{1-\gamma}}\right)\right)ds-\int_x^\beta\frac{s^{2-\gamma}}{\Gamma(3-\gamma)}\ \frac{1}{(s-x)^{1-\gamma}}\left(\frac{d}{dy}\left(\int_\eta^y\frac{dt}{(y-t)^{1-\gamma}}-\int_y^\theta\frac{dt}{(t-y)^{1-\gamma}}\right)\right)ds\right) \quad (221)$$

and

$$v=\frac{M}{(2\Gamma(\gamma))^2}\frac{d}{dx}\left(\int_\alpha^x\frac{1}{(x-s)^{1-\gamma}}\left(\frac{d}{dy}\left(\int_\eta^y\frac{t^{2-\gamma}}{\Gamma(3-\gamma)}\ \frac{dt}{(y-t)^{1-\gamma}}-\int_y^\theta\frac{t^{2-\gamma}}{\Gamma(3-\gamma)}\ \frac{dt}{(t-y)^{1-\gamma}}\right)\right)ds-\int_x^\beta\frac{1}{(s-x)^{1-\gamma}}\left(\frac{d}{dy}\left(\int_\eta^y\frac{t^{2-\gamma}}{\Gamma(3-\gamma)}\ \frac{dt}{(y-t)^{1-\gamma}}-\int_y^\theta\frac{t^{2-\gamma}}{\Gamma(3-\gamma)}\ \frac{dt}{(t-y)^{1-\gamma}}\right)\right)ds\right) \quad (222)$$

If we perform the algebra with the help of the Mathematica computerized algebra pack u displacement in the initial space will be computed and is equal to:

$$u=\frac{K\Delta\Psi}{\Omega} \quad (223)$$

with

$$\Delta=\left(2x^{1+\gamma}+\frac{4x^{1+\gamma}}{\gamma}-\frac{2(-x+\beta)^{\gamma}(2x+\beta\gamma)}{\gamma}+(-x+\beta)^{-1+\gamma}\left(2x^2+2x\beta\gamma+\beta^2\gamma(1+\gamma)\right)\right) \quad (224)$$

$$\Psi=((\theta-y)^{-1+\gamma}+y^{-1+\gamma}) \quad (225)$$
$$\Omega=4(1+\gamma)(2+\gamma)\Gamma(3-\gamma)\Gamma(\gamma)^2 \quad (226)$$

The x-displacement u in the initial space for the specific values (α=η=0, β=3, θ=2, γ=0.6) is shown in Figure 47:

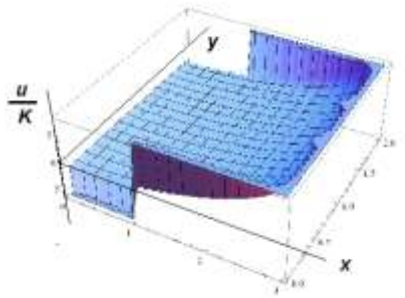

Fig. 47: The x-displacement u in the initial space for the specific values (α=η=0, β=3, θ=2, γ=0.6)

Furthermore,

$$v=\frac{M\Pi P}{\Omega} \quad (227)$$

with $\Pi=(x^{-1+\gamma}+(-x\beta)^{-1+\gamma})$ (228)

$$P=(2y^{1+\gamma}+\frac{4y^{1+\gamma}}{\gamma}-\frac{2(-y+\theta)^{\gamma}(2y+\gamma\theta)}{\gamma}+(-y+\theta)^{-1+\gamma}(2y^2+2y\gamma\theta+\gamma(1+\gamma)\theta^2)) \quad (229)$$

The y-displacement v in the initial space for the specific values (α=η=0, β=3, θ=2, γ=0.6) is shown in Figure 48:

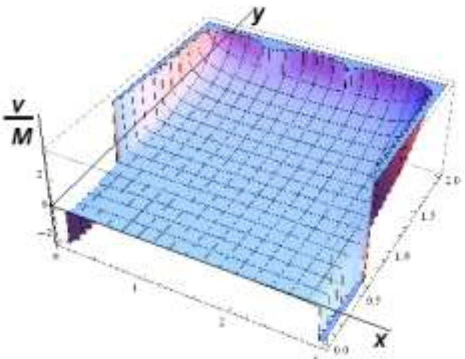

Fig. 48: The y-displacement v in the initial space for the specific values (α=η=0, β=3, θ=2, γ=0.6)

Let us point out that we may not transfer strains in the initial space since geometry and derivatives do not exist in that space.

The present section demonstrates how a problem with both sides of fractional derivatives may be formulated. We introduce that mathematical procedure in the present section, and it may serve as a model for solving problems with Λ-fractional derivatives of both sides.

22 The Continuum Mechanics Hydrocephalus Model in the Λ-Fractional Space

Let's consider a point in the cylindrical tube, at the initial placement, with Lagrangian cylindrical coordinates (R,Θ,Z), which takes the current placement (r, θ, z) with, (Figure 49)

$$r=f(t,R) \quad , \theta=\Theta \quad , \quad z=Z \quad (230)$$

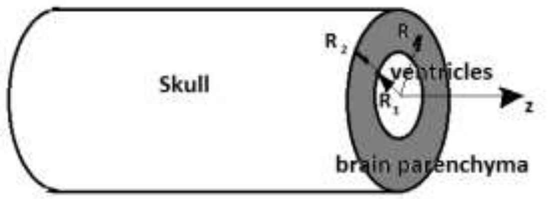

Fig. 49: The hydrocephalus model

Then, we define the deformation gradient $\boldsymbol{F}$, in the cylindrical system (r,θ,z) by:

$$\boldsymbol{F}(t,R)=\begin{vmatrix}\frac{\partial f(t,R)}{\partial R} & 0 & 0\\ 0 & \frac{f(t,R)}{R} & 0\\ 0 & 0 & 1\end{vmatrix}. \quad (231)$$

where the material strain energy density function depends upon the left Cauchy-Green deformation tensor **B** defined by:

$$\boldsymbol{B}(t,R)=\boldsymbol{F}(t,R)\cdot\boldsymbol{F}^T(t,R)=\begin{vmatrix}(\frac{\partial f(t,R)}{\partial R})^2 & 0 & 0\\ 0 & (\frac{f(t,R)}{R})^2 & 0\\ 0 & 0 & 1\end{vmatrix}. \quad (232)$$

The principal invariants of ***B****(t, R)* are defined by:

$$I_1=(\frac{\partial f(t,R)}{\partial R})^2+\left(\frac{f(t,R)}{R}\right)^2+1, \quad (233)$$

$$I_2=(\frac{\partial f(t,R)}{\partial R})^2\left(\frac{f(t,R)}{R}\right)^2+(\frac{\partial f(t,R)}{\partial R})^2+\left(\frac{f(t,R)}{R}\right)^2, \quad (234)$$

$$I_3=(\frac{\partial f(t,R)}{\partial R})^2\left(\frac{f(t,R)}{R}\right)^2. \quad (235)$$

The third invariant I_3=1, since we assume incompressibility of the brain tissue, which yields:

$$h(T,R)=\frac{df(t,R)}{dR}=\frac{R}{f(t,R)} \quad . \quad (236)$$

The solution of the Eq.(236) is defined by:

$$f(t,R)=\sqrt{R^2+k(t)} \quad . \quad (237)$$

Then, the left Cauchy-Green deformation tensor $\boldsymbol{B}(t,R)$ yields,

$$\boldsymbol{B}(t,R)=\begin{vmatrix}\frac{R^2}{R^2+k(t)} & 0 & 0\\ 0 & \frac{R^2+k(t)}{R^2} & 0\\ 0 & 0 & 1\end{vmatrix}. \quad (238)$$

Moreover, the rate of strain tensor is defined in the present case by:

$$\boldsymbol{D}(t,R)=\begin{vmatrix}\frac{\partial \dot{f}(t,R)}{\partial f} & 0 & 0\\ 0 & \frac{\dot{f}(t,R)}{f} & 0\\ 0 & 0 & 0\end{vmatrix}. \quad (239)$$

The stress tensor $\boldsymbol{\sigma}$, $\boldsymbol{f}$or an incompressible Kelvin-Voigt material, is defined by:

$$\boldsymbol{\sigma}=-q\boldsymbol{I}+2\left[\left(\frac{\partial W}{\partial I_1}+I_1\frac{\partial W}{\partial I_2}\right)\mathbf{B}-\frac{\partial W}{\partial I_2}\mathbf{B^2}\right]+\eta\boldsymbol{D}(t,R), \quad (240)$$

As q we denote the incompressibility pressure term, and η is a coefficient that has to do with the visco-elastic behavior of the material.

We adapt the strain energy model, following [21], as a Mooney-Rivlin model with strain energy density:

$$W=c_{10}(I_1-3)+c_{01}(I_2-3). \quad (241)$$

Therefore, the stresses are expressed by:

$$\Sigma_r(t,R)=-q+2\left(c_{10}+c_{01}\left(h^2(t,R)+\frac{f^2(t,R)}{R^2}+1\right)\right)h^2(t,R)-2c_{01}h^4(t,R)+\eta\frac{\partial^2 f(t,R)}{\partial t\partial R}. \quad (242)$$

$$\Sigma_\theta(t,R)=-q+2\left(c_{10}+c_{01}\left(h^2(t,R)+\frac{f^2(t,R)}{R^2}+1\right)\right)\frac{h^2(t,R)}{R^2}-2c_{01}\frac{h^4(t,R)}{R^4}+\eta\frac{\partial}{\partial t}\left(\frac{f(t,R)}{R}\right) \quad (243)$$

$$\Sigma_z(t,R)=-q+2\left(c_{10}+c_{01}\left(h^2(t,R)+\frac{f^2(t,R)}{R^2}+1\right)\right)-2c_{01}. \quad (244)$$

The equilibrium equation, excluding body forces, is expressed by:

$$\frac{\partial\Sigma_R}{\partial R}+\frac{1}{R}(\Sigma_R-\Sigma_\theta)=0, \quad (245)$$

with the b.cs.

$$\Sigma_R(T,R_1)=-P_0(T) \text{ and } \Sigma_R(T,R_2)=0 \quad (246)$$

Further,

$$P_o(T)=\frac{\mu}{2}\int_1^b\left(\frac{1}{x}-\frac{x}{(x+B(T))^2}\right)dx+\frac{\eta}{2R_1}\int_1^b\left(\frac{\dot{B}(T)}{\sqrt{x}\sqrt{(x+B(T))}}+\frac{\dot{B}(T)}{\left(\sqrt{(x+B(T))}\right)^3}\right)dx. \quad (247)$$

with $B(T)=R_1^{-2}k(T)$.

