On Kantorovich-type Operators in L_p Spaces

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Abstract: - This note is devoted to the study of a linear positive sequence of operators representing an integral form in Kantorovich's sense. We prove that this sequence converges to the identity operator in $L_p([0,1])$, $p \geq 1$, spaces. By using the r-th order ($r = 1$ and $r \geq 3$) modulus of smoothness measured in these spaces, we establish an upper bound of the approximation error. Also, we point out a connection between the smoothness of α-Hölder ($0 < \alpha \leq 1$) functions and the local approximation property.

Key-Words: - Positive linear operator, Kantorovich-type operator, L_p space, rate of convergence, Bohman-Korovkin theorem, r-modulus of smoothness, K-functional, Hardy-Littlewood maximal operator.

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1 Introduction

The interest in the study of approximation processes has emerged with growing evidence. In this direction, the investigation of the linear methods of approximation, which are given by sequences of linear and positive operators, has become a firmly rooted part.

In this note, we focus on integral operators in Kantorovich sense. We remind that the genuine Kantorovich polynomials [1] are given as follows:

$$(K_n f)(x) = (n+1)\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt, \quad x \in [0,1], \tag{1}$$

for every $n \geq 1$ and $f \in L_p([0,1])$. $K_n f$ is a modified version of the famous Bernstein polynomial:

$$(B_n f)(x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right)$$

$$p_{n,k}(x) = \binom{n}{k}x^k(1-x)^{n-k}, \; 0 \leq k \leq n, \tag{2}$$

replacing the values of the function over the net $(k/n)_{0\leq k\leq n}$ by means of an integral mean. The use of the integral is welcome because, in practical situations, more information is usually known around a point than exactly at that point.

The main approximation property of the operators defined by (1) is the following: $(K_n f)_{n\geq 1}$ converges to f in $L_p([0,1])$ for every f belonging to the Lebesgue spaces $L_p([0,1])$, $p \geq 1$.

Using as a reference model for this kind of construction, over time numerous discrete linear approximation processes were extended in the same way. For illustration, we mention only a few classic constructions: Szász-Mirakjan-Kantorovich operators [2], Baskakov-Kantorovich operators, [3], Stancu-Kantorovich operators [4], Chlodovsky-Kantorovich operators [5].

In the last decades, various generalizations of Kantorovich operators have also been designed. They target the network of used nodes and the basis of functions incorporated in the construction. Their usefulness lies in the ability to approximate functions from several function spaces such as polynomial weighted function spaces, exponential spaces, BV-spaces, L^p-spaces ($p \geq 1$), or Orlicz spaces. Among the most recent significant papers (years 2023-2024) we mention [6], [7], [8], the selection is subjective.

The main goal of our work is to study a general class of Kantorovich-type operators in $L_p([0,1])$, $1 \leq p < \infty$, the space of all p-th power integrable functions on [0,1]. The speed of convergence to the identity operators is achieved.

2 The Operators

Since any compact $[a,b]$ is isomorphic to [0,1], in what follows we take into account only the interval $I=[0,1]$. Let $(x_{n,k})_{k\in I_n}$ be a net on I, where $I_n \subseteq \mathbf{N}$ is a set of indices. We also consider that the net has equidistant nodes, meaning that for each $n\in\mathbf{N}$ and $\{x_{n,k}, x_{n,k+1}\}\subset I$, $k\in I_n$,

$$x_{n,k+1}-x_{n,k}=p_n,\ n\in I_n, \tag{3}$$

where $\lim_n p_n = 0$. The most encountered case is described by $x_{n,k}=k/n$. We indicate some other variants used in the choice of nodes to define Kantorovich type operators, not necessary for functions defined on a compact.

(i) For integral form of K. Balazs operators in [9] was considered $x_{n,k}=kn^{-\beta}$, where $0<\beta<1$ is fixed. We notice that condition (3) is fulfilled.

(ii) In [10] Altomare and Leonessa illustrate their main results by using in an example the subintervals $\left[\frac{k+a_n}{n+1},\frac{k+b_n}{n+1}\right]$, $0\le k\le n$ and $(a_n)_{n\ge1}$, $(b_n)_{n\ge1}$ are real sequences satisfying $0\le a_n<b_n<1$.

(iii) Since Quantum Calculus began to be widely used in the construction of linear positive operators, for q-Kantorovich-Bernstein operators in [11] the authors use the nodes $([k]_q/[n+1]_q)_{k=\overline{0,n}}$, $q\in(0,1)$. We recall $[0]_q=0$ and $[k]_q=\sum_{j=0}^{k-1}q^j$, $k\ge1$.

