Exploring Models with Generalized Gamma Density in Statistical Theory

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Abstract: - In the paper, we present a comprehensive exploration of the mathematical properties, applications, and scholarly contributions associated with the generalized gamma family. We study connections between cumulants and central moments and analyze the parameter estimation of the Generalized Gamma distribution in mixed models.

Key-Words: - Estimation, Generalized gamma distribution, Generalized standardized gamma distribution, Symmetry.

Received: July 23, 2024. Revised: November 22, 2024. Accepted: December 18, 2024. Published: December 31, 2024.

1 Introduction

The primary aim of this article is to thoroughly examine and analyze the significant contributions made by various authors and researchers in the field of the Generalized Gamma (GG) family, [1]. Through the provision of a comprehensive and detailed overview, the primary objective of this article is to illuminate and bring forth a greater understanding of the vast and extensive body of work that has been generated and produced on this subject matter. Additionally, this research also aims to offer a comprehensive and in-depth comprehension of the moments and cumulants associated with the generalized standardized gamma distribution. These mathematical expressions and calculations serve as a vital component in facilitating a more profound and all-encompassing comprehension of the various properties and characteristics exhibited by this distribution. Moreover, they also serve as a fundamental and indispensable foundation upon which further analysis, exploration, and applications can be built and conducted. The gamma distribution may be used in place of the normal distribution as the basis distribution in expansions of the Gram-Charlier type. In applied work, gamma distributions give useful representations of many physical situations. They have been used to make realistic adjustments to exponential distributions in representing lifetimes and it is very important in the theory of random counters and other topics associated with random processes in time, in meteorological precipitation processes. The GG family, encompassing Exponential, Gamma, and Weibull as subgroups, and Lognormal as a boundary distribution, has been warmly embraced in the realm of economics, [2]. The authors in [3] have limited the applicability of the GG model. The estimation of parameters for its subgroup (two-parameter gamma distribution) using maximum-likelihood and quasi-maximum likelihood estimators can be found in [4]. The authors of the manuscript [5] have introduced a unique moment estimation method for the parameters of the GG distribution by using its characterization. In statistical modeling, the choice of probability distribution plays a pivotal role in accurately capturing the underlying characteristics of the data.

This paper presents parameter estimation, focusing on the versatile gamma distribution and its generalized forms to address various modeling challenges. It provides a more comprehensive examination of the significant works on the GG family. It explores the relationship between cumulants and central moments, providing a comprehensive view of the distribution's characteristics. The paper is organized as follows: section 2 introduces the gamma distribution and an alternative parametrization along with some special cases when different values are assigned to the parameters. Section 3 explores the estimation of parameters with the method of moments, and likelihood estimation and provides an in-depth exploration of the Moment Generating Function. Section 4 extends our exploration to the continuous three-parameter GG distribution, namely basic properties, particular cases, shapes, CDF, the method of moments, maximum likelihood estimation, and Moment Generating Function. In Section 4 we present the definition of location families, scale families, and location-scale families and we consider Generalized standardized gamma distributions emphasizing that such models achieve a harmony between simplicity and flexibility, facilitating ease of estimation and interpretation, especially in cases where the data's specific shape is known or expected. Section 5 presents Model Adjustment in mixed models and how to obtain a Generalized Least Squares Estimator, the Moments Generating Function for GG distribution, and the final section of this paper turns its attention to standard cumulants and due to the existing relationship between high order cumulants and these cumulants. We also consider mixed Models, where the components of the random part are characterized by r^{th} cumulants. We explore the orthogonality structure of these models, providing a framework for estimating the coefficient parameters as well as cumulants of any order. In Section 6 we study the case considering generalized standardized gamma distribution but the components distribution of the random portion of the model can belong to different types.

2 Gamma Distribution

The gamma distribution is a two-parameter family of continuous probability distributions. The exponential, Erlang, and chi-squared distributions are special cases of the gamma distribution. This distribution can be parameterized in terms of a shape parameter α and an inverse scale parameter $\beta = 1/\theta$, called the rate parameter. A random variable X that is gamma-distributed with shape α and rate β is denoted by $X \sim \Gamma(\alpha, \beta)$, [6]. The corresponding PDF, in the shape-rate parametrization, is given by

$$f(x;\alpha,\beta) = \frac{\beta^{\alpha}x^{\alpha-1}e^{-\beta x}}{\Gamma(\alpha)}, x > 0,\ \alpha,\beta > 0, \quad (1)$$

where $\Gamma(\alpha)$ is the gamma function. Alternatively, the PDF can be expressed in terms of the shape parameter α and the scale parameter θ

$$f(x;\alpha,\theta) = \frac{x^{\alpha-1}e^{\frac{-x}{\theta}}}{\theta^{\alpha}\Gamma(\alpha)}, x > 0,\ \alpha,\theta > 0 \quad . \quad (2)$$

Both parameterizations are common because either can be more convenient depending on the situation. Alternative parameterizations are used when $\alpha = d/p$, with d and p being shape parameters. The CDF of the gamma distribution is the regularized gamma function:

$$F(x;\alpha,\theta) = \int_0^x f(t;\alpha,\theta)dt = \frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)} \quad (3)$$

where γ $(\alpha, \beta x)$ is the lower incomplete gamma function. In [7], they considered the GG function:

$$\Gamma_\lambda(x;\alpha,k) = \int_0^\infty x^{\alpha-1}(x+k)^{-\lambda}e^{-x}dx, \quad (4)$$

where λ is a non-negative integer and $\alpha, \beta > 0$. Sometimes it is interesting to study the following function so-called Modified GG function, MGGF, as

$$\Gamma_\lambda(x;\alpha,k,b) = \int_0^\infty x^{\alpha-1}(x+k)^{-\lambda}e^{-bx}dx, b \geq 0, \quad (5)$$

$x_1, x_2, \ldots, x_n,$ with $\lambda \geq 0$, and [8] state that this function can have special cases, namely

a) If $b = 1$, (5) reduces to the exact form of Kobayashi's function given by (4).

b) If $b = 1$, $\lambda = 0$, (5) becomes the standard form of the gamma function:

$$\Gamma(x;\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx \quad ; \quad (6)$$

c) If $b = 0$, $k = 1$, $\lambda = \alpha + h$, (5) yields the standard form of the beta function of the second type. The PDF of the MGGF distribution is with $x \geq 0$, $\varphi = (\alpha, k, b, \lambda, \theta, \beta)^t$, $\alpha, k, \theta, \beta \geq 0$, $\lambda, \beta > 0$, where α, β are the shape parameters, θ, b *is* the scale parameters, k is the displacement parameter and λ is the parameter of intensity of the effect of the corresponding displacement parameter;

d) When $b = 1$, $\lambda = 0$, (2) reduces to the GG distribution, given in [6] and [9] with PDF

$$f(x;\beta,\alpha,\theta) = \frac{\beta}{\theta\,\Gamma(\alpha)}\left(\frac{x}{\theta}\right)^{\alpha\beta-1} e^{-\left(\frac{x}{\theta}\right)^{\beta}} \quad (7)$$

with $x \geq 0, \beta, \alpha, \theta > 0$.