Thus, it shows up the equation,

$$B'(T)=\frac{-\frac{\mu}{2}\left(\frac{(-1+b)B(T)}{(1+B(T))(b+B(T))}+\mathrm{Ln}\left(\frac{b(1+B(T))}{b+B(T)}\right)\right)+P_0(T)}{\eta\left(\frac{1}{(1+B(T))^{0.5}}-\frac{2}{(b+B(T))^{0.5}}\right)}, \quad (248)$$

with the b.c., $B(0)=0$. (249)

The deformation of the hydrocephalus cylinder is extracted from Eq.(248, 249)

Hence, the stresses in the Λ-fractional space are extracted from Eqs.(242-244). We adopt the parameters $\mu=207$, $\eta=0.66$, $b=9$, $R_1=2$ *cm*. In that case, the radial stress $\Sigma_r(T,R)$ in the Λ-fractional space is shown in Figure 50.

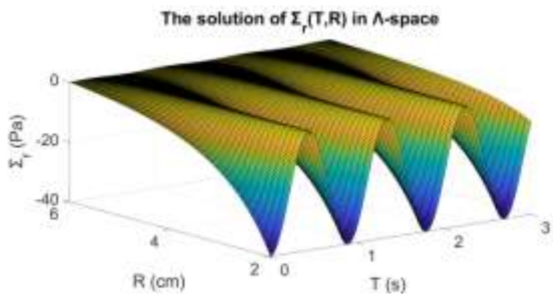

Fig. 50: The radial stress $\Sigma_r(T,R)$ of the ventricular cylinder

Figure 51 shows the distribution of the stress $\Sigma_\theta(T,R)$ in the Λ-fractional space.

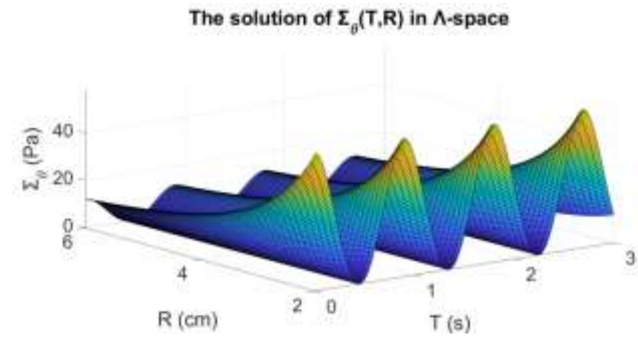

Fig. 51: The stress $\Sigma_\theta(T,R)$ in the Λ-fractional space

These assumptions influence this fluid flow (through the deformation of the hydrocephalus cylindrical tube).

The stresses in the initial space

a. Fractional response with respect to time

We should recall the related equations by transferring the stresses in the initial space. In the present section, we only consider the fractional behavior of time, which is related to the viscoelastic behavior of the model, [38], [39]. In that case, time T in the Λ-space and time t in the initial space are given by:

$$T = \frac{1}{\Gamma(1-\gamma)} \int_0^t \frac{s}{(t-s)^{\gamma}} = \frac{t^{2-\gamma}}{\Gamma(3-\gamma)}. \qquad (250)$$

Hence, we relate time t in the initial space to time T in Λ-space by:

$$t = (\Gamma(3-\gamma)T)^{\frac{1}{2-\gamma}}. \qquad (251)$$

Further, the various stresses $\sigma(t,r)$ in the initial space are defined by:

$$\sigma(t,r) = \frac{1}{\Gamma(\gamma)} \frac{d}{dt} \int_0^t \frac{\Sigma(\frac{s^{2-\gamma}}{\Gamma(3-\gamma)}, r)}{(t-s)^{1-\gamma}} ds. \qquad (252)$$

Figure 52 shows the distribution of the radial stress $\sigma_r(t,r)$ in the initial space for γ=0.9.

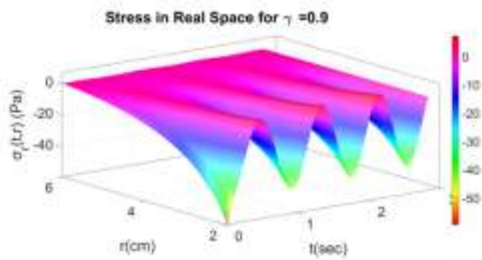

Fig. 52: Distribution of the radial stress $\sigma_r(t,r)$ in the initial space for γ=0.9

Further, Figure 53 shows the stress $\sigma_\theta(t,r)$ in the initial space for *γ=0.9*.

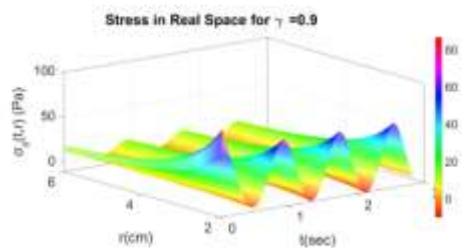

Fig. 53: Distribution of the stress $\sigma_\theta(t,r)$ in the initial space for γ=0.9

In addition, Figure 54 shows the distribution of the radial stress $\sigma_r(t,r)$ in the initial space for γ=0.8.

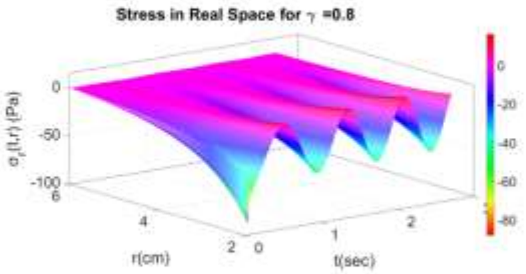

Fig. 54: Distribution of the radial stress $\sigma_r(t,r)$ in the initial space for γ=0.8

Similarly, Figure 55. shows the distribution of the radial stress $\sigma_\theta(t,r)$ in the initial space for γ=0.8.

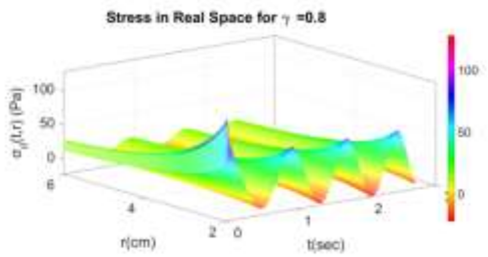

Fig. 55: Distribution of the stress $\sigma_\theta(t,r)$ in the initial space for γ=0.8

If the fractional order is reduced to γ=0.7, the distribution of the radial stress $\sigma_r(t,r)$ in the initial space is shown in Figure 56, and the distribution of the tangential stress $\sigma_\theta(t,r)$ is shown in Figure 57.

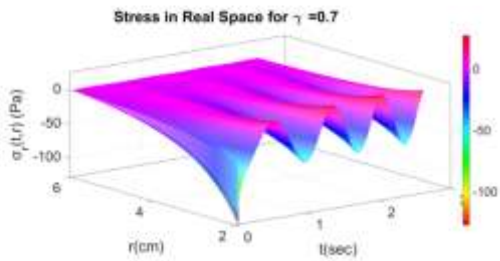

Fig. 56: Distribution of the radial stress $\sigma_r(t,r)$ in the initial space for γ=0.7

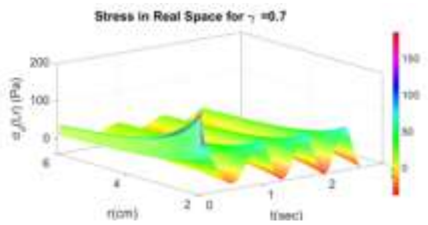

Fig. 57: Distribution of the stress $\sigma_\theta(t,r)$ in the initial space for γ=0.8

Finally, for the fractional order γ=0.6, we show the radial stress distribution $\sigma_r(t,r)$ in the initial space in Figure 58 and the distribution of the stress $\sigma_\theta(t,r)$ In the initial space is figured in Figure 59.

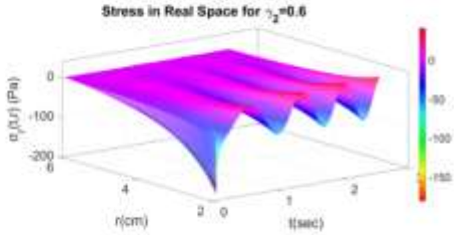

Fig. 58: Distribution of the radial stress $\sigma_r(t,r)$ in the initial space for γ=0.6

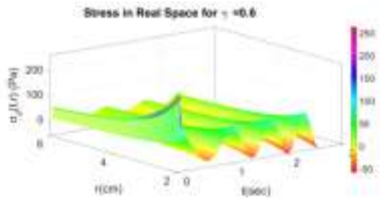

Fig. 59: Distribution of the stress $\sigma_\theta(r,t)$ in the initial space for γ=0.6

The influence of the time (viscosity) fractional order on stresses is shown in Figure 52, Figure 53, Figure 54, Figure 55, Figure 56, Figure 57, Figure 58 and Figure 59. The less the fractional time-fractional order, the higher the stresses.

b. Fractional response with respect to time and space

The fractional response with respect to time corresponds to the viscoelastic reaction of the material; moreover, the fractional response with respect to the special variable *R* corresponds to the porosity of the material. In that case, space variable transferring and time variable transferring should be adopted.