For q-Kantorovich operators which generalize the discrete Stancu operators, in [12] the nodes $(([k]_q+\alpha)/([n+1]_q+\beta))_{k=\overline{0,n}}$, $q\in(0,1)$ and $0\le\alpha\le\beta$ were used. In both cases the nodes are not equidistant.

Our aim is to investigate the operators:

$$(L_n^* f)(x)=\frac{1}{p_n}\sum_{k\in I_n}\lambda_{n,k}(x)\int_{x_{n,k}}^{x_{n,k+1}} f(t)dt,\ x\in I, \tag{4}$$

where $\lambda_{n,k}\in C(I)$, $\lambda_{n,k}\ge0$ for each $(n,k)\in\mathbf{N}\times I_n$ and $f\in L_p(I)$.

Obviously, these integral operators are associated with the discrete operators defined as follows:

$$(L_n f)(x)=\sum_{k\in I_n}\lambda_{n,k}(x) f(x_{n,k}),\ x\in I. \tag{5}$$

The reference standard of a such construction is given by the operators indicated at (2).

Regarding these operators we consider that the following conditions are met:

$$\sum_{k\in I_n}\lambda_{n,k}=e_0,\ \sum_{k\in I_n}x_{n,k}\lambda_{n,k}=e_1,$$
$$\left|\sum_{k\in I_n}x_{n,k}^2\lambda_{n,k}-e_2\right|\le q_n, \tag{6}$$

where e_j, $j\in\mathbf{N}_0=\{0\}\cup\mathbf{N}$, is the monomial of degree j and $(q_n)_{n\ge1}$ is a sequence of positive numbers.

The first two above conditions ensure that the operators L_n, $n\in\mathbf{N}$, defined by (5) reproduce the affine functions, a characteristic common to the numerous classes of linear positive operators of the discrete type. To become $(L_n)_{n\ge1}$ an approximation process, it is enough to impose

$$\lim_{n\to\infty} q_n = 0. \tag{7}$$

Thus, based on the Bohman-Korovkin theorem, [13], [14], the uniform convergence of the sequence $(L_n f)_{n\ge1}$ to f is ensured for any $f\in C(I)$.

In [15] two classes of Kantorovich type operators were investigated achieving a comparison of the approximation error between them in the particular case of Banach space $C(I)$. One of the classes is indicated at (4). In this note we study approximation properties of L_n^*, $n\ge1$, operators for functions belonging to $L_p(I)$, $p\ge1$. Based on (6), we easily deduce the following identities:

$$(L_n^* e_0)(x)=1, \tag{8}$$

$$(L_n^* e_1)(x)=x+\frac{p_n}{2},$$
$$(L_n^* e_2)(x)=(L_n e_2)(x)+p_n x+\frac{p_n^2}{3},\ x\in[0,1]. \tag{9}$$

Considering the function:

$$\varphi_x(t)=t-x,\ (t,x)\in I\times I,$$

for each $n\in\mathbf{N}$, the first and second-order central moments are respectively given by:

$$\begin{cases}\mu_{n,1}^*(x):=(L_n^*\varphi_x)(x)=\dfrac{p_n}{2},\\ \mu_{n,2}^*(x):=(L_n^*\varphi_x^2)(x)=(L_n e_2)(x)-x^2+\dfrac{p_n^2}{3}\\ \qquad\le q_n+\dfrac{p_n^2}{3},\ x\in I.\end{cases} \tag{10}$$

Based on relations (8), (9) and (7), the Bohman-Korovkin criterion ensures:

$$\lim_{n\to\infty} \| L_n^* f - f \|_{C(I)} = 0, \ f \in C(I), \quad (11)$$

where $\|\cdot\|_{C(I)}$ is the usual sup-norm,

$$\| h \|_{C(I)} = \sup_{t\in I} | h(t) |, \ h \in C(I).$$

3 Results

The operator norm of L_n^* will be denoted by $\| L_n^* \|$, the operator being considered from $L_p(I)$ to $L_p(I)$. The key assumption in the study of this class is the existence of a constant $M > 0$ such that

$$\| L_n^* \| \le M, \ n \in \mathbf{N}. \quad (12)$$

Remark 1. For the genuine Kantorovich operator defined by (1), we have $M = 1$, see, e.g., [16] or [17].