Different special cases arise when different values are assigned to the parameters. For instance, by selecting specific parameter values, we can derive various distributions such as the Generalized Beta distribution of the second kind, the Weibull distribution, and so on.

3 Estimation of Parameters

3.1 Method of Moments

We will use the notation $X \sim \Gamma(\alpha, \beta)$ to denote that a random variable X with density

$$f(x;\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-\frac{x}{\beta}},\ 0 < x < \infty, \tag{8}$$

with $\alpha > 0, \beta > 0$, follows a gamma distribution with parameters α, β. The r^{th} moment for the gamma distribution is given by:

$$\mu_1^r = E(X^r) = \frac{\beta^r\Gamma(\alpha+r)}{\Gamma(\alpha)},\ \alpha > 0,\ \beta > 0, \tag{9}$$

r a positive integer. Let us consider $X_1, X_2, ..., X_n \sim \Gamma(\alpha, \beta)$, and $x_1, x_2, ..., x_n$ a random sample from a gamma distribution. If we observe a particular value each $x_1, x_2, ..., x_n$, then the sample moments are given by $\frac{\sum_{i=1}^{n} x_i}{n}$.

The method moments consist of setting those population moments equal to the sample moments. Considering the cases $r = 1,2, ...$ we must solve the equations:

$$\frac{\sum_{i=1}^{n} x_i}{n} = \frac{\beta\ \Gamma(\alpha+1)}{\Gamma(\alpha)}, \tag{10}$$

and

$$\frac{\sum_{i=1}^{n} x_i^2}{n} = \frac{\beta^2\ \Gamma(\alpha+2)}{\Gamma(\alpha)}, \tag{11}$$

for α and β. By using the properties for the gamma function and equations (10) and (11), we obtain the estimators:

$$\hat{\alpha} = \frac{\bar{x}^2 n}{\sum_{i=1}^{n} x_i^2 - \bar{x}^2 n} \tag{12}$$

and

$$\hat{\beta} = \frac{\bar{x}}{\hat{\alpha}}, \tag{13}$$

using the notation $\hat{\beta}$ to denote the maximum likelihood estimator for β.

3.2 Maximum Likelihood Estimation

Maximum likelihood estimation is a method of estimating the parameters of a distribution by maximizing a likelihood function so that under the assumed statistical model the observed data is most probable. The point in the parameter space that maximizes the likelihood function is called the maximum likelihood estimate.

Let us consider X_i, $i = 1, ..., n$ independent GG random variables. If $x_1, x_2, ..., x_n$ is a random sample from a gamma distribution with parameters α and β, it is possible to make inferences about the population that is most likely to have generated the sample, specifically the probability distribution corresponding to the population.

Associated with each probability distribution is a unique vector, say $\theta = [\theta_1, \theta_2, ..., \theta_n]^t$ of parameters that index the probability distribution within a parametric family. As θ changes in value, different probability distributions are generated. The likelihood function for the gamma distribution is given by:

$$\mathcal{L}(\alpha,\beta \mid x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f(x_i \mid \alpha, \beta) = \frac{1}{\Gamma(\alpha)^n \beta^{n\alpha}} (x_1, x_2, \ldots, x_n)^{\alpha-1} e^{-\sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)} \tag{14}$$

The maximum likelihood estimation method involves finding the values for α and β that minimize equation (14). By taking the natural logarithm of both sides of (14) and applying the properties of logarithms, we have:

$$\ell(\alpha,\beta \mid x_1, x_2, \ldots, x_n) = -n\ln(\Gamma(\alpha)) - n\alpha\ln(\beta) + (\alpha-1)\ln(x_1, x_2, \ldots, x_n) - \sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right) \tag{15}$$

Taking the derivative concerning α in (15) and setting it equal to zero gives the corresponding equation:

$$\frac{\partial\ell}{\partial\alpha} = -\frac{n\Gamma'(\alpha)}{\Gamma(\alpha)} - ln(\beta) + \sum_{i=1}^{n} ln(x_i) = 0$$

$$\Leftrightarrow -\frac{n\Gamma'(\alpha)}{\Gamma(\alpha)} + \sum_{i=1}^{n} ln\left(\frac{x_i}{\beta}\right) = 0. \tag{16}$$

Differentiating concerning β, we have:

$$\frac{\partial\ell}{\partial\beta} = -\frac{-n\alpha}{\beta} + \frac{\sum_{i=1}^{n} x_i}{\beta^2} = 0 \Leftrightarrow \frac{\sum_{i=1}^{n} x_i}{\beta} = n\alpha, \tag{17}$$

from which we derive:

$$\hat{\beta} = \frac{\sum_{i=1}^{n} x_i}{n\alpha} = \frac{\bar{x}}{\alpha}. \tag{18}$$

Solving for α the equation (16) is quite complicated because of the function $\Gamma(\alpha)$. There is no closed way to solve for α in this equation, but in [10] they show how to obtain the maximum likelihood estimation for α.

3.3 Moment-generating Function for the Gamma Distribution

We shall start by considering the two-parameter gamma distribution that is frequently a probability model for waiting times; for instance, in life testing, the waiting time until death is a random variable that is frequently modeled with a gamma distribution. Its importance is largely due to its relation to exponential and normal distributions. The PDF is given by:

$$f(x;\alpha,\beta)=\frac{1}{\Gamma(\alpha)\beta^{\alpha}}x^{\alpha-1}e^{-\frac{x}{\beta}} \quad (19)$$

with Γ(.) the gamma function, the parameter α is referred to as the shape parameter, as it primarily affects the distribution's kurtosis, [11], while the parameter β is called the scale parameter since most of its influence is on the spread of the distribution and $0 < x < \infty$, $\alpha > 0$, $\beta > 0$. In some applied fields, the parametrization with shape α and rate θ, which is the inverse of β, as mentioned before, is more common.