Hence, let us assume that the fractional order of time and space are represented by γ_2 and γ_1, respectively. Furthermore, let us define the fractional space order of the radius *r*, corresponding to the radius R in the Λ-fractional space. Then, we may define the transformation from the Λ-fractional space *(T,R)* to the initial space *(t,r)*, through the relations:

$$R = \frac{1}{\Gamma(1-\gamma_1)}\int_0^r \frac{q}{(r-q)^{\gamma_1}} = \frac{t^{2-\gamma_1}}{\Gamma(3-\gamma_1)} \qquad (253)$$

$$r = (\Gamma(3-\gamma_1)R)^{\frac{1}{2-\gamma_1}} \qquad (254)$$

and

$$T = \frac{1}{\Gamma(1-\gamma_2)}\int_0^t \frac{\tau}{(t-\tau)^{\gamma_2}} = \frac{r^{2-\gamma_2}}{\Gamma(3-\gamma_2)} \qquad (255)$$

$$t = (\Gamma(3-\gamma_2)T)^{\frac{1}{2-\gamma_2}} \qquad (256)$$

Thus,

$$\sigma(t,r) = {}^{RL}_{0}D_t^{1-\gamma_2}\left({}^{RL}_{r_1}D_r^{1-\gamma_1}(\Sigma(\tau,q))\right) = \frac{1}{\Gamma(\gamma_2)\cdot\Gamma(\gamma_1)}\cdot\frac{d}{dt}\int_0^t \frac{1}{(t-\tau)^{1-\gamma_2}}\left(\frac{d}{ds}\int_0^r \frac{\Sigma(\tau,q)}{(s-q)^{1-\gamma_1}}dq\right)d\tau \qquad (257)$$

When the stresses $\Sigma_r(T,R)$, $\Sigma_\theta(T,R)$ *are transferred* from the Λ-fractional space to the initial one, we can define the radial stress $\sigma_r(t,r)$ and the stress $\sigma_\theta(t,r)$ *if* Eq.(257) is applied.

Indeed, the stresses for space fractional order γ_1=0.9 (which indicates the material's porosity) and the time fractional order γ_2=0.9, are shown in Figure 60 and Figure 61.

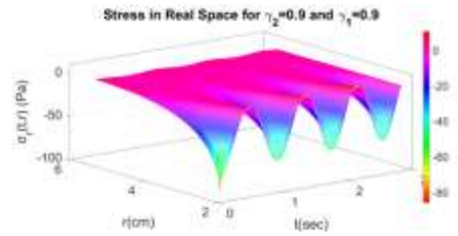

Fig. 60: Distribution of the radial stress $\sigma_r(t,r)$ in the initial space with space (porosity) fractional order γ_1=0.9 and time fractional order γ_2=0.9

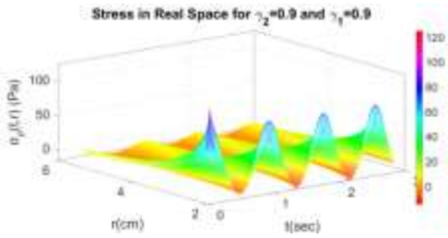

Fig. 61: Distribution of the stress $\sigma_\theta(t,r)$ in the initial space with space (porosity) fractional order γ_1=0.9 and time fractional order γ_2=0.9

If the space (porosity) order is decreased to γ_1=0.7, while leaving the time (viscosity) fractional order the same, γ_2=0.9, the stress distribution for the radial $\sigma_r(t,r)$ and the stress $\sigma_\theta(t,r)$ can be seen in Figure 62 and Figure 63.

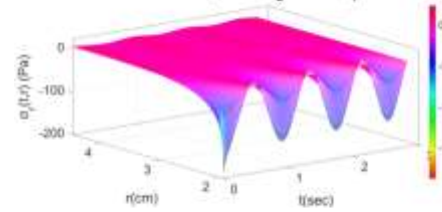

Fig. 62: Distribution of the radial stress $\sigma_r(t,r)$ in the initial space with space (porosity) fractional order γ_1=0.7 and time fractional order γ_2=0.9

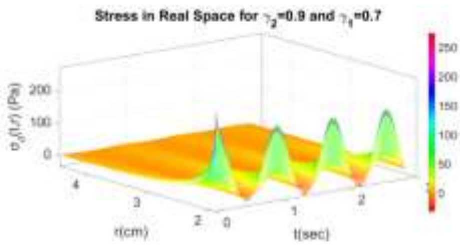

Fig. 63: Distribution of the stress $\sigma_\theta(t,r)$ in the initial space with space (porosity) fractional order γ_1=0.7 and time fractional order γ_2=0.9

If the space (porosity) order is γ_1=0.9, and if we decrease the time (viscosity) fractional order to γ_2=0.7, then we can show the stress distribution for the radial $\sigma_r(t,r)$ and the stress $\sigma_\theta(t,r)$ in Figure 64 and Figure 65.

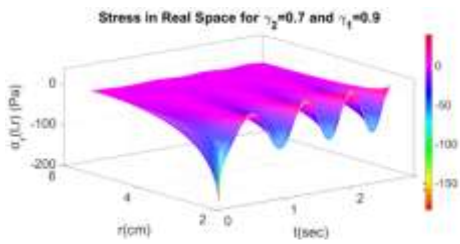

Fig. 64: Distribution of the radial stress $\sigma_r(t,r)$ in the initial space with space (porosity) fractional order γ_1=0.9 and time fractional order γ_2=0.7

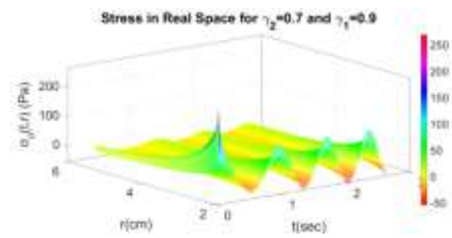

Fig. 65: Distribution of the stress $\sigma_\theta(t,r)$ in the initial space with space (porosity) fractional order γ_1=0.9 and time fractional order γ_2=0.7

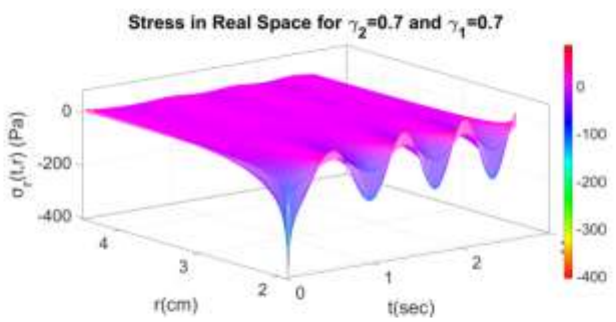

Fig. 66: Distribution of the radial stress $\sigma_r(t,r)$ in the initial space with space (porosity) fractional order γ_1=0.7 and time fractional order γ_2=0.7

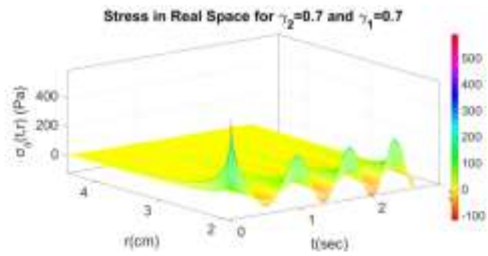

Fig. 67: Distribution of the stress $\sigma_\theta(t,r)$ in the initial space with space (porosity) fractional order γ_1=0.7 and time fractional order γ_2=0.7

The influence of the fractional orders of time and space on the stresses is shown in Figure 60, Figure 61, Figure 62, Figure 63, Figure 64, Figure 65, Figure 66 and Figure 67. In these figures, the less the fractional time-fractional order or the space (porosity) fractional order, the higher the stresses.

23 Λ-Fractional Calculus Dendrites and Axons Study

Potential electric signals of potential V are transferred by dendrites and axons. It is very common to model these minute parts of the neural system using fractional calculus. Moreover, it is assumed that these cables have a constant radius R_0. Fractional derivatives are most suitable to describe this non-local phenomenon. We use Λ-fractional derivatives to model the electric current passing through these building blocks of the neural system while Λ-transform and Λ-space are also participating. The voltage of the electric current inside the cable is governed by the equation ([40]):

$$C_M \frac{\partial V(x,t)}{\partial t} = \frac{d_0}{4r_L}\frac{\partial^2 V(x,t)}{\partial x^2} - i_{ion}. \qquad (258)$$

In Eq.(258) d_0, V(x,t), C_M are the constant diameter of the cable, the voltage of the current passing through the cable, and the specific membrane capacitance accordingly; at the same time r_L denotes the longitudinal resistance, and i_{ion} is the ionic current per unit area into and out of the cable. In the passive cable case, that is, when i_{ion}= V/r_M, with r_M the specific membrane resistance, we have this equation, which is processed geometrically in [40]; Therefore, we can extract the final cable equation:

$$\frac{\partial V(s,t)}{\partial t} = \frac{1}{r_L C_M \int_0^{2\pi} d\theta \sqrt{detg(\theta,s)}}\frac{\partial}{\partial s}\left(a(s)\frac{\partial V(s,t)}{\partial s}\right) - \frac{V(s,t)}{r_M C_M} \qquad (259)$$

In Eq.(259) s is the length of the cable; θ is the angle in the cross-section of the cable; a(s) is the cross-sectional area of the cable, and g(θ,s) is the metric of the cable. It is important to underline that we solved this equation using the Caputo derivative in [40].

According to Lazopoulos' approach, we execute the necessary transformation of the equation to Λ-space with the ordinary derivatives, resulting in the following solution for the voltage in Λ-space ([40]):

$$V^{\Lambda}(T,S) = V_0 l_0 \sqrt{\frac{r_L\cdot c_M}{2\cdot\pi\cdot R_0\cdot T}}\cdot e^{-\frac{r_L\cdot c_M\cdot S^2}{2R_0\cdot T}}\cdot e^{-\frac{T}{r_L\cdot c_M}}, \tag{260}$$

where T, S is the time and arc length in Λ-space. We connect these variables with the ones in real space with the relations for fractional order γ:

$$t = [\Gamma(3-\gamma)\cdot T]^{\frac{1}{2-\gamma}}, s = [\Gamma(3-\gamma)\cdot S]^{\frac{1}{2-\gamma}} \tag{261}$$

The other parameters in Eq. (260)are constants and take the values:

$$c_M = 0.001F/cm^2, r_M = 3000\cdot\Omega\cdot cm^2, r_L = 100\cdot\Omega\cdot cm R_0 = 10^{-4}cm, V_0 = 1.3\times10^{-6}V, l_0 = 0.13cm$$

Initially, the case where the values of arc lengths S in Λ-space are constants will be examined. In order to find the values of the voltage V(t,s) in the initial space, the following inverse transformation is imposed:

$$V(t,s) = {}^{RL}_{0}D_t^{1-\gamma}(V^{\Lambda}(t,s)) = \frac{1}{\Gamma(\gamma)}\frac{d}{dt}\int_0^t \frac{V^{\Lambda}(\tau,s)}{(t-\tau)^{1-\gamma}}d\tau. \tag{262}$$

We can see the voltage V(t, s) for various values of s and fractional order γ in real space in Figure 68, Figure 69, Figure 70 and Figure 71. These figures show that as the value of arc length s is increased, the voltage's maximum to higher time values is shifted. It is believed that we expect this delay in maximum response due to increased cable length. Moreover, we have a decrease in the maximum value of voltage and broadness of the voltage curve as the arc length s increases, denoting an inertial behavior across the cable.