Our first result shows that the Kantorovich operators are an approximation process on the space $L_p(I)$ endowed with the norm $\|\cdot\|_{L_p(I)}$, $1 \le p < \infty$,

$$\| h \|_{L_p(I)} = \left(\int_0^1 | h(t) |^p \, dt \right)^{1/p}.$$

Theorem 1. *Let* L_n^*, $n \ge 1$, *be defined by* (4) *such that* (6), (7) *and* (12) *are fulfilled. For any* $f \in L_p(I)$, $1 \le p < \infty$,

$$\lim_{n\to\infty} \| L_n^* f - f \|_{L_p(I)} = 0$$

holds.

Proof. Let f a function belonging to the space $L_p(I)$. We will prove

$$\forall \varepsilon > 0, \ \exists n_0 \in \mathbf{N}, \ \forall n \ge n_0, \ \| L_n^* f - f \|_{L_p(I)} < \varepsilon. \quad (13)$$

Let $\varepsilon > 0$ be arbitrarily fixed. Since the space $C(I)$ is dense in $L_p(I)$ with respect to the natural norm, for the function f there is $g_f \in C(I)$, such that:

$$\| f - g_f \|_{L_p(I)} < \varepsilon.$$

Also, relation (11) implies:

$$\exists n_0 \in \mathbf{N}, \ \forall n \ge n_0, \ \| L_n^* g_f - g_f \|_{C(I)} < \varepsilon.$$

Consequently, for any $n \ge n_0$, we can write:

$$\| L_n^* f - f \|_{L_p(I)} \le \| L_n^* f - L_n^* g_f \|_{L_p(I)}$$
$$+ \| L_n^* g_f - g_f \|_{L_p(I)} + \| g_f - f \|_{L_p(I)}$$
$$< \| L_n^* (f - g_f) \|_{L_p(I)} + \| L_n^* g_f - g_f \|_{C(I)} + \varepsilon$$
$$< \| L_n^* \| \, \| f - g_f \|_{L_p(I)} + 2\varepsilon < (M + 2)\varepsilon.$$

We used (12) and the inequality:

$$\| h \|_{L_p(I)} \le \| h \|_{C(I)}, \ h \in C(I).$$

The proof of (13) is finished. □

Further, set $W_{p,r}(I)$, $r \in \mathbf{N}$, the space which consists of those functions defined on I for which the first $r-1$ derivatives are absolutely continuous on I and the r-th derivative belongs to $L_p(I)$.

For the evaluation of the speed of convergence, we recall two notions, with the aim of making the exposition self-explanatory. The r-th order modulus of smoothness of f, $r \in \mathbf{N}$, measured in $L_p(I)$ spaces, $p \ge 1$, is given by:

$$\omega_r(f, t)_p = \sup_{0<h\le t} \| \Delta_h^r f \|_{L_p(I)}, \ f \in L_p(I), \ t > 0,$$

where

$$\Delta_h^r f(x) = (E^h - I)^r f(x),$$

E^h representing the translation operator. For any $k \le r$, $(E^h)^k f(x) = f(x + kh)$ if x, $x + rh$ belong to I and becomes zero otherwise.

The K-functional of $f \in L_p(I) := X$ for each $t > 0$ is defined by:

$$K(t, f; X, Y)$$
$$= \inf\{ \| f - g \|_X + t(\| g \|_X + \| g^{(r)} \|_X); g \in Y \},$$

where $Y := W_{p,r}(I)$, [18]. Also in [19] is considered the modified K'-functional as follows:

$$K'(t, f; X, Y) = \inf\{ \| f - g \|_X + t \| g^{(r)} \|_X ; g \in Y \}.$$

The following connections between these functionals and modulus of smoothness $\omega_r(f, \cdot)_p$ are valid [19]:

$$K'(t, f; X, Y) \le K(t, f; X, Y)$$
$$\le \min\{1, t\} \| f \|_X + 2K'(t, f; X, Y), \quad (14)$$

and

$$c_1 \omega_r(f, t)_p \le K'(t^r, f; X, Y) \le c_2 \omega_r(f, t)_p, \quad (15)$$

$0 < t \le 1$, where $c_1 = c_1(p, r)$, $c_2 = c_2(p, r)$ are positive constants.

Theorem 2. *Let* L_n^*, $n \ge 1$, *be defined by* (4) *such that* (6), (7) *and* (12) *are fulfilled. For any* $f \in L_p(I)$, $p > 1$ *and n sufficiently large:*

$$\| L_n^* f - f \|_{L_p(I)} \le \tilde{C}\omega_1\left(f, \sqrt{\xi_n}\right)_p, \quad (16)$$

where $\tilde{C}$ *is a constant and* $\xi_n = q_n + \frac{p_n^2}{3}$, *see* (10).