According to [12], the moment-generating function, MGF, is given by:

$$M_X(t)=E(e^{tX})=\int_0^{\infty}e^{tx}\frac{1}{\beta^{\alpha}\Gamma(\alpha)}x^{\alpha-1}e^{-\frac{x}{\beta}}dx, \quad (20)$$

$\infty < x < +\infty$.

Doing a change of variable $y = x(1-\beta t)$ we obtain:

$$E(e^{tX})=\frac{1}{(1-\beta t)^{\alpha}},\ \ t<\frac{1}{\beta}. \quad (21)$$

If $t \geq 1/\beta$ then the quantity $1/\beta - t$ in the integrand of the above equation is nonpositive and the integral in the second part is infinite. Thus, the MGF of the gamma distribution exists only if $t < 1/\beta$. Following [12], if X has MGF $M_X(t)$, then

$$\mu_r'=E(X^r)=M_X^r(0)=\frac{d^r}{dt^r}\ M_X(t)|_{t=0}. \quad (22)$$

That is, the r^{th} moment is equal to the r^{th} derivative of $M_X(t)$ evaluated at $t = 0$. It is well known that moments are specific measures that allow a more detailed description of a probability distribution. Central moments μ_r can be expressed in terms of noncentral moments (raw moments) using the following relationship, [13],

$$\mu_r=\sum_{j=0}^{r}C_j^r\ (-1)^{r-j}\mu_j'\ \mu^{r-j}. \quad (23)$$

Putting $r=0$ in (23) gives $\mu_0'=1$, independently of the parameters α, β and τ. Therefore, the moment of order zero does not provide any information about the shape or location of the distribution. This is because the moment of order zero is simply the integral of the PDF over the entire domain, which is always equal to one. In parameter estimation problems, it is important to use informative moments that can help estimate the parameters of the distribution accurately.

4 Generalized Gamma Distribution

4.1 Basic Properties of Generalized Gamma Distribution

We will use the notation $X \sim GG(\alpha, \beta, \tau)$ to denote that a random variable has a GG distribution with three parameters. This distribution has two shape parameters, α and τ, and one scale parameter, β, but no location parameter. It has a fixed lower bound equal to zero and exhibits great flexibility in shape, [14] allowing for various forms commonly observed in hydrological applications, [2] and [6]. The GG (α, β, τ) has GG distribution with PDF

$$f(x;\alpha,\beta,\tau)=\frac{\alpha}{\Gamma(\tau)\beta^{\alpha\tau}}\ \ x^{\alpha\tau-1}e^{-\left(\frac{x}{\beta}\right)^{\alpha}}, \quad (24)$$

where x, β, $\alpha > 0$, and τ can be either positive or negative.

An important property of the GG (α, β, τ) family, [13] and [15], is that the family is closed under power transformation, that is:

$$W=X^s\sim GG\left(\alpha,\beta^s,\frac{\tau}{s}\right),\ \ s>0, \quad (25)$$

with s a positive integer and, if $Z = \eta X$, $\eta > 0$, then GG ($\eta\alpha$, β, τ).

4.2 Particular Cases and Shapes

We now obtain cases and shapes from (24). We have the well-known exponential distribution when $\alpha = \tau = 1$, gamma distribution when $\tau = 1$, and Weibull distribution when $\alpha = 1$. If $\tau = 2$, we obtain a subfamily of GG (α, β, τ) which is known as the generalized normal distribution, that itself includes half normal distribution. By setting $\alpha = 1/2$, $\tau = 2$, we get the half-normal distribution defined by:

$$g(x|\ \beta)=\frac{2}{\beta\sqrt{\pi}}\ \ e^{-\left(\frac{x}{\beta}\right)^2},\ x>0, \quad (26)$$

where we use the fact that $\Gamma(1/2) = \sqrt{\pi}$. In the literature sometimes appears $\beta = \sqrt{2}\ \sigma$, where σ is the standard deviation of a normal random variable. So, equation (24) becomes:

$$g(x|\ \sigma)=\frac{1}{\sigma}\ \sqrt{\frac{2}{\pi}}\ e^{-\left(\frac{x^2}{2\sigma^2}\right)} \quad (27)$$

Examining the behavior of the τ parameter of the GG distribution where τ increases, the shape of the distribution tends to become thinner, and within the interval [0,1], the distribution exhibits right skewness. This skewness becomes more pronounced as τ approaches zero. As τ decreases, the graphical representation of this distribution exhibits significant right skewness.

4.3 The Cumulative Distribution Function

The CDF of a random variable X denoted by $F(x)$, is defined as $F(x) = Pr(X \leq x)$. Using identity for the probability of disjoint events, if X is a discrete random variable, then

$$F(x) = \sum_{k=1}^{n} Pr(X = x_k) \quad , \tag{28}$$

where x_k gives the largest possible value of X that is less than or equal to x. The CDF measures any value up to and including x. If X is a continuous random variable, then

$$F(x) = \int_{-\infty}^{x} f(t)dt, \quad -\infty < x < \infty. \tag{29}$$

The CDF of GG distribution is given by:

$$F(x; \alpha, \beta, \tau) = \frac{\gamma(\beta/\tau), (x/\alpha)^{\tau}}{\beta/\tau} \quad , \tag{30}$$

where $\gamma(.)$ denotes the lower incomplete gamma function, as in Section 2. Another interesting function is the survival function given as

$$S_{GG}(t) = 1 - \Gamma(\alpha^{-2}(e^{-\tau}t)^{\alpha/\sigma};\alpha^{-2}). \tag{31}$$

4.4 Estimation of Parameters - Generalized Gamma Distribution

4.4.1 Method of Moments

We will use:

$$\int_0^{\infty} x^r f(x)dx = \int_0^{\infty} \frac{\tau}{\Gamma(\alpha)\beta^{\alpha\tau}} x^{r+\alpha\tau-1} e^{-\left(\frac{x}{\beta}\right)^{\tau}} dx, \tag{32}$$

to find the moments of the GG distribution. Doing the change of variable $y = (\frac{x}{\beta})^{\tau}$, we get:

$$\frac{\tau}{\Gamma(\alpha)\beta^{\alpha\tau}\tau} \beta^{r+\alpha\tau-1} \beta \int_0^{\infty} y^{\alpha+\frac{r}{\tau}-1} e^{-y} dy. \tag{33}$$

The r^{th} moment (also known as the r^{th} moment of the origin or raw moment) of a GG distribution is given by

$$\mu'_r = E[X^r] = \frac{\beta^r \, \Gamma\left(\alpha + \frac{r}{\tau}\right)}{\Gamma(\alpha)} \quad . \tag{34}$$

It is well known that in the method of moments, we find sample moments and set them equal to their population counterparts, solving for the parameters of the distribution. Thus, the equations are obtained by equating population and sample moments:

$$\overline{x} = \frac{\beta \, \Gamma\left(\alpha + \frac{1}{\tau}\right)}{\Gamma(\alpha)}, \quad \sum_{i=1}^{n} x_i^2 = \frac{\beta^2 \, \Gamma\left(\alpha + \frac{2}{\tau}\right)}{\Gamma(\alpha)}, \tag{35}$$

and

$$\sum_{i=1}^{n} x_i^3 = \frac{\beta^3 \, \Gamma\left(\alpha - \frac{3}{\tau}\right)}{\Gamma(\alpha)} \quad . \tag{36}$$

4.4.2 Maximum likelihood Estimation - Generalized Gamma Distribution

In this section, we obtain the maximum likelihood estimators for the GG distribution.