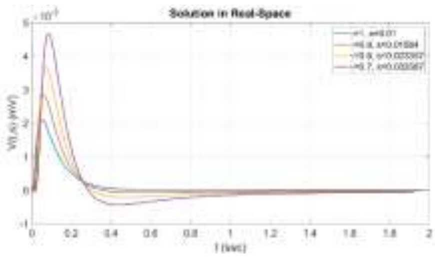

Fig. 68: The voltage V(t,s) for various values of fractional order γ and corresponding values of s, in real space. (S=0.01)

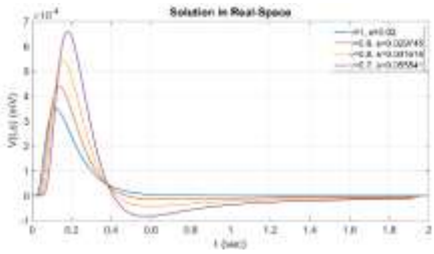

Fig. 69: The voltage V(t,s) for various values of fractional order γ and corresponding values of s, in real space. (S=0.02)

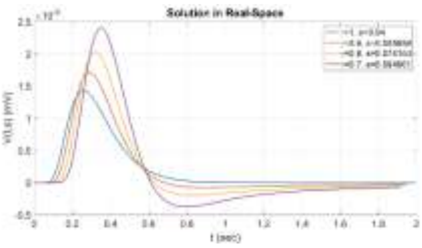

Fig. 70: The voltage V(t,s) for various values of fractional order γ and corresponding values of s, in real space. (S=0.04)

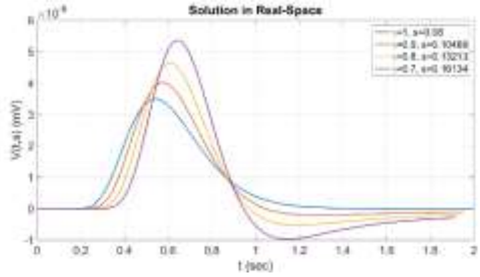

Fig. 71: The voltage V(t,s) for various values of fractional order γ and corresponding values of s, in real space. (S=0.08)

Finally, it must be mentioned that in all cases of arc length values, the decrease of fractional order γ gives greater maximum values in voltage, reversing the polarity of the resulting voltage (from positive to negative ones) as time passes.

Now, we will examine the voltage V^Λ(T,S) (Eq.(260)) as a two-variable function in Λ-space. In order to transform it to the initial space, the following formula will be used for inverse transformation for both t and s, according to the Λ-fractional approach:

$$V(t,s) = {}^{RL}_{0}D_t^{1-\gamma_2}({}^{RL}_{0}D_s^{1-\gamma_1}(V^{\Lambda}(\tau,q))) = \frac{1}{\Gamma(\gamma_2)\cdot\Gamma(\gamma_1)}\cdot\frac{d}{dt}\int_0^t\frac{1}{(t-\tau)^{1-\gamma_2}}\left(\frac{d}{ds}\int_0^s\frac{V^{\Lambda}(\tau,q)}{(s-q)^{1-\gamma_1}}dq\right)d\tau \tag{263}$$

where the relation gives $V^{\Lambda}(\tau,q)$:

$$V^{\Lambda}(\tau,q)=V_0 l_0\sqrt{\frac{r_L\cdot c_M\cdot\Gamma(3-\gamma_2)}{2\cdot\pi\cdot R_0\cdot\tau^{2-\gamma_2}}}\cdot e^{-\frac{r_L\cdot c_M\cdot\Gamma(3-\gamma_2)\cdot q^{4-2\gamma_1}}{2R_0\cdot(\Gamma(3-\gamma_1))^2\cdot\tau^{2-\gamma_2}}}\cdot e^{-\frac{\tau^{2-\gamma_2}}{r_L\cdot c_M\cdot\Gamma(3-\gamma_2)}} \tag{264}$$

In this case, the fractional orders (γ_2,γ_1) for the inverse transformation are different for time t and arc length s. The voltage V(t,s) in real space for various values of fractional orders is presented in Figure 72, Figure 73, Figure 74, Figure 75, Figure 76, Figure 77 and Figure 78. The constants in Eq.(264) take the same values as in Eq.(260).

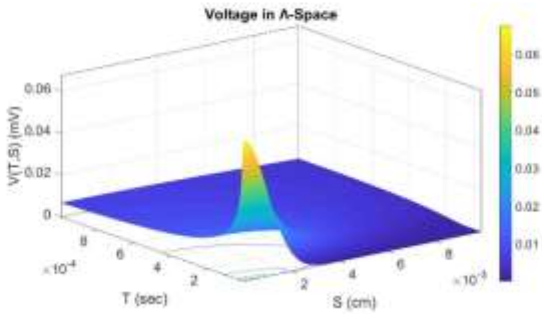

Fig. 72: The voltage V^{Λ}(T,S) in Λ-space as a function of time T and arc length S

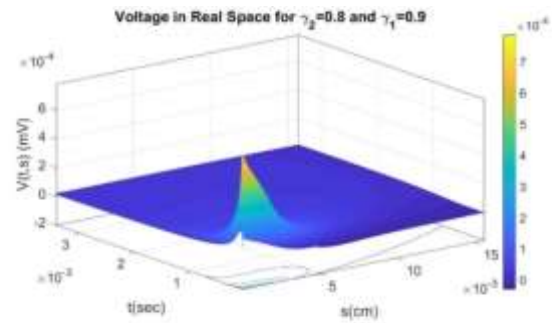

Fig. 73: The voltage V(t,s) in real space as a function of time and arc length s, for fractional orders γ2=0.8 and γ1=0.9

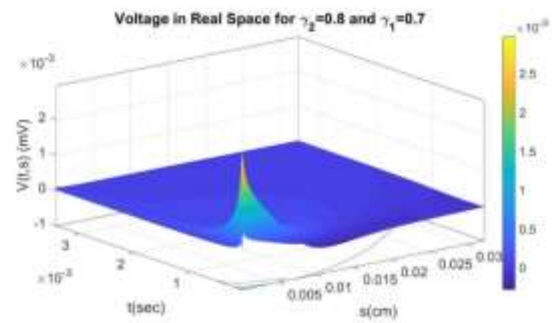

Fig. 74: The voltage V(t,s) in real space as a function of time and arc length s, for fractional orders γ2=0.8 and γ1=0.7

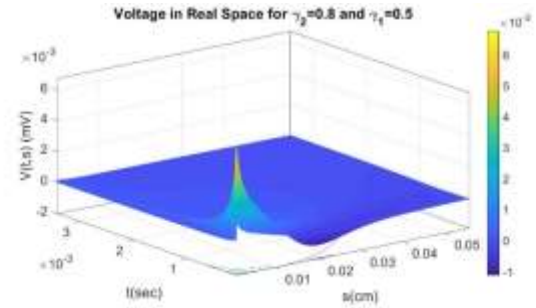

Fig. 75: The voltage V(t,s) in real space as a function of time and arc length s, for fractional orders γ2=0.8 and γ1=0.5

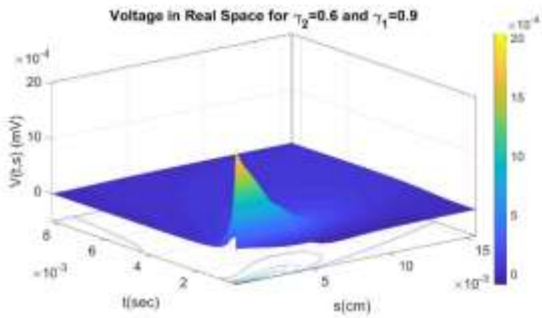

Fig. 76: The voltage V(t,s) in real space as a function of time and arc length s, for fractional orders γ2=0.6 and γ1=0.9

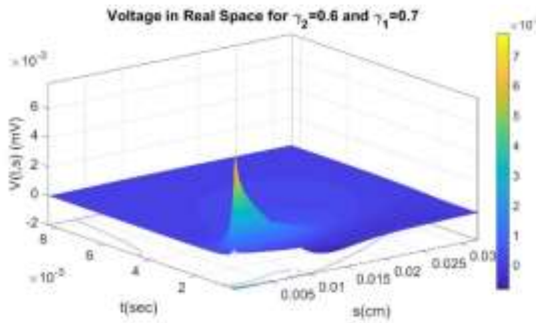

Fig.77. The voltage V(t,s) in real space as a function of time and arc length s, for fractional orders γ2=0.6 and γ1=0.7

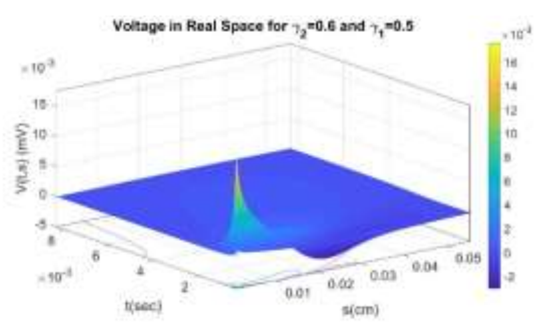

Fig. 78: The voltage V(t,s) in real space as a function of time and arc length s, for fractional orders γ2=0.6 and γ1=0.5

Based on Figure 72, Figure 73, Figure 74, Figure 75, Figure 76, Figure 77 and Figure 78, we can conclude that as we decrease the fractional order for time t (γ_2) or arc length s (γ_1), the maximum value reached by the voltage V(t,s) increases. Also, in all cases, the voltage's polarity (positive to negative) is changed along the cable. Finally, we can observe that as fractional order for time t (γ_2) or arc length s (γ_1) decreases, we have non-zero voltage values for higher values of arc length s (longer cable).

24 The Λ-fractional Special Relativity Theory

Two principles govern the Special Relativity.

a. The first states that the same physical laws govern all physical phenomena in all inertial systems and
b. The light has constant speed *c* in the vacuum in all inertial observers and equal to *c*= 2.99792458 x10^{8}m/sec.