Proof. At first step we prove

$$\| L_n^* g - g \|_{L_p(I)} \le A_p \sqrt{\xi_n} \, \| g' \|_{L_p(I)}, \quad (17)$$

for any $g \in W_{p,1}(I)$ and $p > 1$, where A_p is a constant depending on *p*. To achieve this, we use the Hardy-Littlewood maximal operator *M* defined for any $h \in L_p(I)$ and $p > 1$ as follows:

$$(Mh)(x) = \sup_{\substack{u \in I \\ u \ne x}} \frac{1}{|u - x|} \left| \int_x^u g(t) dt \right|. \quad (18)$$

For $p > 1$ it is bounded in $L_p(I)$,

$$\| Mh \|_{L_p(I)} \le A_p \| h \|_{L_p(I)}, \quad (19)$$

A_p being a constant depending only on *p*, see, e.g., [20]. By using (18) we can write:

$$\begin{aligned} &|(L_n^* g)(x) - g(x)| \\ &= \frac{1}{p_n} \left| \sum_{k \in I_n} \lambda_{n,k}(x) \int_{x_{n,k}}^{x_{n,k+1}} (g(u) - g(x)) du \right| \\ &\le \frac{1}{p_n} \sum_{k \in I_n} \lambda_{n,k}(x) \left| \int_{x_{n,k}}^{x_{n,k+1}} g'(t) dt \right| du \\ &\le \frac{1}{p_n} (Mg')(x) \sum_{k \in I_n} \lambda_{n,k}(x) \int_{x_{n,k}}^{x_{n,k+1}} |u - x| du. \end{aligned}$$

Using Cauchy-Schwarz inequality for both integrals and sums, we get:

$$\begin{aligned} &|(L_n^* g)(x) - g(x)| \\ &\le (Mg')(x) \left(\sum_{k \in I_n} \lambda_{n,k}(x) \right)^{1/2} \\ &\times \left(\frac{1}{p_n} \sum_{k \in I_n} \lambda_{n,k}(x) \int_{x_{n,k}}^{x_{n,k+1}} (u - x)^2 dx \right)^{1/2} \\ &= (Mg')(x) (\mu_{n,2}^*(x))^{1/2}, \end{aligned}$$

and consequently

$$\| L_n^* g - g \|_{L_p(I)} \le \| Mg' \|_{L_p(I)} \left(q_n + \frac{1}{3} p_n^2 \right)^{1/2},$$

see (10). The relation (19) leads us to the inequality (17).

At the second step, by using (17) and (12) for any $f \in L_p(I)$ and $g \in W_{p,q}(I)$, we can write

$$\begin{aligned} &\| L_n^* f - f \|_{L_p(I)} \\ &\le \| L_n^* (f - g) - (f - g) \|_{L_p(I)} + \| L_n^* g - g \|_{L_p(I)} \\ &\le M \| f - g \|_{L_p(I)} + A_p \sqrt{\xi_n} \| g' \|_{L_p(I)} \\ &\le C(\| f - g \|_{L_p(I)} + \sqrt{\xi_n} \| g' \|_{L_p(I)}) \|, \end{aligned}$$

where $C = \max\{M, A_p\}$. Taking the infimum over all $g \in W_{p,1}(I)$ and using (15) for $r = 1$, we obtain

$$\begin{aligned} \| L_n^* f - f \|_{L_p(I)} &\le CK'\left(\sqrt{\xi_n}, f; L_p(I), W_{p,1}(I)\right) \\ &\le \tilde{C}\omega_1\left(f, \sqrt{\xi_n}\right)_p, \end{aligned}$$

where $\tilde{C}$ is a constant depending on *M* and *p*. Since $\lim_{n \to \infty} \xi_n = 0$, for *n* large enough the condition $\xi_n \le 1$ is met.

The theorem is completely proved. □

Remark 2. The approach of the above proof is not valid for $p = 1$ because the bounding of the maximal operator indicated in (19) fails. For the case, $p = 1$ or $p > 1$, $\omega_r(f, \cdot)_p$ can be used, where $r \ge 3$. Our statement is based on the following result.

Proposition 1. [21] *Let* $\{L_n\}$ *be a uniformly bounded sequence of positive linear operators from* $L_p[a,b]$ *into* $L_p[c,d]$, $1 \le p < \infty$, $a \le c < d \le b$. *If* $r \ge 3$ *is an integer, then, for* $f \in L_p[a,b]$,

$$\| f - L_n f \|_p \le C_p (\| f \|_p \lambda_{n,p} + \omega_r(f, \lambda_{n,p}^{1/r})_p), \quad (20)$$

where the L_p *norm of the left is taken over* $[c,d]$, $C_p > 0$ *is independent of f and n and* $\omega_r(f, \cdot)_p$ *is the r-th order modulus of smoothness of f measured in* $L_p[a,b]$.