The likelihood function is the product of the PDFs for each observation, so:

$$\mathcal{L}(\alpha, \beta, \tau \mid x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f(x_i \mid \alpha, \beta, \tau). \tag{37}$$

Now, using the PDF of the GG distribution, the likelihood function for n *iid* observations x_1, x_2, ..., x_n is:

$$\mathcal{L}(\alpha, \beta, \tau \mid x_1, x_2, \ldots, x_n) =$$

$$= \frac{\tau^n}{\Gamma(\alpha)^n \beta^{n\alpha\tau}} (x_1, x_2, \ldots, x_n)^{\alpha\tau-1} e^{-\sum_{i=1}^{n} (\frac{x_i}{\beta})^{\tau}}, \tag{38}$$

from which we calculate the log-likelihood function:

$$\ell(\alpha, \beta, \tau \mid x_1, x_2, \ldots, x_n) =$$

$$= n \ln(\tau) - n \ln(\Gamma(\alpha)) - n\alpha\tau \ln(\beta) +$$

$$+(\alpha\tau - 1) \ln(x_1, x_2, \ldots, x_n) - \sum_{i=1}^{n} \left(\frac{x_i}{\beta}\right)^{\tau}. \tag{39}$$

The first-order conditions for finding the optimal values of parameters α, β and τ are obtained by differentiating the log-likelihood function to these parameters. Differentiating for α, we obtain:

$$\frac{\partial \ell}{\partial \alpha} = 0 \Leftrightarrow \frac{\partial \ell}{\partial \alpha} = -\frac{n\Gamma'(\alpha)}{\Gamma(\alpha)} + \tau \sum_{i=1}^{n} (ln(x_i) - ln(\beta))$$

$$= -\frac{n\Gamma'(\alpha)}{\Gamma(\alpha)} + \tau \sum_{i=1}^{n} ln\left(\frac{x_i}{\beta}\right). \tag{40}$$

Differentiating to β, we obtain:

$$\frac{\partial \ell}{\partial \beta} = 0 \Leftrightarrow -n\alpha + \sum_{i=1}^{n} ln\left(\frac{x_i}{\beta}\right)^{\tau} = 0 \quad . \tag{41}$$

From this last equation, β turns out to be:

$$\beta(\alpha, \tau) = \frac{\left(\sum_{i=1}^{n} x_i^{\tau}\right)^{\frac{1}{\tau}}}{(n\alpha)^{\frac{1}{\tau}}} \tag{42}$$

depending on α and τ. Differentiating to τ

$$\frac{\partial \ell}{\partial \tau} = \frac{n}{\tau} - n\alpha ln(\beta) + \alpha \sum_{i=1}^{n} ln(x_i) - \sum_{i=1}^{n} \left[\left(\frac{x_i}{\beta}\right)^{\tau} ln\left(\frac{x_i}{\beta}\right) \right]. \quad (43)$$

Now considering $-nln(\beta) = -\sum_{i=1}^{n} ln(\beta)$ and using the properties of the logarithm function we have:

$$\frac{n}{\tau} + \alpha \sum_{i=1}^{n} ln\left(\frac{x_i}{\beta}\right) - \sum_{i=1}^{n} \left[\left(\frac{x_i}{\beta}\right)^{\tau} ln\left(\frac{x_i}{\beta}\right) \right]. \quad (44)$$

Putting the value of β in (42) into (44) and rearranging terms we get an expression for α in terms of τ

$$\alpha(\tau) = \frac{1}{\tau\left[\sum_{i=1}^{n} ln\left(\frac{x_i}{\beta}\right) - \frac{\sum_{i=1}^{n} z_i^{\tau} ln(x_i)}{\sum_{i=1}^{n} x_i^{\tau}}\right]}, \quad (45)$$

where $z_i = ln\left(\frac{x_i}{\beta}\right)^{\tau}$, [20].

If we substitute (45) and (42) into (45) we will have an equation in terms of τ only. This equation is:

$$M(\tau) = -\psi(\alpha) + \tau\frac{\sum_{i=1}^{n} ln(x_i)}{n} - ln(\sum_{i=1}^{n} x_i^{\tau}) + ln(n\alpha), \quad (46)$$

where $\psi(\alpha) = \frac{\Gamma'(\alpha)}{\Gamma(\alpha)}$ is the digamma function, the derivative of the logarithm of the gamma function, and α is given by (50). It is not always possible to find a solution for $M(\tau)$, [16], to estimate the parameters for the GG distribution.

4.5 Moments and Cumulants Generating Function

Moments and cumulants are the expected values of certain functions of a random variable. They serve to numerically describe the variable concerning given characteristics, e.g., location, variation, skewness, and kurtosis. The moments about zero play a key role for all kinds of moments because the latter can be-easily expressed by zero-moments.

Given a random variable X, the moment generating function, if it exists, (i.e., is finite) is given by:

$$\varphi_X(t) = E(e^{tX}), \quad (47)$$

when E means mean value. The r^{th} order derivative at the origin of $\phi_X(t)$, when defined, is termed the r^{th} moment about zero (relative to the origin) of the random variable X

$$\mu'_r(X) = \varphi^r_X(0),\ r = 1, 2, ..., \quad (48)$$

where r *is* any real number (but for the most part, r is taken as a non-negative integer).