An inertial system Σ (x,y,z,t) and another one Σ'(x΄,y΄,z΄, t΄) are considered moving along the axis x with constant relative velocity V. Then the special theory of relativity is characterized by two basic properties:

p1. Dilation of time $\Delta t'$ $\Delta t = \frac{\Delta t}{\sqrt{1-\frac{V^2}{c^2}}}$ (265)

p2. Contraction of length along the x-axis, $\Delta L' = \Delta L\sqrt{1-\frac{V^2}{c^2}}$. (266)

Further, we express the Lorentz transformation between the two inertial systems by:

$$\begin{pmatrix} c\Delta t' \\ \Delta x' \\ \Delta y' \\ \Delta z' \end{pmatrix} = \Lambda(V) \begin{pmatrix} c\Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} \quad (267)$$

with $\Lambda(V)$ the 4x4 matrix,

$$\Lambda(V) = \begin{pmatrix} \gamma(V) & -\beta(V)\gamma(V) & 0 & 0 \\ -\beta(V)\gamma(V) & \gamma(V) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (268)$$

where,

$$\beta(V) = \frac{V}{c}, \ \gamma(V) = \frac{1}{\sqrt{1-\beta^2(V)}}. \quad (269)$$

Further,

$$\Lambda(V)\Lambda(-V) = \mathbf{1} \quad (270)$$

where **1** is the unit matrix of the 4x4 matrices.

Moreover, (u_x ,u_y ,u_z) are the components of velocity in the Σ inertial coordinate system and (u'_x , u'_y, u'_z) the corresponding coordinates in the Σ' inertial coordinate system as we apply the Lorenz transformation, we have:

$$u'_x = \frac{u_x - V}{1 - Vu_x/c^2} \quad (271)$$

$$u'_y = \sqrt{1-\frac{V^2}{c^2}} \ \frac{u_y}{1-Vu_x/c^2} \quad (272)$$

$$, \ u'_z = \sqrt{1-\frac{V^2}{c^2}} \ \frac{u_z}{1-Vu_x/c^2} \ . \quad (273)$$

The corresponding components of the acceleration in the inertial coordinate system Σ are (α_x, α_y, α_z), therefore the corresponding components (a'_x , a'_y, a'_z) in the Σ' system are defined by:

$$a'_x = a_x \frac{(1-V/c^2)^{3/2}}{(1-Vu_x/c^2)^3}, \quad (274)$$

$$a'_y = \frac{(1-V/c^2)^{3/2}}{(1-Vu_x/c^2)^3}\left(a_y + \frac{V}{c^2}(a_x u_y - a_y u_x)\right) \quad (275)$$

$$a'_z = \frac{(1-V/c^2)^{3/2}}{(1-Vu_x/c^2)^3}\left(a_z + \frac{V}{c^2}(a_x u_z - a_z u_x)\right) \quad (276)$$

Further, the mass of the body is not constant but it depends upon its speed *u* with:

$$m = \frac{m_0}{\sqrt{1-(u^2/c^2)}} \quad (277)$$

where m_0 is the rest mass for the inertial observer.
Also, Newton's second law is defined by:

$$\boldsymbol{f} = \frac{d\boldsymbol{p}}{dt} = \frac{d}{dt}\left[\frac{m_0\boldsymbol{u}}{\sqrt{1-(u^2/c^2)}}\right] = \frac{d}{dt}(m\boldsymbol{u}), \quad (278)$$

where $\boldsymbol{f}$ is the force, *m* is the current mass, m_0 is the inertial observer's rest mass, p is the momentum vector, and u is the velocity vector. Moreover, we define the kinetic energy of a mass point with mass *m* and speed *u* with u=$|\boldsymbol{u}|$ by the formula:

$$K(u,m) = \frac{mc^2}{\sqrt{1-(u^2/c^2)}} - mc^2 \ . \quad (279)$$

Yet, the total energy $E(u,m)$ of the system is expressed by:

$$E(u,m) = \frac{mc^2}{\sqrt{1-(u^2/c^2)}} = mc^2 + K(u,m) \quad (280)$$

According to Λ-fractional theory, a fractional derivative that corresponds to a differential exists only in the Λ-space, where only fractional differential geometry may be established. Hence, with its corresponding laws, physics may be established in that space. We may simply transfer the various results as functions in the initial space. Consider *V* the constant relative speed of the inertial system ${}^{\Lambda}\Sigma'$ for the initial inertial coordinate system ${}^{\Lambda}\Sigma$ in the Λ-space and c be the speed of light.

Then, we express the relation between the time intervals ΔT in the ${}^{\Lambda}\Sigma$ and $\Delta T'$in the ${}^{\Lambda}\Sigma'$ inertial systems by the basic property of dilation of time, which we consider valid in the Λ-space with:

$$\Delta T' = \frac{\Delta T}{\sqrt{1-\frac{V^2}{c^2}}}. \tag{281}$$

where according to transformation,

$$T = {}_0I_t^{1-\gamma}t = \frac{t^{2-\gamma}}{\Gamma(3-\gamma)} \quad \text{and} \quad F(T) = {}_0I_t^{1-\gamma}f(t) \tag{282}$$

Further ΔT and $\Delta T'$ may be transferred in the initial spaces.

Then, we transfer the dilation relation Eq.(281) into the initial spaces Σ and Σ' by the relation:

$${}^{RL}_{0}D_t^{1-\gamma}\Delta T' = \frac{{}^{RL}_{0}D_t^{1-\gamma}\Delta T}{\sqrt{1-\frac{V^2}{c^2}}} \quad . \tag{283}$$

Hence, the dilation relation,Eq.(281) yields a similar contraction relation in the initial spaces Σ and Σ' by the relation:

$$\Delta t' = \frac{\Delta t}{\sqrt{1-\frac{V^2}{c^2}}} \quad . \tag{284}$$

Eq.(284) is similar to Eq.(265), apart from that constant speed V refers to the relative constant speed of the ${}^{\Lambda}\Sigma'$ inertial frame concerning the ${}^{\Lambda}\Sigma$ inertial frame in the Λ-space.

Proceeding to the contraction of lengths along the X-axis of the Λ-space, we get:

$$\Delta L' = \Delta L\sqrt{1-\frac{V^2}{c^2}}, \tag{285}$$

where, $\Delta L' = {}_0I_{t'}^{1-\gamma}\Delta l'$ and $\Delta L = {}_0I_t^{1-\gamma}\Delta l$. (286)

Since,

$$\Delta l = {}^{RL}_{0}D_t^{1-\gamma}\Delta L \quad \text{and} \quad \Delta l' = {}^{RL}_{0}D_t^{1-\gamma}\Delta L', \tag{287}$$

transferring Eq. (287) into the initial spaces, we get a similar contraction of lengths relation:

$$\Delta l' = \Delta l\sqrt{1-\frac{V^2}{c^2}} \quad , \tag{288}$$

where again V is the relative speed of the inertial frame ${}^{\Lambda}\Sigma'$ concerning ${}^{\Lambda}\Sigma$.

Further, Lorentz transformation between the two inertial systems is expressed by:

$$\begin{pmatrix} c\Delta T' \\ \Delta X' \\ \Delta Y' \\ \Delta Z' \end{pmatrix} = \Lambda(V)\begin{pmatrix} c\Delta T \\ \Delta X \\ \Delta Y \\ \Delta Z \end{pmatrix}, \tag{289}$$

with $\Lambda(V)$ the 4x4 matrix, Eq.(268), and Eqs. (269, 270). Then,

$$\begin{pmatrix} c\Delta t' \\ \Delta x' \\ \Delta y' \\ \Delta z' \end{pmatrix} = \Lambda(V)\ {}^{RL}_{0}D_t^{1-\gamma}\begin{pmatrix} c\Delta T \\ \Delta X \\ \Delta Y \\ \Delta Z \end{pmatrix}. \tag{290}$$

The fractional Lorenz transformation is expressed by Eq.(290). Furthermore, the inertial frames Σ and Σ'correspond to the ${}^{\Lambda}\Sigma$ and ${}^{\Lambda}\Sigma'$ in the corresponding fractional Λ-space; (${}^{\Lambda}\Sigma'$ is moving with speed V concerning the inertial system ${}^{\Lambda}\Sigma$ along the axis, X). If (U_x, U_y, U_z) are the components of velocity in the ${}^{\Lambda}\Sigma$ inertial coordinate system and (U'_x , U'_y, U'_z) the corresponding coordinates in the ${}^{\Lambda}\Sigma'$ inertial coordinate system applying Lorenz transformation, then following the Eqs. (271),(272),(273):

$$U'_x = \frac{U_x - V}{1 - VU_x/c^2} \tag{291}$$

$$U'_y = \sqrt{1-\frac{V^2}{c^2}}\ \frac{U_y}{1-VU_x/c^2} \tag{292}$$

$$U'_z = \sqrt{1-\frac{V^2}{c^2}}\ \frac{U_z}{1-VU_x/c^2}. \tag{293}$$

Since in the fractional Λ-space,

$$U_x = {}^{\Lambda}_{0}D_t^{\gamma}x(t) = \frac{dX(t)}{dT(t)} \quad \text{with} \tag{294}$$

$$X(t) = {}_0I_t^{1-\gamma}x(t) \tag{295}$$

$$\text{and} \quad T(t) = {}_0I_t^{1-\gamma}t \tag{296}$$

similar expressions are valid for the speed components U_y, U_z in the Λ-space. If we transfer the components of the velocity from the Λ-space to the initial one, the components in the Σ inertial frame are given by:

$$u_x = {}^{RL}_{0}D_t^{1-\gamma}U_x = {}^{RL}_{0}D_t^{1-\gamma}({}^{\Lambda}_{0}D_t^{\gamma}x(t)) = {}^{RL}_{0}D_t^{1-\gamma}(\frac{dX(t)}{dT(t)}) \tag{297}$$

$$u_y = {}^{RL}_{0}D_t^{1-\gamma}U_y = {}^{RL}_{0}D_t^{1-\gamma}({}^{\Lambda}_{0}D_t^{\gamma}y(t)) = {}^{RL}_{0}D_t^{1-\gamma}(\frac{dY(t)}{dT(t)}) \tag{298}$$

$$u_z = {}^{RL}_{0}D_t^{1-\gamma}U_z = {}^{RL}_{0}D_t^{1-\gamma}({}^{\Lambda}_{0}D_t^{\gamma}z(t)) = {}^{RL}_{0}D_t^{1-\gamma}(\frac{dZ(t)}{dT(t)}) \tag{299}$$