We specify, in [12], $\lambda_{n,p} = \max_{i=0,1,2} \| L_n e_i - e_i \|_p$ and in [21] it was assumed that $\lambda_{n,p} \to 0$ as $n \to \infty$.

In our case, $(L_n^*)_{n \ge 1}$ is a uniformly bounded sequence, see (12). Also,

$$\lambda_{n,p} \le \max\left\{ \frac{1}{2} p_n, q_n + \frac{1}{3} p_n^2 \right\} := \eta_n,$$

see (10), and according to our assumptions we have

$$\lim_{n \to \infty} \eta_n = 0.$$

Since $\omega_r(f;t)_p$ is a non-decreasing function in t for each f, as in (20), a similar degree of L_p approximation $(p \geq 1)$ for L_n^*, $n \geq 1$, takes place.

$$\| L_n^* f - f \|_{L_p(I)} \leq \tilde{C}(\| f \|_{L_p(I)} \eta_n + \omega_r(f, \eta_n^{1/r})_p),$$

$f \in L_p(I)$, where $\tilde{C}$ is a positive constant depending on M, r, p and n is sufficiently large.

The proof is performed via K and K' functionals with their connections to $\omega_r(f,\cdot)_p$ ($r \geq 3$ integer, $p \geq 1$) and well as the formula [22].

The next objective is to investigate the relation between the local smoothness of function and the local approximation. A function $f \in C(I)$ is locally α-Hölder continuous $(0 < \alpha \leq 1)$ on $E \subset [0,1]$ if it satisfies the condition:

$$| f(x) - f(y) | \leq M | x - y |^\alpha, \ (x, y) \in I \times E, \tag{21}$$

where M is a constant depending only on α and f.

In what follows we denote this class of functions by $H(\alpha; I, E)$.

Set $d(x, E) = \inf\{| x - t |: t \in E\}$, the distance between x and E.

Theorem 3. *Let* L_n^*, $n \geq 1$, *be defined by* (4) *such that* (6), (7) *and* (12) *are fulfilled. For a given* $\alpha \in (0,1]$ *and* $E \subset I$,

$$|(L_n^* f)(x) - f(x)| \leq M(\xi_n^{\alpha/2} + 2d^\alpha(x, E)),$$
$$x \in I, \ f \in H(\alpha; I, E),$$

where ξ_n *is given at Theorem 2.*

Proof. Hölder's inequality corroborated with (8) and (10) imply:

$$L_n^*(| e_1 - xe_0 |^\alpha, x) \leq (\mu_{n,2}^*)^{\alpha/2}(x) \leq \xi_n^{\alpha/2}, \tag{22}$$

$x \in [0,1]$.

Since $f \in C(I)$, (21) holds for any $x \in I$ and $y \in \bar{E}$, the closure of E. Let $(x, x_0) \in I \times \bar{E}$ such that $d(x, E) = | x - x_0 |$. We get:

$$|(L_n^* f)(x) - f(x)|$$
$$\leq L_n^*(| f - f(x_0)e_0 |^\alpha, x) + | f(x) - f(x_0) |$$
$$\leq M(L_n^*(| e_1 - x_0 e_0 |^\alpha, x) + | x - x_0 |^\alpha).$$

Since $\alpha \in (0,1]$, for any $t \in I$,

$$| t - x_0 |^\alpha \leq | t - x |^\alpha + | x - x_0 |^\alpha$$

takes place. Based on the above inequalities and relation (22), the conclusion follows. □

Remark 3. If $E = I$, we obtain

$$\| L_n^* f - f \|_{L_p(I)} \leq M\xi_n^{\alpha/2}, \ f \in H(\alpha; I).$$

4 Conclusion

A class of general Kantorovich-type operators was examined in this paper. Due to the generality of the construction derived from fixing the nodes and the functions that compose operators, the results do not seem spectacular at first sight, but it could be proved that the sequence forms an approximation process in the $L_p([0,1])$, $p \geq 1$, spaces. At the same time, the approximation error was established by using smoothness moduli of order r, $\omega_r(f,\cdot)_p$, $r = 1$ and $r \geq 3$, respectively. Usually, for classical operators, the magnitude of error is of the order $O(1/\sqrt{n})$. Through our construction, the magnitude is more flexible, namely $O(\eta_n^{1/r})$. The rate of convergence of operators on the class of α-Hölder functions was also examined.

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