Besides moments about zero, $\mu'_r(X),\ r = 1, 2, ...,$ we will have central moments which are related to the above moments, having:

$$\mu_r(X) = E((X - E(X))^r),\ r = 1,2,.... \quad (49)$$

We point out that $\mu_2(X)$ is the variance, $\sigma^2(X)$, and the relations between central moments and moments about zero are:

$$\begin{cases} \mu_2(X) = \mu'_2(X) - \mu'^2_1(X) \\ \mu_3(X) = \mu'_3(X) - 3\mu'_1(X)\mu'_2(X) + 2\mu^3_1(X) \\ \mu_4(X) = \mu'_4(X) - 4\mu'_1(X)\mu'_3(X) + 6\mu'^2_1(X)\mu'_2(X) - 3\mu'^4_1(X) \end{cases}. \quad (50)$$

Let's now consider the case where the distribution F_X of a random variable X has location, dispersion, and shape parameters, namely α, β and τ, that will be part of moments. It is well established, as documented in [17], that a distribution can be expressed in terms of a standard distribution:

$$\begin{cases} \mu_1(\ |\alpha, \beta, \tau) = \alpha + \beta\mu_1(\ |0, 1, \tau) \\ \mu_2(\ |\alpha, \beta, \tau) = \beta^2\mu_2(\ |0, 1, \tau) \\ \mu_3(\ |\alpha, \beta, \tau) = \beta^3\mu_3(\ |0, 1, \tau) \end{cases}. \quad (51)$$

Additionally, we can express:

$$h(\tau) = \frac{\mu_3(\ |\lambda, \delta, \tau)}{\mu_2(\ |\lambda, \delta, \tau)^{\frac{3}{2}}} = \frac{\mu_3(\ |0, 1, \tau)}{\mu_2(\ |0, 1, \tau)^{\frac{3}{2}}}, \quad (52)$$

and this relation only depends on τ. If we know $\mu_2(\ |\lambda, \delta, \tau)^{\frac{3}{2}}$ and $\mu_3(\ |\lambda, \delta, \tau)$ we also know $h(\tau)$ and we obtain τ.

In addition to the moment-generating function, our attention is now drawn to the cumulant generation function. Considering (47), we have $\psi_X(t)$, the cumulant generation function:

$$\psi_X(t) = ln(\varphi_X(t)). \quad (53)$$

It is known that, according to [18],

$$\varphi(t\ |\alpha, \beta, \tau) = e^{\alpha t}\varphi(\beta\ t\ |0, 1, \tau), \quad (54)$$

so we have:

$$\psi(t\ |\alpha,\beta,\tau) = \alpha t + \psi(\beta t\ |0,1,\tau). \quad (55)$$

The r^{th} cumulant is the r^{th} derivative of the cumulant generation function about zero:

$$\begin{cases} \psi'(0\ |\alpha, \beta, \tau) = \alpha + \beta\ \psi'(0\ |0, 1, \tau) \\ \psi^r(0\ |\alpha, \beta, \tau) = \beta^r\ \psi^r(0\ |0, 1, \tau),\ r > 1 \end{cases}, \quad (56)$$

coming when $r = 2$

$$\beta = \sqrt{\frac{\chi_2(\ |\alpha, \beta, \tau)}{\chi_2(\ |0, 1, \tau)}}. \quad (57)$$

Following [7],

$$\begin{cases} \chi_1(\ |0,1,\tau) = \psi'(0\ |0,1,\tau) = \mu_1(\ |0,1,\tau) \\ \chi_2(\ |0,1,\tau) = \psi^2(0\ |0,1,\tau) = \mu_2(\ |0,1,\tau) = \sigma^2(\ |0,1,\tau) \\ \chi_3(\ |0,1,\tau) = \psi^3(0\ |0,1,\tau) = \mu_3(\ |0,1,\tau) \end{cases}, \quad (58)$$

and the fourth-order cumulant is defined as

$$\chi_4(\ |0,1,\tau) = \psi^4(0\ |0,1,\tau) = \mu_4 - 3(\sigma^2)^2(\ |0,1,\tau). \quad (59)$$

The cumulants above enable us to obtain the skewness and kurtosis coefficients

$$\begin{cases} \gamma_1 = \frac{\chi_3(\ |0,1,\tau)}{\chi_2(\ |0,1,\tau)^{\frac{3}{2}}} \\ \gamma_2 = \frac{\chi_4(\ |0,1,\tau)}{\chi_2(\ |0,1,\tau)^2} - 3 \end{cases}. \quad (60)$$

The skewness of a random variable is the third 'standardized' moment (about zero).

5 Model Adjustment

We will now direct our attention to mixed models and on exploration of their relevant parameters. Mixed models are described by:

$$\boldsymbol{Y} = \boldsymbol{X\beta} + \sum_{i=1}^{w} \boldsymbol{X}_i \boldsymbol{Z}_i + e_i. \quad (61)$$

or omitting e_i to lighten the model. These models are given by the sum of $X\beta$, with β the vector of fixed effects and w independent random terms $Z_i = (Z_{i,1},...,Z_{i,ci})$, $i = 1,...,w$, with $E(Z_i) = 0$, $i = 1,...,w$. Matrices X and X_i, $i = 1,...,w$, are design matrices, and we admit that the components of Z_i are *iid*, having r^{th} cumulants that now, for notation purposes, we will call $\chi_{r,i}$, $i = 1,...,w$, $r > 2$.

In particular $\chi_{1,i} = 0$, $i = 1,...,w$, when the mean values of the components vector are zero and $\sigma_i^2 = \chi_{2,i}$, $i = 1,...,w$. We will integrate the location parameters in the vector of coefficients β, as in [18], so when we estimate β, we are estimating $\lambda_1, ... ,\lambda_w$. Note that here we present an important and substantial advancement in distribution research, that is it can be perfectly assumed that the components distribution of the $Z_1,...,Z_w$, belong to different types, [18].

Let us proceed with our analysis considering the orthogonal complement, $\Omega^\perp$, of the range space $\Omega = R(X)$ of matrix X. The dimension of $\Omega^\perp$ is $n^{\cdot} = n - k$ and

$$Y_l^0 = \alpha_l^\top \boldsymbol{Y},\ l = 1,...,\dot{n}, \quad (62)$$

with an orthonormal basis $(\alpha_1 ,..., \alpha_{n^{\cdot}})$.