Therefore, Eqs. (297),(298),(299) are defined by:

$$U'_x = \frac{{}_0^\Lambda D_t^\gamma x(t) - V}{1 - \frac{V {}_0^\Lambda D_t^\gamma x(t)}{c^2}} \tag{300}$$

$$U'_y = \sqrt{1 - \frac{V^2}{c^2}} \; \frac{{}_0^\Lambda D_t^\gamma y(t)}{1 - \frac{V {}_0^\Lambda D_t^\gamma x(t)}{c^2}} \tag{301}$$

$$U'_z = \sqrt{1 - \frac{V^2}{c^2}} \; \frac{{}_0^\Lambda D_t^\gamma z(t)}{1 - V {}_0^\Lambda D_t^\gamma x(t)/c^2} \tag{302}$$

We transfer the components of the velocity from the Λ-space to the initial one; then the components in the Σ inertial frame are expressed by:

$$u_x = {}_0^{RL}D_t^{1-\gamma} U_x = {}_0^{RL}D_t^{1-\gamma} ({}_0^\Lambda D_t^\gamma x(t)) = {}_0^{RL}D_t^{1-\gamma} (\frac{dX(t)}{dT(t)}) \tag{303}$$

$$u_y = {}_0^{RL}D_t^{1-\gamma} U_y = {}_0^{RL}D_t^{1-\gamma} ({}_0^\Lambda D_t^\gamma y(t)) = {}_0^{RL}D_t^{1-\gamma} (\frac{dY(t)}{dT(t)}) \tag{304}$$

$$u_z = {}_0^{RL}D_t^{1-\gamma} U_z = {}_0^{RL}D_t^{1-\gamma} ({}_0^\Lambda D_t^\gamma z(t)) = {}_0^{RL}D_t^{1-\gamma} (\frac{dZ(t)}{dT(t)}) \tag{305}$$

Furthermore, we can transfer the speed components in the ${}^\Lambda\Sigma'$ inertial coordinate system in the initial space only as functions, not as derivatives, by the relations:

$$u'_x = {}_0^{RL}D_t^{1-\gamma} U'_x = {}_0^{RL}D_t^{1-\gamma} (\frac{{}_0^\Lambda D_t^\gamma x(t) - V}{1 - V {}_0^\Lambda D_t^\gamma x(t)/c^2}) \tag{306}$$

$$u'_y = {}_0^{RL}D_t^{1-\gamma} U'_y = \sqrt{1 - \frac{V^2}{c^2}} \; {}_0^{RL}D_t^{1-\gamma} \; (\frac{{}_0^\Lambda D_t^\gamma y(t)}{1 - \frac{V {}_0^\Lambda D_t^\gamma x(t)}{c^2}}) \tag{307}$$

$$u'_z = {}_0^{RL}D_t^{1-\gamma} U'_z = \sqrt{1 - \frac{V^2}{c^2}} \; {}_0^{RL}D_t^{1-\gamma} \; (\frac{{}_0^\Lambda D_t^\gamma z(t)}{1 - \frac{V {}_0^\Lambda D_t^\gamma x(t)}{c^2}}) \tag{308}$$

Following a similar procedure, the corresponding components (A'_x , A'_y, A'_z) are defined in the system and ${}^\Lambda\Sigma'$ by the corresponding components of the acceleration (A_x, A_y, A_z) in the inertial coordinate system ${}^\Lambda\Sigma$, see Eqs. (274),(275),(276).

$$A'_x = A_x \frac{\left(1 - V/c^2\right)^{3/2}}{(1 - V U_x/c^2)^3}, \tag{309}$$

$$A'_y = \frac{\left(1 - V/c^2\right)^{3/2}}{(1 - V U_x/c^2)^3} \left(A_y + \frac{V}{c^2}\left(A_x U_y - A_y U_x\right)\right) \tag{310}$$

$$A'_z = \frac{\left(1 - V/c^2\right)^{3/2}}{(1 - V U_x/c^2)^3} \left(A_z + \frac{V}{c^2}\left(A_x U_z - A_z U_x\right)\right) . \tag{311}$$

Similar expressions to Eqs. (274),(275),(276) yield

$$A_x = {}_0^\Lambda D_t^\gamma ({}_0^\Lambda D_t^\gamma x(t)) = \frac{d^2 X(t)}{dT^2} \tag{312}$$

$$X(t) = {}_0I_t^{1-\gamma} x(t) \text{ and } T(t) = {}_0I_t^{1-\gamma} t \tag{313}$$

$$A_y = {}_0^\Lambda D_t^\gamma ({}_0^\Lambda D_t^\gamma y(t)) = \frac{d^2 Y(t)}{dT^2}, \tag{314}$$

$$A_z = {}_0^\Lambda D_t^\gamma ({}_0^\Lambda D_t^\gamma z(t)) = \frac{d^2 Z(t)}{dT^2} \tag{315}$$

Therefore, Eqs. (312),(313),(314),(315) yield:

$$A'_x = {}_0^\Lambda D_t^\gamma \left({}_0^\Lambda D_t^\gamma x(t)\right) \left(\frac{\left(1 - V/c^2\right)^{3/2}}{\left(1 - V({}_0^\Lambda D_t^\gamma x(t))/c^2\right)^3}\right) \tag{316}$$

$$A'_y = \left(\frac{\left(1 - V/c^2\right)^{3/2}}{\left(1 - V({}_0^\Lambda D_t^\gamma x(t))/c^2\right)^3}\right) \Bigg({}_0^\Lambda D_t^\gamma \left({}_0^\Lambda D_t^\gamma y(t)\right) + \frac{V}{c^2}\Big({}_0^\Lambda D_t^\gamma \left({}_0^\Lambda D_t^\gamma x(t)\right)\left({}_0^\Lambda D_t^\gamma y(t)\right) - {}_0^\Lambda D_t^\gamma \left({}_0^\Lambda D_t^\gamma y(t)\right)\left({}_0^\Lambda D_t^\gamma x(t)\right)\Big)\Bigg) \tag{317}$$

$$A'_z = \left(\frac{\left(1 - V/c^2\right)^{3/2}}{\left(1 - V({}_0^\Lambda D_t^\gamma x(t))/c^2\right)^3}\right) \Bigg({}_0^\Lambda D_t^\gamma \left({}_0^\Lambda D_t^\gamma z(t)\right) + \frac{V}{c^2}\Big({}_0^\Lambda D_t^\gamma \left({}_0^\Lambda D_t^\gamma x(t)\right)\left({}_0^\Lambda D_t^\gamma z(t)\right) - ({}_0^\Lambda D_t^\gamma \left({}_0^\Lambda D_t^\gamma z(t)\right))\left({}_0^\Lambda D_t^\gamma x(t)\right)\Big)\Bigg) \tag{318}$$

If we transfer the acceleration components from the Λ-space to the initial one, we have the components in the Σ inertial frame given by,

$$a_x = {}_0^{RL}D_t^{1-\gamma} A_x = {}_0^{RL}D_t^{1-\gamma} ({}_0^\Lambda D_t^\gamma ({}_0^\Lambda D_t^\gamma) x(t)) = {}_0^{RL}D_t^{1-\gamma} (\frac{d^2 X(t)}{dT^2}) \tag{319}$$

$$a_y = {}_0^{RL}D_t^{1-\gamma} A_y = {}_0^{RL}D_t^{1-\gamma} ({}_0^\Lambda D_t^\gamma ({}_0^\Lambda D_t^\gamma) y(t)) = {}_0^{RL}D_t^{1-\gamma} (\frac{d^2 Y(t)}{dT^2}) \tag{320}$$

$$a_z = {}_0^{RL}D_t^{1-\gamma} A_z = {}_0^{RL}D_t^{1-\gamma} \left({}_0^\Lambda D_t^\gamma ({}_0^\Lambda D_t^\gamma) z(t)\right) = {}_0^{RL}D_t^{1-\gamma} \left(\frac{d^2 Z(t)}{dT^2}\right) \tag{321}$$

Furthermore, the acceleration components in the ${}^\Lambda\Sigma'$ inertial coordinate system may be transferred in the initial space as functions, not as derivatives, by the relations,

$$a'_x = {}^{RL}_{0}D_t^{1-\gamma} A'_x \quad , \tag{322}$$
$$a'_y = {}^{RL}_{0}D_t^{1-\gamma} A'_y \quad , \tag{323}$$
$$a'_x = {}^{RL}_{0}D_t^{1-\gamma} A'_z \quad , \tag{324}$$
where, A'_x , A'_y , A'_z are defined by Eqs. (316),(317),(318).

However, the body's mass depends upon its velocity ***u*, so it** is not constant, see Eq.(277), in the conventional special relativity theory. Therefore, in the Λ-space where the mass M corresponds to the rest mass M_0 is defined by:

$$M = \frac{M_0}{\sqrt{1-\left(\left(\left({}^{\Lambda}_{0}D_t^{\gamma}x(t)\right)^2+\left({}^{\Lambda}_{0}D_t^{\gamma}y(t)\right)^2+\left({}^{\Lambda}_{0}D_t^{\gamma}z(t)\right)^2\right) \ /c^2\right)}} \tag{325}$$

where we define M_0 as the rest mass regarding the inertial observer in the Λ-space. Let us point out that,

$$M_0 = {}_0I_t^{1-\gamma} m_0 = \frac{m_0 t^{1-\gamma}}{\Gamma(2-\gamma)} \tag{326}$$

and m_0 is the rest mass in the inertial system Σ of the initial space. Therefore, we define the current mass m in the inertial frame Σ of the initial space:

$$m = {}^{RL}_{0}D_t^{1-\gamma} M = {}^{RL}_{0}D_t^{1-\gamma}\left(\frac{m_0 t^{1-\gamma}}{\Gamma(2-\gamma)\sqrt{1-\left(\left(\left({}^{\Lambda}_{0}D_t^{\gamma}x(t)\right)^2+\left({}^{\Lambda}_{0}D_t^{\gamma}y(t)\right)^2+\left({}^{\Lambda}_{0}D_t^{\gamma}z(t)\right)^2\right) \ /c^2\right)}}\right) \tag{327}$$

Also, Newton's second law is defined in the fractional Λ-space, by:

$$\boldsymbol{F} = \frac{d\boldsymbol{P}}{dT} = \frac{d}{dT}\left[\frac{M_0\boldsymbol{U}}{\sqrt{1-(U^2/c^2)}}\right] = \frac{d}{dT}(M\boldsymbol{U}) \tag{328}$$

where $\boldsymbol{F}$ is the force in the Λ-space, M is the current mass in the Λ-space, M_0 is the rest mass in the Λ-space concerning the inertial observer, $\boldsymbol{P}$ is the momentum vector, and $\boldsymbol{U}$ the velocity vector. Educing that, Eqs.(291),(292),(293) becomes:

$$\boldsymbol{U} = {}^{\Lambda}_{0}D_t^{\gamma}x(t)\,\boldsymbol{i} + {}^{\Lambda}_{0}D_t^{\gamma}y(t)\,\boldsymbol{j} + {}^{\Lambda}_{0}D_t^{\gamma}z(t)\,\boldsymbol{k} \tag{329}$$

and M is the corresponding mass in the Λ-space defined by Eqs. (326, 327); the force ***F*,** according to the second Newton's law, is defined by:

$$\boldsymbol{F} = \frac{d\boldsymbol{P}}{dT} = \frac{d}{dT}\left[\frac{M_0({}^{\Lambda}_{0}D_t^{\gamma}x(t)\,\boldsymbol{i}+{}^{\Lambda}_{0}D_t^{\gamma}y(t)\,\boldsymbol{j}+{}^{\Lambda}_{0}D_t^{\gamma}z(t)\,\boldsymbol{k})}{\sqrt{1-((({}^{\Lambda}_{0}D_t^{\gamma}x(t))^2+({}^{\Lambda}_{0}D_t^{\gamma}y(t))^2+({}^{\Lambda}_{0}D_t^{\gamma}z(t))^2)/c^2)}}\right] = \frac{d}{dT}(M\boldsymbol{U}). \tag{330}$$

Hence we transfer back the force $\boldsymbol{F}$ to the initial space by:

$$\boldsymbol{f} = {}^{RL}_{0}D_t^{1-\gamma}\boldsymbol{F} \quad . \tag{331}$$

Moreover, we express the kinetic energy of a mass point with mass M and speed U with $U=|\boldsymbol{U}|$ in the Λ-space with the help of the relation, Eq.(277),

$$K(U,M) = \frac{Mc^2}{\sqrt{1-(U^2/c^2)}} - Mc^2 \tag{332}$$

with $U^2 = ({}^{\Lambda}_{0}D_t^{\gamma}x(t))^2 + ({}^{\Lambda}_{0}D_t^{\gamma}y(t))^2 + ({}^{\Lambda}_{0}D_t^{\gamma}z(t))^2$ (333)

Transferring the kinetic energy into the initial space with the inertial frame of reference:

$$K(u,m) = {}_0I_t^{1-\gamma}K(U,M) \tag{334}$$

Also,we express the total energy $E(U,M)$ of the system in the Λ-space with the help of Eq.(278):

$$E(U,M) = \frac{Mc^2}{\sqrt{1-(U^2/c^2)}} = Mc^2 + K(U,M) \tag{335}$$

And the total energy in the initial space is defined by:

$$E(u,m) = {}_0I_t^{1-\gamma}E(U,M) = {}_0I_t^{1-\gamma}(Mc^2 + K(U,M)). \tag{336}$$

25 Fractional Special Relativity and Classical Electromagnetism

a. relativity and electromagnetism

We have already discussed the influence of special relativity theory on classical electromagnetism. The various fields are related by (outlining that influence upon the electric field intensity **E**, electric flux density **D**, the magnetic field strength **H,** and the magnetic flux density **B** between two inertial frames with primed frame moving relative to the unprimed velocity **v,** and $\hat{\mathbf{v}}$ the velocity unit vector):

$$\mathbf{E}' = \gamma(\mathbf{E} + \mathbf{v}\times\mathbf{B}) - (\gamma-1)(\mathbf{E}\cdot\hat{\mathbf{v}})\hat{\mathbf{v}} \tag{337}$$
$$\mathbf{B}' = \gamma(\mathbf{B} - \mathbf{v}\times\mathbf{B}/c^2) - (\gamma-1)(\mathbf{B}\cdot\hat{\mathbf{v}})\hat{\mathbf{v}} \tag{338}$$
$$\mathbf{D}' = \gamma(\mathbf{D} + \mathbf{v}\times\mathbf{H}/c^2) + (1-\gamma)(\mathbf{D}\cdot\hat{\mathbf{v}})\hat{\mathbf{v}} \tag{339}$$
$$\mathbf{H}' = \gamma(\mathbf{H} - \mathbf{v}\times\mathbf{D}) + (1-\gamma)(\mathbf{H}\cdot\hat{\mathbf{v}})\hat{\mathbf{v}} \tag{340}$$

when a particle of charge q moves with velocity **u** concerning frame Σ, the Lorenz force in the inertial frame is defined by:

$$\mathbf{F} = q\mathbf{E} + q\,\mathbf{u} \times \mathbf{B} \tag{341}$$

Further, in the inertial frame Σ', the Lorenz force is:

$$\mathbf{F}' = q\mathbf{E}' + q\,\mathbf{u} \times \mathbf{B}'. \tag{342}$$

Further, the equations for the charge density ρ and the current density **J** are defined by:

$$\mathbf{J}' = \mathbf{J} - \gamma\rho\mathbf{v} + (\gamma-1)(\mathbf{J}\cdot\hat{\mathbf{v}})\hat{\mathbf{v}} \quad (343)$$

$$\rho' = \gamma\left(\rho - \frac{\mathbf{J}\cdot\hat{\mathbf{v}}}{c^2}\right) \quad (344)$$

b. Fractional relativity and electromagnetism

The fractional analysis has already proved as a mathematical tool for expressing relativistic phenomena; the present paragraph introduces the fractional Λ-derivative in the electromagnetism theory under the special relativity theory. We must remember that only the fractional Λ-derivative may generate differential geometry, whereas all the other fractional derivatives do not correspond to differentials and are unsuitable for formulating equations for physical problems. We also remind that differential geometry, along with the fractional Λ-derivatives, are valid only in the Λ-space, where the various derivatives exhibit conventional behavior and are local. The various functions may be transferred from the Λ-space to the original one by using the equation:

$$f(t) = {}^{RL}_{0}D_t^{1-\gamma}\left({}_{0}I_t^{1-\gamma}F(T(t))\right) = {}^{RL}_{0}D_t^{1-\gamma}\left({}_{0}I_t^{1-\gamma}f(t)\right) \quad (345)$$

where *F(T)* is the various functions expressed in Eqs. (337)-Eqs. (344) in the Λ-space. All those functions depend upon the variable T, see Eqs. (266, 267). Therefore, all the equations Eq.(342-344) should be expressed with the variable T. For example, Eq.(341) corresponding to the Lorenz force in the Λ-space may be expressed by:

$$\mathbf{F}(\mathrm{T}) = q\mathbf{E}(\mathrm{T}) + q\,\mathbf{u}(\mathrm{T})\times\mathbf{B}(\mathrm{T}). \quad (346)$$

where,

$$\mathbf{u}(\mathrm{T}) = \frac{\mathrm{d}(\mathbf{X}(\mathrm{T}))}{\mathbf{d}\mathrm{T}}\ . \quad (347)$$

Then, considering Eq.(345), Lorenz force ***f(t)*** is transferred into the initial space by the transformation:

$$\boldsymbol{f(t)} = {}^{RL}_{0}\boldsymbol{D}_t^{1-\gamma}\left({}_{0}\boldsymbol{I}_t^{1-\gamma}\boldsymbol{F}(\boldsymbol{T}(t))\right) = {}^{RL}_{0}D_t^{1-\gamma}\left(q\mathbf{E}(\mathrm{T}) + q\,\mathbf{u}(\mathrm{T})\times\mathbf{B}(\mathrm{T})\right) \quad (348)$$

Furthermore, Eq.(342), if we express the corresponding Lorenz force **F′** in the inertial system *Σ′*, with Eqs. (337, 338)

$$\mathbf{F}' = q(\gamma(\mathbf{E}+\mathbf{v}\times\mathbf{B}) - (\gamma-1)(\mathbf{E}\cdot\hat{\mathbf{v}})\hat{\mathbf{v}} + q\,\mathbf{u}\times(\gamma(\mathbf{E}+\mathbf{v}\times\mathbf{B}) - (\gamma-1)(\mathbf{E}\cdot\hat{\mathbf{v}})\hat{\mathbf{v}}). \quad (349)$$

Hence, according to the proposed theory, we transfer the Lorenz force corresponding to the inertial system *Σ′* into the initial system by:

$$\boldsymbol{f}'(t) = {}^{RL}_{0}\boldsymbol{D}_t^{1-\gamma}\left({}_{0}\boldsymbol{I}_t^{1-\gamma}\boldsymbol{F}'(\boldsymbol{T}(t))\right) \quad (350)$$

Therefore, we may follow the described procedure for the definition of any physical quantity related to fractional relativistic electromagnetism.

26 The Maxwell's Equations

a. Theory

We consider the following quantities in order to derive the conventional Maxwell's electromagnetic equations of integer order:

H=magnetic field
D=dielectric displacement
E=Electric field
J=density of electric current
B=magnetic induction ρ= charge density.

Those functions are the space-depended variables x,y,z, and the time t. The conventional Maxwell's equations are,

Ampere's law:

$$\nabla\times\mathbf{H} = \mathbf{J} + \frac{\partial\mathbf{B}}{\partial t} \quad (351)$$

Faraday's law:

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \quad (352)$$

The continuity equation:

$$\nabla\cdot\mathbf{D} = \rho \quad (353)$$

And the non-existence of monopole magnetic:

$$\nabla\cdot\mathbf{B} = 0 \quad (354)$$

Also, the wave equations:

$$v^2\nabla^2\mathbf{B} - \frac{\partial^2\mathbf{B}}{\partial t^2} = 0 \quad (355)$$

$$v^2\nabla^2\mathbf{E} - \frac{\partial^2\mathbf{E}}{\partial t^2} = 0 \quad (356)$$

where *v* is the wave velocity.