Now, with

$$\boldsymbol{a}_{l,i}^{0\top} = \boldsymbol{\alpha}_l^\top \boldsymbol{X}_i = (a_{l,i,1}, ..., a_{l,i,c_i}), \quad (63)$$

$l = 1, ..., n$, $i = 1, ..., w$, and, according to [17] and [18], the cumulant vector has components

$$\chi_r = (\chi_{r,1}, ..., \chi_{r,w}), i = 1,...,w,\ r = 2,3, \quad (64)$$

and we will call $O_r(Y_l^0)$, $l = 1, ..., n$, $r = 2,3$, the r^{th} cumulant of Y_l^0 as

$$\boldsymbol{O}_r(Y_l^0) = \sum_{i=1}^{w} b_{r,l,i}\chi_{r,i},\ \ l = 1,...,\dot{n},\ r = 2,3, \quad (65)$$

with

$$b_{r,l,i} = \sum_{h=1}^{c_i} a_{l,i,h}^r, l = 1,...,\dot{n}, i = 1,...,w, r = 2,3. \quad (66)$$

Considering r=2,3, $B(r) = [b_{r,l,i}]$,i=1, ..., w, and the vector O_r with components:

$$\boldsymbol{O}_r = (\boldsymbol{O}_r(Y_1^0), ..., \boldsymbol{O}_r(Y_{\dot{n}}^0)),\ r = 2,3, \quad (67)$$

we can write

$$O_r = B(r)\,\chi_r,\ r = 2,3. \quad (68)$$

For O_r we have the estimator:

$$\widetilde{\boldsymbol{O}}_r = (\boldsymbol{O}_r(Y_1^0), ..., \boldsymbol{O}_r(Y_{\dot{n}}^0)),\ \ r = 2,3, \quad (69)$$

what gives rise to LSE

$$\widetilde{\boldsymbol{\chi}}_r = (\boldsymbol{B}(r)^\top\ \boldsymbol{B}(r))^+ \boldsymbol{B}(r)^\top\ \dot{\boldsymbol{Y}}[r],\ r = 2,3, \quad (70)$$

where + stands for Moore-Penrose inverse of a matrix. Let us consider $\dot{\boldsymbol{Y}}[r] = (\dot{Y}_1^r, ..., \dot{Y}_{\dot{n}}^r)$, since the mean vector of $\dot{\boldsymbol{Y}}[r]$ is O_r and

$$\boldsymbol{O}_r = \boldsymbol{B}(r)\widetilde{\boldsymbol{\chi}}_r. \quad (71)$$

In particular $\chi_{1,i} = 0$, i=1,...,w, when the mean values of the components are null and we have:

$$\widetilde{\chi}_2 = (\widetilde{\sigma}_1^2\ ...\ \widetilde{\sigma}_w^2), \quad (72)$$

considering $\sigma_i^2 = \chi_{2,i}^2,\ i = 1,...,w$ the variance components of Z_i, i=1, ..., w. Taking the estimators $\widetilde{\sigma}_i^2 = \widetilde{\chi}_{2,i}^2,\ i = 1,...,w,$ we have for Y the estimated variance-covariance matrix, [19]

$$\Sigma(\boldsymbol{Y}) = \sum_{i=1}^{w} \sigma_i^2\, \boldsymbol{M}_i, \quad (73)$$

for which we have the estimator:

$$\tilde{\Sigma}(\boldsymbol{Y}) = \sum_{i=1}^{w} \tilde{\sigma}_i^2\, \boldsymbol{M}_i, \quad (74)$$

and, according to [19], gives the Generalized Least Squares Estimator, GLSE, for β

$$\widetilde{\boldsymbol{\beta}} = (\boldsymbol{X}^\top \tilde{\Sigma}(\boldsymbol{Y})^{-1}\boldsymbol{X})^+ \boldsymbol{X}^\top \tilde{\Sigma}(\boldsymbol{Y})^{-1}\boldsymbol{Y}. \quad (75)$$

If we take

$$\begin{cases} \boldsymbol{X}^0 = [\boldsymbol{X}\ \ \boldsymbol{X}_1\boldsymbol{1}_{c_1}\ ...\ \boldsymbol{X}_w\boldsymbol{1}_{c_w}] \\ \boldsymbol{\beta}^0 = [\boldsymbol{\beta}^\top\ \ \lambda_1\ ...\ \lambda_w]^\top \end{cases}, \quad (76)$$

we have

$$\boldsymbol{Y} = \boldsymbol{X}^0\,\boldsymbol{\beta}^0 + \sum_{i=1}^{w} \boldsymbol{X}_i \boldsymbol{Z}_i^0,\ i = 1, ..., w, \quad (77)$$

with the components of $Z^0_i = Z - \mathbf{1}_{ci}\,\lambda_i$, i=1, ..., w, having null location parameters.

If the components of vectors Z_i, i=1, ..., w, have location, dispersion and shape parameters their r^{th} order cumulants will be, following equation (51), we have:

$$\chi_{r,i}(\ |0,\delta_i,\tau_i) = \delta_i^r \chi_{r,i}(\ |0,1,\tau_i), \quad (78)$$

r = 2,3,4, i=1, ..., w. For r=2, we have $\chi_{2,i}(\ |0,\delta_i,\tau_i) = \delta_i^2\chi_{2,i}(\ |0,1,\tau_i)$, i = 1,...,w, therefore

$$\delta_i = \sqrt{\frac{\chi_{2,i}(\ |\lambda_i,\delta_i,\tau_i)}{\chi_{2,i}(\ |0,1,\tau_i)}} = \sqrt{\frac{\chi_{2,i}(\ |0,\delta_i,\tau_i)}{\chi_{2,i}(\ |0,1,\tau_i)}}, \quad (79)$$

To estimate τ_i we consider $g_{r,i}(\tau_i) = \chi_{r,i}(\ |0,1,\tau_i)$ and

$$h_i(\tau_i) = \frac{\chi^3_{2,i}(\ |0,\delta_i,\tau_i)}{\chi^2_{3,i}(\ |0,1,\tau_i)} = \frac{\chi^3_{2,i}(\ |0,1,\tau_i)}{\chi^2_{3,i}(\ |0,1,\tau_i)} = \frac{g^2_{3,i}(\tau_i)}{g^3_{2,i}(\tau_i)}, \quad (80)$$

thus

$$\tau_i = h_i^{-1}\left(\frac{g^2_{3,i}(\tau_i)}{g^3_{2,i}(\tau_i)}\right) \quad (81)$$

and

$$\widetilde{\tau}_i = h_i^{-1}\left(\frac{g^2_{3,i}(\widetilde{\tau}_i)}{g^3_{2,i}(\widetilde{\tau}_i)}\right). \quad (82)$$

It should be noted that, after estimating τ_i, the estimator $\ddot{\delta}_i$ is:

$$\widetilde{\delta}_i = \sqrt{\frac{\widetilde{\chi}_{2,i}(\ |0,\delta_i,\tau_i)}{\chi_{2,i}(\ |0,1,\widetilde{\tau}_i)}} = \sqrt{\frac{\widetilde{\chi}_{2,i}(\ |0,\delta_i,\tau_i)}{g_{2,i}(\ |\widetilde{\tau}_i)}}, \quad (83)$$

$i = 1,...,w$, where $\chi_{2,i}(\ |0,1,\widetilde{\tau}_i) = g_{2,i}(\ |\widetilde{\tau}_i), i = 1,...,w$, is obtained replacing $\tau_i, i = 1,...,w$, by $\widetilde{\tau}_i, i = 1,...,w$, in $\chi_{2,i}(\ |\lambda_i,\delta_i,\widetilde{\tau}_i), i = 1,...,w$, with $\delta_i = 1, i = 1,...,w$. After estimating $\widetilde{\tau}_i, i = 1,...,w$, and $\widetilde{\delta}_i, i = 1,...,w$ we obtain the 4^{th} cumulants estimators, and superior, through

$$\widetilde{\chi}_{r,i}(\ |0,\delta_i,\tau_i) = \widetilde{\delta}_i^{\,r}\,\chi_{r,i}(\ |0,1,\widetilde{\tau}_i), i = 1,...,w, r \geq 4. \quad (84)$$

If the distribution does not have a shape parameter, expression (79) lightens into:

$$\delta_i = \sqrt{\frac{\chi_{2,i}(\ |\lambda_i,\delta_i)}{\chi_{2,i}(\ |0,1)}}, i = 1,...,w, \quad (85)$$

and the estimator is:

$$\widetilde{\delta}_i = \sqrt{\frac{\widetilde{\chi}_{2,i}(\ |\lambda_i,\delta_i)}{\chi_{2,i}(\ |0,1)}},\ i = 1,...,w. \quad (86)$$

If the distribution has no location parameter but has two shape parameters, ρ_i and δ_i, for example, we consider $\chi_{2,i}(\ |\rho_i,\ \delta_i,\ \tau_i) = \delta_i^2\,\chi_{2,i}(\ |1,1,\tau_i)$, i=1,...,w. The components of Z^0_i,i=1,...,w, see equation (77), will have parameters (0,δ_i,τ_i) or (λ_i,δ_i), i = 1,...,w. In the last case, we do not consider the shape parameter.

The vectors Z^0_i,i=1,...,w,have the same cumulants of order $r > 1$ as Z_i, i=1,...,w, and we can estimate these cumulants from equations (69) and (70). Considering equation (76) we see that λ_1, ..., λ_w are estimated by estimating β^0.

On the other hand, proceeding as in equation (70), we only have to estimate the cumulants $\chi_{r,i}$, i=1,...,w, r=2,3,... to obtain the GLSE for

$$\widetilde{\boldsymbol{\beta}} = (\boldsymbol{X}^\top \widetilde{\boldsymbol{\Sigma}}(\boldsymbol{Y})^{-1}\boldsymbol{X})^{+}\boldsymbol{X}^\top \widetilde{\boldsymbol{\Sigma}}(\boldsymbol{Y})^{-1}\boldsymbol{Y}, \quad (87)$$

as seen earlier in equation (75).

5.1 Moment-generating Function for Generalized Gamma Distribution

In the following, we shall operate as before, in (19), but now consider three parameters. Under the assumption that $\mu'_r(x)$ exists, [14], i.e., $\alpha + \tau/\beta > 0$.

$$M_X(t) = E(e^{tX}) = \frac{1}{(1-\beta t)^{\tau}},\ \ t < \frac{1}{\beta}. \quad (88)$$

Using r=1 in (34), the mean $\mu'_1(x)$ of the GG (α, β, τ) is:

$$E[X] = \mu'_1(x) = \frac{\beta\,\Gamma\left(\alpha+\frac{1}{\tau}\right)}{\Gamma(\alpha)}, \quad (89)$$

and the variance of X is given by:

$$Var[X] = \frac{\beta^2\,\Gamma\left(\alpha+\frac{2}{\tau}\right)}{\Gamma(\alpha)} - \left(\frac{\beta\,\Gamma\left(\alpha+\frac{1}{\tau}\right)}{\Gamma(\alpha)}\right)^2. \quad (90)$$

6 Standard Distributions

Regarding PDFs, there are three types of groups: location groups, scale groups, and location-scale groups. Each group is formed by defining a single PDF, denoted as $f(x)$, known as the standard PDF for that group. Other PDFs within the group are then produced by altering the standard PDF in a specific manner. A basic theorem regarding PDFs states

Theorem 1 [12] Let $f(x)$ be any PDF and let μ and $\sigma > 0$ be any given constants. Then the function:

$$g(x|\mu,\sigma) = \frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right) \quad (91)$$

is a PDF.

The location parameter μ shifts the PDF $f(x)$ and the shape of the graph is unchanged, so the family of PDFs $f(x-\mu)$, indexed by the parameter μ, $\infty < x < \infty$, is called the location family with standard PDF $f(x)$ and μ is called the location parameter for the family. The scale parameter effect, considering σ that parameter, is either to stretch, when $\sigma > 1$ or contract, when $\sigma < 1$, the graph of $f(x)$ while still maintaining the same basic shape of the graph. So, we have

Definition 1 [12] Let $f(x)$ be any PDF. Then for any $\sigma > 0$, the family of the PDFs $\frac{1}{\sigma}f\left(\frac{x}{\sigma}\right)$, *indexed* by the parameter σ, is called the scale family with standard PDF $f(x)$ and σ is called the scale parameter of the family.

Most often when scale parameters are used, $f(x)$ is either symmetric about 0 or positive only for $x > 0$. In these cases, the stretching is either symmetric about 0 or only in the positive direction. But, in the definition, any PDF may be used as the standard.

If we introduce both the location and scale parameters, we shift the graph and the point that was above 0 is now above μ and we have stretching ($\sigma > 1$) or contracting ($\sigma < 1$). The normal and double exponential families are examples of location-scale families. The following theorem, which appears in [12] relates the transformation of the PDF $f(x)$ that defines a location-scale family to the transformation of a random variable Z with PDF z.

Theorem 2 Let $f(x)$ be any PDF, μ any real number, and let σ be any positive real number. Then X is a random variable with PDF $\frac{1}{\sigma}f\left(\frac{x-\mu}{\sigma}\right)$ if and only if there exists a random variable Z with PDF $f(z)$ and $X = \sigma Z + \mu$. An important fact to extract from Theorem 2 is that the distribution of the random variable $Z = \frac{X-\mu}{\sigma}$ is a member of the location-scale family corresponding to $\mu = 0$, $\sigma = 1$.