At this point, we should know that the aforementioned Maxwell's equations are valid in the Λ-space, where differential exists and fractional differential geometry is developed. Afterward, the equations are transferred from the fractional Λ-space to the initial one, applying the law:

f(x,y,z,t)=

$${}^{RL}_{0}D_z^{1-\gamma}\left({}^{RL}_{0}D_y^{1-\gamma}\left({}^{RL}_{0}D_x^{1-\gamma}\left({}^{RL}_{0}D_t^{1-\gamma}\left(F(x,y,z,t)\right)\right)\right)\right) \quad (357)$$

Therefore, we can transfer all the quantities concerning Maxwell's equations from the Λ-fractional space to the initial one. The application that follows explains the various stages of the proposed method.

b. Application.

Let us consider in the Λ-fractional space,

$$\boldsymbol{E} = E_m \, sin(\omega T - \beta X)\boldsymbol{k} \qquad (358)$$

in free space. Find D, B, H in the original space.

Sketch E and H in the original space with $\omega=\beta=1$ and $T=0.2$.

The Maxwell equation $\nabla \times \mathbf{E} = \partial \mathbf{B}/\partial T$ yields in the Λ-fractional space,

$$\boldsymbol{B} = \frac{\beta E_m}{\omega}\sin(\omega T - \beta X)\boldsymbol{k} \qquad (359)$$

with a neglected constant of integration**.** Then,

$$\boldsymbol{H} = -\frac{\beta H_m}{\omega}\sin(\omega T - \beta X)\boldsymbol{j} \ . \qquad (360)$$

Considering $\beta=\omega=1$ we get

$$E(\mathrm{T}, X) = E_m \cdot \sin(\mathrm{T} - \mathrm{X}) \qquad (361)$$

$$\mathrm{H}(\mathrm{T}, X) = -\mathrm{H}_m \cdot \sin(\mathrm{T} - X) \qquad (362)$$

with $H_m=E_m/\mu_0$.

Those fields valid in the Λ-fractional space, expressed in the variables of the initial space, are:

$$E(t,x) = E_m \cdot \sin\left(\frac{t^{2-\gamma_2}}{\Gamma(3-\gamma_2)} - \frac{x^{2-\gamma_1}}{\Gamma(3-\gamma_1)}\right) \qquad (363)$$

$$\mathrm{H}(t,x) = -H_m \cdot \sin\left(\frac{t^{2-\gamma_2}}{\Gamma(3-\gamma_2)} - \frac{x^{2-\gamma_1}}{\Gamma(3-\gamma_1)}\right) \qquad (364)$$

If we transfer the various quantities into the initial space *(x,y,z,t)* with the spatial fractional order γ_1 and time fractional order γ_2, the fields *E(t,x)* and *H(t,x)* in the initial space may be computed through the relations:

$$E(\mathrm{t},x) = {}_0^{RL}D_t^{1-\gamma_2}\left({}_0^{RL}D_x^{1-\gamma_1}(E(\mathrm{t},x))\right) = \frac{1}{\Gamma(\gamma_2)\cdot\Gamma(\gamma_1)}\cdot\frac{d}{dt}\int_0^t \frac{1}{(\mathrm{t}-\tau)^{1-\gamma_2}}\left(\frac{d}{dx}\int_0^x \frac{E(\tau,s)}{(x-s)^{1-\gamma_1}}ds\right)d\tau \qquad (365)$$

$$H(t,x) = {}_0^{RL}D_t^{1-\gamma_2}\left({}_0^{RL}D_x^{1-\gamma_1}(H(t,x))\right) = \frac{1}{\Gamma(\gamma_2)\cdot\Gamma(\gamma_1)}\cdot\frac{d}{dt}\int_0^t \frac{1}{(t-\tau)^{1-\gamma_2}}\left(\frac{d}{dx}\int_0^x \frac{H(\tau,s)}{(x-s)^{1-\gamma_1}}ds\right)d\tau \qquad (366)$$

We consider for the present application the fields *E(T,X)* and *H(T,X)* with:

$$E(\mathrm{T}, X) = E_m \cdot \sin(\mathrm{T} - X) \qquad (367)$$

$$\mathrm{H}(\mathrm{T}, X) = -\mathrm{H}_m \cdot \sin(\mathrm{T} - X) \qquad (368)$$

and $E_m=H_m=2$.

For $T=0.2$, the diagram of *E(X)* is shown in Figure 79.

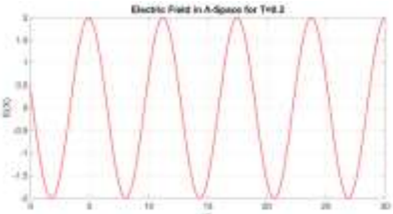

Fig. 79: The electric field *E(X)* in the Λ-fractional space for time $T=0.2$

Further, for fractional time order $\gamma_2=0.4$, time $T=0.2$ corresponds to $t=0.45725$.

If we consider Eqs.(367,368), the electric fields in the original space have been computed and are shown in Figure 80 for fractional space order $\gamma_1=0.3$, 0.5, 0.7 and 0.9. The same is accurate regarding the increase of width and frequency on space distribution of the electric field *E(x,t)* in the present case.

Fig. 80: The electric field *E(X)* in the initial space for time $t=0.457$ and $\gamma_2=0.4$ for various γ_1.

It is evident that the wave's width of the electric field increases when the fractional space order decreases. Furthermore, when decreasing the fractional space order, the width and the frequency with respect to space of the electric field distribution is increased.

When we increase the time fractional order $\gamma_2=0.6$, the real-time t corresponding to $T=0.2$ in the Λ- fractional space is $t=0.36983$. We have computed the diagrams of the electric field *E(x)* in the true initial space for various space fractional orders γ_1; they are shown in Figure 81:

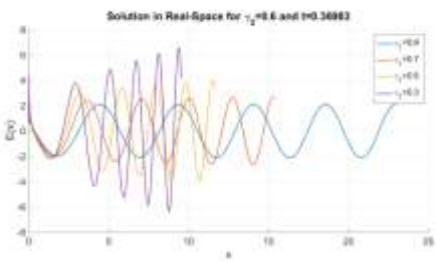

Fig. 81: The electric field *E(X)* in the initial space for time $t=0.37$ and $\gamma_2=0.6$ for various γ_1.

In the present case, the exact comment on the increase of width and frequency on space distribution of the Electric field *E(x,t)* in the initial space is also valid. As we increase time-fractional order γ_2=0.8, the real-time t corresponding to *T*=0.2 in the Λ- fractional space goes to *t*=0.28354. The diagrams of the electric field *E(x)* in the true initial space for various space fractional orders γ_1 are computed and are shown in Figure 82:

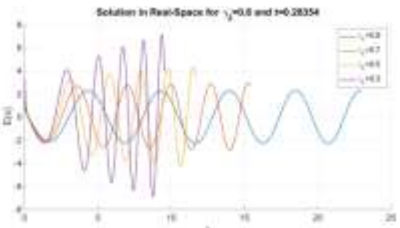

Fig. 82: The electric field *E(X)* in the initial space for time *t*=0.28 and $\gamma_2=0.8$ for various γ_1.

From Figure 80, Figure 81 and Figure 82, it is evident that the width of the electric field E(x,t) increases by increasing the time fractional order. Proceeding to the corresponding magnetic field in the Λ-fractional space for T=0.2 we get the diagram, Figure 83.

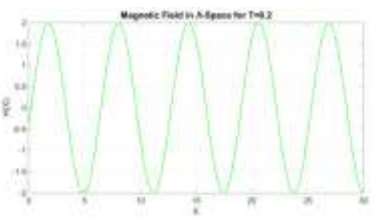

Fig. 83: The Magnetic field *H(T)* in the Λ-fractional space for *T=0.2*

The magnetic field distribution has been computed by transferring the magnetic field into the initial space for various time fractional orders γ2 and space fractional orders γ1. For the time fractional order γ_2=0.4 , the time *T*=0.2, in the Λ-fractional space corresponds to *t*=0.45725 in the initial space (Figure 83). Figure 84 shows the distribution of the magnetic field *H(t)* for various space fractional orders γ_1.

Fig. 84: The magnetic field *H(X)* in the initial space for time *t*=0.457 and $\gamma_2=0.4$ for various γ_1

If we transfer the magnetic field into the initial space for γ_2=0.6, the time *T*=0.2 in the Λ- fractional space corresponds to *t*=0.37 in the initial space. The distribution of the magnetic field *H(t)* in the initial space for various space fractional orders γ_1 is shown in Figure 85.

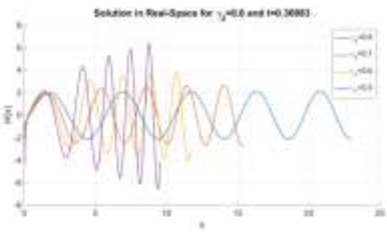

Fig. 85: The magnetic field *H(X)* in the initial space for time *t*=0.37 and $\gamma_2=0.6$ for various γ_1

If we transfer the magnetic field into the initial space for γ_2=0.8, the time *T*=0.2, in the Λ-fractional space, will correspond to *t*=0.28354 in the initial space. The distribution of the magnetic field *H(t)* in the initial space for various space fractional orders γ_1 is shown in Figure 86.

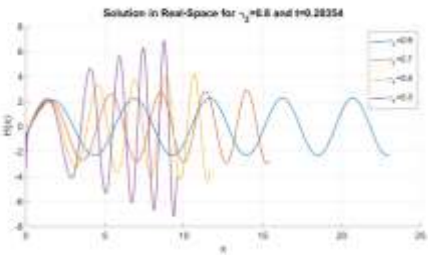

Fig. 86: The magnetic field *H(X)* in the initial space for time *t*=0.29 and $\gamma_2=0.8$ for various γ_1

As a general comment, it is concluded that the width of the electric and magnetic fields are increased with increasing fractional time orders, while they are decreased with fractional space orders.

27 Conclusion

Λ-fractional analysis is presented based upon the introduced Λ-fractional derivative; the only fractional derivative conforming with the prerequisites of differential topology for being a mathematical derivative is presented. Hence, it is the only fractional derivative that generates differential geometry. So, it is a unique fractional analysis that describes non-local phenomena in physics, mechanics, biology, economy, and others. The present review paper presents the basic theory and some critical applications in mathematics, mechanics, and physics. An almost complete catalog of Λ-fractional applications is presented in the references for better information.

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