6.1 Generalized Standardized Gamma Distribution

We thus aim to consider the PDF of the generalized standardized gamma distribution (GSGD) defined by the following PDF, setting $\tau = 1$ and $\beta = 1$ in (5), we have:

$$f(x;1,1,\tau) = \frac{1}{\Gamma(\tau)}x^{\tau-1}e^{-x},\ x \geq 0,\ \tau > 0. \quad (92)$$

This explicit expression allows us to calculate the PDF for any given value of x and τ.

It is desirable to develop models that have a small number of parameters while maintaining a high degree of flexibility for modeling data. In our case the distribution having fewer parameters than the standard gamma distribution (only τ as the shape parameter), it might be easier to estimate and interpret the distribution, particularly when the specific shape is known or expected.

The MGF for the GSGD distribution is given now by:

$$\varphi(t|1,1,\tau) = M_X(t) = E[e^{tX}] = \frac{1}{(1-t)^\tau}, t < 1. \quad (93)$$

The mean of GSGD distribution is obtained by differentiating the MGF to t and then evaluating it at $t = 0$, so μ is given by:

$$\mu = \mu_1 = \frac{d}{dt}\varphi(t|1,1,\tau)|_{t=0} = \frac{d}{dt}\frac{1}{(1-t)^\tau}|_{t=0} = \tau. \quad (94)$$

The second moment of GSGD distribution can be calculated by taking the second derivative of the MGF at $t = 0$

$$\mu_2 = \frac{d^2}{dt^2}\varphi(t|1,1,\tau)|_{t=0} = \frac{d^2}{dt^2}\frac{1}{(1-t)^\tau}|_{t=0} = \tau(\tau+1). \quad (95)$$

The variance of GSGD distribution is given by:

$$Var(X) = \sigma^2 = \mu_2 - \mu^2 = \tau(\tau+1) - \tau^2 = \tau. \quad (96)$$

We also have $\mu_3 = 2\tau$ and $\mu_4 = 3\tau^2 + 6\tau$. For the GSGD distribution, the skewness is $2\tau^{-1/2,}$ and kurtosis can also be expressed in terms of the shape parameter τ as $K = 3 + 6/\tau$.

The characteristic function is:

$$E[e^{itX}] = (1- it)^{-\tau}, \quad (97)$$

and the cumulant generation function is defined as the logarithm of the MGF

$$\psi(t|1,1,\tau) = ln(M_X(t)) = ln\left(\frac{1}{1-t}\right)^\tau. \quad (98)$$

Using the properties of logarithms, we can obtain the cumulant generation function given by:

$$\psi(t|1,1,\tau) = -\tau\ ln(1-t),\ t < 1. \quad (99)$$

The moments μ_r of a distribution can be expressed in terms of cumulants, [2] and [21], considering $k_r = \psi^r(0|1,1,\tau)$, which gives us r^{th} cumulants:

$$k_r = (r-1)!\tau,\ r = 1,...,\ \tau = 1,... . \quad (100)$$

Now, following subsection 4.5, to reduce the distribution to a single shape parameter while maintaining its essential properties, we consider the power transformation $\tau_i' = \tau_i^{1/\alpha_i}$ that combines the original shape parameters. We have a well-known relation between the cumulants of a distribution with parameters $\delta_i, \beta_i, \tau_i'$ in terms of cumulants:

$$\begin{cases} \chi_{1,i}(\ |\delta_i,\beta_i,\tau_i') = \delta_i + \beta_i\,\chi_{1,i}(\ |0,1,\tau_i) \\ \chi_{r,i}(\ |\delta_i,\beta_i,\tau_i') = \beta_i^r\,\chi_{r,i}(\ |0,1,\tau_i),\ \ r=2,3 \end{cases} \quad (101)$$

where $\chi_{1,i}(\ |0,1,\tau_i^0) = g_{1,i}(\tau_i^0)$ and $\chi_{r,i}(\ |0,1,\tau_i^0) = g_{r,i}(\tau_i^0)$ are known. For r=2 we obtain:

$$\chi_{2,i}(\ |\delta_i,\beta_i,\tau_i') = \beta_i^2\,\chi_{2,i}(\ |0,1,\tau_i'), \quad (102)$$

which means that, according to [17], we can express standard cumulants involving three parameters as a linear combination of cumulants involving only one parameter. So, the estimator for β_i is:

$$\widetilde{\beta}_i = \sqrt{\frac{\widetilde{\chi}_{2,i}(0,\delta_i,\tau_i')}{g_{2,i}(\widetilde{\tau'}_i)}},\ \ i=1,\dots,w, \quad (103)$$

where $g_{2,i}(\tau_e{}^0{}_i) = \chi_{2,i}(0,1,\tau_e{}^0{}_i)$ is obtained substituting τ_i' by $\widetilde{\tau'}_i$ in the expression of the second cumulant with δ = 1. Once we estimate β_i and τ_i' we can obtain estimators $\widetilde{\chi}_{r,i}(\ \ |0,\delta_i,\tau_i') = \delta_i^r\,\chi_{r,i}(\ \ |0,1,\widetilde{\tau_i'})$ for $\widetilde{\chi}_{r,i}(\ \ |0,\delta_i,\tau_i'), r \geq 4$, not only for the second and third cumulants but for cumulants of all orders as well.

7 Conclusion

This paper has undertaken a comprehensive exploration of the Generalized Gamma (GG) family shedding light on significant contributions made by various authors and researchers in this field. The GG family, with its inherent versatility and flexibility, is a valuable tool in various domains including economics and meteorology. Explicit formulas for the higher order cumulants have been derived and a number of these findings are novel, namely, the components of the random part of the model can belong to different types, while others serve to complement and augment existing results found within the literature.

Acknowledgment:
This work is funded by national funds through the FCT - Fundação para a Ciência e Tecnologia, I.P., under the scope of the projects: UIDP/MAT/00212/2020, UIDP/00297/2020 and UIDB/04630/2020.

Declaration of Generative AI and AI-assisted Technologies in the Writing Process
The authors wrote, reviewed and edited the content as needed and they have not utilised artificial intelligence (AI) tools. The author(s) take full responsibility for the content of the publication.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The authors equally contributed in the present research, at all stages from the formulation of the problem to the final findings and solution.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

This work is funded by national funds through the FCT - Fundação para a Ciência e Tecnologia, I.P., under the scope of the projects: UIDP/MAT/00212/2020, UIDP/00297/2020 and UIDB/04630/2020.

Conflict of Interest

The authors have no conflicts of interest to declare.